Chapter III

CHAPTER – III
INTERVAL FUZZY NUMBERS AND INTERVAL FUZZY NUMBER MATRICES

Definition : 3.1

An **interval number** is defined as $\hat{A} = [a_L, a_R] = \{a : a_L \leq a \leq a_R\}$ where, a_L and a_R are the real numbers called the left end point and right end point respectively of the interval $\hat{A}$.

An **interval fuzzy number** is defined as $\hat{A} = [a_L, a_R] = \{a : a_L \leq a \leq a_R\}$ where $a_L, a_R \in [0, 1]$.

Definition : 3.2

A crisp real number k may be considered as a degenerate interval [k, k].

Let $\hat{A} = [a_L, a_R]$ and $\hat{B} = [b_L, b_R]$ be two interval fuzzy numbers. Different **binary operations** between $\hat{A}$ and $\hat{B}$ are defined as below :

(i) Addition :

$$\hat{A} + \hat{B} = [a_L + b_L, a_R + b_R]$$

(ii) Multiplication :

The product of two interval fuzzy numbers $\hat{A} = [a_L, a_R]$ and $\hat{B} = [b_L, b_R]$ is given by

$$\hat{A}\hat{B} = [\min\{a_L.b_L, a_R.b_L, a_L.b_R, a_R.b_R\}, \max\{a_L.b_L, a_R.b_L, a_L.b_R, a_R.b_R\}]$$

If $\hat{A}$ and $\hat{B}$ both are positive then $\hat{A}\hat{B} = [a_L.b_L, a_R.b_R]$.

The negative of an interval fuzzy number $\hat{A} = [a_L, a_R]$ is given by $-A = [-a_R, -a_L]$.

(iii) Subtraction :

The subtraction of two interval fuzzy numbers $\hat{A} = [a_L, a_R]$ and $\hat{B} = [b_L, b_R]$ is given by

$$\hat{A} - \hat{B} = [a_L - b_R, a_R - b_L]$$

Definition : 3.3

A matrix of order n x n is said to be an **interval fuzzy number matrix (IFNM)** if all its elements are the interval fuzzy numbers.

Definition : 3.4

Let $A = (\hat{a}_{ij})$ and $B = (\hat{b}_{ij})$ be two interval fuzzy number matrices of same order. Then we have the following,

(i) $A \oplus B = (\hat{a}_{ij} + \hat{b}_{ij})$

(ii) $A \ominus B = (\hat{a}_{ij} - \hat{b}_{ij})$

(iii) If $A = (\hat{a}_{ij})_{mxn}$ and $B = (\hat{b}_{ij})_{nxp}$, then $AB = (\hat{c}_{ij})_{mxp}$, where $\hat{c}_{ij} = \sum_{k=1}^{n} \hat{a}_{ik}.\hat{b}_{kj}$ i = 1, 2, ..., m and j = 1, 2, ..., p.

(iv) $A^T = (\hat{a}_{ji})$ (the transpose of A)

(v) $kA = (k\,\hat{a}_{ij})$, where k is a scalar.

Definition : 3.5

An interval fuzzy number matrix is said to be a **pure null interval fuzzy number matrix** if all its elements are zero. i.e., if all its elements are $< 0, 0 >$. This matrix is denoted by 0.

Definition : 3.6

An IFNM is said to be a **fuzzy null interval fuzzy number matrix** if all its elements are of the form $a_{ij} = < 0, \xi >$, where $\xi \neq 0$.

Definition : 3.7

A square IFNM is said to be a **pure unit interval fuzzy number matrix** if $a_{ii} = [1, 1] = < 1, 0 >$ and $a_{ij} = [0, 0] = < 0, 0 >$, $i \neq j$, for all i, j. It is denoted by I.

Definition : 3.8

A square IFNM is said to be a **fuzzy unit interval fuzzy number matrix** if $a_{ii} = < 1, \xi_1 >$ and $a_{ij} = < 0, \xi_2 >$, for $i \neq j$ and $i, j \in 1, 2, \ldots, n$ where ξ_1, $\xi_2 \neq 0$.

Definition : 3.9

Let B be an interval fuzzy number matrix of order n x n. Then B is said to be **orthogonal** if $B.B^T = B^T.B = I_n$, where I_n denotes pure unit interval fuzzy number matrix of order n x n and the operation '.' denotes the multiplication of interval fuzzy number matrices.

Theorem : 3.10

There exists no purely orthogonal interval fuzzy number matrix other than the pure unit interval fuzzy number matrix.

Definition : 3.11

Let B be an interval fuzzy number matrix of order n x n. Then if $B.B^T = I_n$ or $B^T.B = I_n$, where I_n is fuzzy unit IFNM of n^{th} order then B is called a **fuzzy orthogonal interval fuzzy number matrix**, i.e., If $B.B^T$ or $B^T.B$ is of the form $(a_{ij})_{nxn}$ where $a_{ii} = < 1, \xi_1 >$ and $a_{ij} = < 0, \xi_2 >$ for all $i \neq j$; $i, j \in 1, 2, \ldots, n$ and where $\xi_1 \neq 0$ and $\xi_2 \neq 0$.

Theorem : 3.12

If A and B be any interval fuzzy number matrices then det(A) det(B) $\neq$ det(AB).

Lemma : 3.13

If a and b are two real numbers and x be an interval, then $(a + b)x = ax + bx$, if $a \geq 0$, $b \geq 0$ and $(a + b)x \neq ax + bx$, otherwise.

i.e., in general, for n real numbers a_i, i = 1, 2, ..., n and for an interval x, $\left(\sum_{i=1}^{n} a_i\right)x = \sum_{i=1}^{n} a_i\, x$ if $a_i \geq 0$ for all i = 1, 2, ..., n ; and $\left(\sum_{i=1}^{n} a_i\right)x \neq \sum_{i=1}^{n} a_i\, x$, otherwise.

Theorem : 3.14

If A be a real square matrix, such that $AA^T = cI$, where c is a non-zero real number and I is the unit matrix of order equal to the order of A ; and x be an interval fuzzy number such that $x^2 = \frac{1}{c}[1 - \xi, 1 + \xi]$, $0 \leq \xi \leq 1$, then $\overline{A} = xA$ is a fuzzy orthogonal interval fuzzy number matrix.

Proof

Let, $A = (a_{ij})_{nxn}$

Then $A.A^T = (b_{ij})_{nxn}$ where, $b_{ij} = \sum_{k=1}^{n} a_{ik}\, a_{jk}$.

Since, $A.A^T = cI$

$$\text{then } b_{ij} = \begin{cases} c, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Thus, for a non-diagonal element b_{pq} of $A.A^T$, there exist two index sets $\wedge_1$ and $\wedge_2$ such that $b_{pq} = \sum_{k=\wedge_1} a_{pk}\ a_{qk} - \sum_{k=\wedge_2} a_{pk}\, a_{qk}$ where

$\sum_{k=\wedge_1} a_{pk}\ a_{qk} = \sum_{k=\wedge_2} a_{pk}\, a_{qk}$.

Now, $\overline{A} = x\,A = (x\,a_{ij})_{nxn} = (\overline{a}_{ij})_{nxn}$ (say) and $\overline{A}.\overline{A}^T = (\overline{b}_{ij})_{nxn}$ where,

$$\overline{b}_{ij} = \sum_{k=1}^{n} \overline{a}_{ik}\,\overline{a}_{jk}.$$

Now, for a diagonal element $\overline{b}_{ii}$ of $\overline{A}.\overline{A}^T$,

$$\overline{b}_{ii} = \sum_{k=1}^{n} x\,a_{ik}.x\,a_{ik} = \sum_{k=1}^{n} x^2\,a_{ik}^2 = x^2 \sum_{k=1}^{n} a_{ik}^2$$

(using lemma 3.13, since $a_{ik}^2 \geq 0$)

$= x^2\,b_{ii}$

$= \frac{1}{c}\,[1-\xi,\,1+\xi]\,.\,c$

$= [1-\xi,\,1+\xi]$

$\therefore\ \overline{b}_{ii} = [1-\xi,\,1+\xi].$

For the non-diagonal element $\overline{b}_{pq}$ of $\overline{A}.\overline{A}^T$

$$\overline{b}_{pq} = \sum_{k=1}^{n} x\,a_{pk}\ x\,a_{qk}\,.\,x\,a_{pk}\ a_{qk}$$

$$= \sum_{k=1}^{n} x^2\,a_{pk}^2\,a_{pk}^2$$

$$= \sum_{k=\wedge_1}^{n} x^2\,a_{pk}\,a_{qk} - \sum_{k=\wedge_2}^{n} x^2\,a_{pk}\,a_{qk}$$

Let, $\sum_{k=\wedge_1} x^2\,a_{pk}\,a_{qk} = [\gamma,\,\delta]$

Then $\sum_{k=\wedge_2} x^2\,a_{pk}\,a_{qk}$ is also equal to $[\gamma,\,\delta]$, so that

$\overline{b}_{pq} = [\gamma,\,\delta] - [\gamma,\,\delta]$

$= [\gamma-\delta,\,\delta-\gamma]$

$= [-\xi_1,\,\xi_1]$, where, $\xi_1 > 0$.

Hence proved.