Chapter III

CHAPTER – III

CONNECTEDNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

In this chapter we discuss the concept of (r, s)-connectedness in intuitionistic fuzzy topological spaces introduced by Kim and Abbas [21]. The authors [21] have introduced the concept of (r, s)-separated fuzzy sets in an intuitionistic fuzzy topological spaces using which they [21] have defined the concept of (r, s)-connected fuzzy sets. Some interesting characterizations of these sets are obtained. Moreover, properties of (r, s)-connected fuzzy sets and (r, s)-components which are analogous to the corresponding properties in general topological spaces are discussed. Finally, stratification of an intuitionistic fuzzy topological space is obtained and the authors [21] have shown that every (r, s)-component in an intuitionistic fuzzy topological space is a (r, s)-component in the stratification of it. First let us give the preliminary definitions and results needed for our discussion.

Section 3.1

Preliminary definitions and results

Let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X is denoted by I^X.

Definition : 3.1.1

An **intuitionistic gradation of openness (IGO,** for short) on X is an ordered pair $(\mathcal{T}, \mathcal{T}^*)$ of mappings from I^X to I such that

(IGO1) $\mathcal{T}(\lambda) + \mathcal{T}^*(\lambda) \leq 1$ for all $\lambda \in I^X$

(IGO2) $\mathcal{T}(\bar{0}) = \mathcal{T}(\bar{r}) = 1$ and $\mathcal{T}^*(\bar{0}) = \mathcal{T}^*(\bar{1}) = 0$,

(IGO3) $\mathcal{T}(\lambda_1 \wedge \lambda_2) \geq \mathcal{T}(\lambda_1) \wedge \mathcal{T}(\lambda_2)$ and

$\mathcal{T}^*(\lambda_1 \wedge \lambda_2) \leq \mathcal{T}^*(\lambda_1) \vee \mathcal{T}^*(\lambda_2)$, for each $\lambda_i \in I^X$, i = 1, 2

(IGO4) $\mathcal{T}(\vee_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} \mathcal{T}(\lambda_i)$ and

$\mathcal{T}^*(\vee_{i\in\Delta} \lambda_i) \leq \vee_{i\in\Delta} \mathcal{T}^*(\lambda_i)$ for each $\lambda_i \in I^X$, $i \in \Delta$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called an **intuitionistic fuzzy topological space** (IFTS, for short). $\mathcal{T}$ and $\mathcal{T}^*$ may be interpreted as **gradation of openness** and **gradation of nonopenness**, respectively.

An IFTS $(X, \mathcal{T}, \mathcal{T}^*)$ is called **stratified** if

(IS) $\mathcal{T}(\bar{\alpha}) = 1$ and $\mathcal{T}^*(\bar{\alpha}) = 0$ for each $\alpha \in I$.

Let $(\mathcal{U}, \mathcal{U}^*)$ and $(\mathcal{T}, \mathcal{T}^*)$ be IGO's on X. We say $(\mathcal{U}, \mathcal{U}^*)$ is **finer** than $(\mathcal{T}, \mathcal{T}^*)$, ($(\mathcal{T}, \mathcal{T}^*)$ is **coarser** than $(\mathcal{U}, \mathcal{U}^*)$) if $\mathcal{T}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*(\lambda) \leq \mathcal{U}*(\lambda)$ for all $\lambda \in I^X$.

Definition : 3.1.2

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS. A function $C : I^X \times I_0 \times I_1 \to I^X$ is called an **intuitionistic closure operator** if for $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$ with $r + s \leq 1$, it satisfies the following conditions :

(C1) $C(\bar{0}, r, s) = \bar{0}$

(C2) $\lambda \leq C(\lambda, r, s)$

(C3) $C(\lambda, r, s) \vee C(\mu, r, s) = C(\lambda \vee \mu, r, s)$

(C4) $C(\lambda, r, s) \leq C(\lambda, r_1, s_1)$ if $r \leq r_1$, $s \geq s_1$ with $r_1 + s_1 \leq 1$

(C5) $C(C(\lambda, r, s), r, s) = C(\lambda, r, s)$.

Theorem : 3.1.3 (Chang [11])

Let C be an intuitionistic closure operator on X. Define the functions $\mathcal{T}_C, \mathcal{T}_C^* : I^X \to I$ by

$$\mathcal{T}_C(\lambda) = \vee\{r \in I_0 \,/\, C(\bar{1} - \lambda, r, s) = \bar{1} - \lambda\},$$

$$\mathcal{T}_C^*(\lambda) = \wedge\{s \in I_1 \,/\, C(\bar{1} - \lambda, r, s) = \bar{1} - \lambda\}$$

Then $(\mathcal{T}_C, \mathcal{T}_C^*)$ is an IGO on X.

Remark : 3.1.4

$C_{\mathcal{T},\mathcal{T}^*}$ is an increasing function since $\lambda_1 \leq \lambda_2$

$$\Rightarrow C_{\mathcal{T},\mathcal{T}^*}(\lambda_1, r, \rho) \leq C_{\mathcal{T},\mathcal{T}^*}(\lambda_2, r, \rho).$$

Theorem : 3.1.5

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS. Then for each $r \in I_0$, $s \in I_1$, $\lambda \in I^X$, we define an operator $C_{\mathcal{T},\mathcal{T}^*} : I^X \times I_0 \times I_1 \to I^X$ as follows :

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \wedge\{\mu \in I^X / \lambda \leq \mu, \mathcal{T}(\bar{1} - \mu) \geq r, \mathcal{T}^*(\bar{1} - \mu) \leq s\}$$

Then (i) $C_{\mathcal{T},\mathcal{T}^*}$ is an intuitionistic closure operator

(ii) $\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}$ and $\mathcal{T}^*_{C_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}^*$

Proof

(i) Conditions (C1), (C2), (C3), (C4) and (C5) can be verified using the definition of $C_{\mathcal{T},\mathcal{T}^*}$. Hence $C_{\mathcal{T},\mathcal{T}^*}$ is an intuitionistic closure operator.

(ii) By definition

$\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}}(\mu) = \vee\{r \in I_0 / C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s) = \bar{1} - \mu\}$

Let $\mathcal{T}(\mu) = r_0$ and $\mathcal{T}^*(\mu) = s_0$

Then $\mathcal{T}(\bar{1} - (\bar{1} - \mu)) = r_0$ and

$\mathcal{T}^*(\bar{1} - (\bar{1} - \mu)) = s_0$

Also $\bar{1} - \mu \leq \bar{1} - \mu$

$\therefore \bar{1} - \mu \in \{\lambda \in I^X / \bar{1} - \mu \leq \lambda, \mathcal{T}(\bar{1} - \lambda) \geq r_0, \mathcal{T}^*(\bar{1} - \lambda) \leq s_0\}$

$\therefore \wedge\{\lambda \in I^X / \bar{1} - \mu \leq \lambda, \mathcal{T}(\bar{1} - \lambda) \geq r_0, \mathcal{T}^*(\bar{1} - \lambda) \leq s_0) = \bar{1} - \mu.$

i.e. $C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r_0, s_0) = \bar{1} - \mu$ (1)

$\therefore r_0 \in \{r \in I_0 / C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s_0) = \bar{1} - \mu\}$

$\therefore \vee\{r \in I_0 / C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s_0) = \bar{1} - \mu\} \geq r_0$

i.e., $\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}}(\mu) \geq r_0$

$= \mathcal{T}(\mu)$ (2)

Similarly by (1),

$s_0 \in \{s \in I_0 \,/\, C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r_0, s) = \bar{1} - \mu\}$

$\therefore \wedge\{s \in I_0 \,/\, C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r_0, s) = \bar{1} - \mu\} \leq s_0$

i.e., $\mathcal{T}^*_{c_{\mathcal{T},\mathcal{T}^*}}(\mu) \leq s_0$

$= \mathcal{T}^*(\mu)$ (3)

From (2) and (3), we get

$\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}} \geq \mathcal{T}$ and $\mathcal{T}^*_{c_{\mathcal{T},\mathcal{T}^*}} \leq \mathcal{T}^*$ (4)

Claim : $\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}} \leq \mathcal{T}$

Suppose not, then $\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}} \not\leq \mathcal{T}$

$\therefore$ there exists μ such that $\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}}(\mu) > \mathcal{T}(\mu)$

i.e., $\vee\{r \,/\, C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s) = \bar{1} - \mu\}$

$> \mathcal{T}(\mu)$

By supremum property,

there exists r_0 such that $C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r_0, s) = \bar{1} - \mu$ (5)

and $\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}}(\mu) \geq r_0 > \tau(\mu)$ (6)

By definition of $C_{\mathcal{T},\mathcal{T}^*}$, we get from (5)

$\mathcal{T}(\bar{1} - (\bar{1} - \mu)) \geq r_0$

i.e. $\mathcal{T}(\mu) \geq r_0$ (7)

(6) and (7) contradict each other

$\therefore \mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}} \leq \mathcal{T}$ (8)

From (4) and (8), we get

$\mathcal{T}_{c_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}$.

Similarly if can be shown that

$$\mathcal{T}^*_{c_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}^*$$

Definition : 3.1.6 (Samanta and Mondal [30])

Let (X, $\mathcal{T}$, $\mathcal{T}^*$) and (Y, $\mathcal{U}$, $\mathcal{U}^*$) be IFTS's and f : X → Y a function. Then, f is called **intuitionistic continuous** if $\mathcal{U}(\lambda) \leq \mathcal{T}(f^{-1}(\lambda))$ and $\mathcal{U}^*(\lambda) \geq \mathcal{T}^*(f^{-1}(\lambda))$ for all $\lambda \in I^Y$.

Theorem : 3.1.7

Let (X, $\mathcal{T}$, $\mathcal{T}^*$) and (Y, $\mathcal{U}$, $\mathcal{U}^*$) be IFTS's and f : (X, $\mathcal{T}$, $\mathcal{T}^*$) → (Y, $\mathcal{U}$, $\mathcal{U}^*$) a function. Then the following statements are equivalent, for each $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$, $s \in I_1$.

(1) f is intuitionistic continuous

(2) $f(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)) \leq C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s)$

(3) $C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s))$.

Proof (1) ⇒ (2)

Let f be intuitionistic continuous. Then $\mathcal{U}(\lambda) \leq \mathcal{T}(f^{-1}(\lambda))$ and $\mathcal{U}^*(\lambda) \geq \mathcal{T}^*(f^{-1}(\lambda))$ for all $\lambda \in I^Y$.

Assume that $\lambda = \bar{1} - \mu$

$\therefore \mathcal{U}(\bar{1} - \mu) \leq \mathcal{T}(f^{-1}(\bar{1} - \mu))$

$\mathcal{U}(\bar{1} - \mu) \leq \mathcal{T}(\bar{1} - f^{-1}(\mu))$ (i)

and $\mathcal{U}^*(\bar{1} - \mu) \geq \mathcal{T}^*(f^{-1}(\bar{1} - \mu))$

$= \mathcal{T}^*(\bar{1} - f^{-1}(\mu))$

$\therefore \mathcal{U}^*(\bar{1} - \mu) \geq \mathcal{T}^*(\bar{1} - f^{-1}(\mu))$ (ii)

Consider $C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s)$

$= \wedge\{\mu \in I^Y / f(\lambda) \le \mu,\ \mathcal{U}(\bar{1} - \mu) \ge r,$

$\mathcal{U}^*(\bar{1} - \mu) \le s$

$\ge \wedge\{\mu \in I^Y / \lambda \le f^{-1}(\mu),\ \mathcal{T}(\bar{1} - f^{-1}(\mu)) \ge r,$

$\mathcal{T}^*(\bar{1} - f^{-1}(\mu)) \le s$ (by (i) and (ii))

$\ge \wedge\{f(f^{-1}(\mu) \in I^Y / \lambda \le f^{-1}(\mu),\ \mathcal{T}(\bar{1} - f^{-1}(\mu)) \ge r,$

$\mathcal{T}^*(\bar{1} - f^{-1}(\mu)) \le s\}$ since $\mu \ge f(f^{-1}(\mu))$

$\ge f(\wedge\{f^{-1}(\mu) \in I^Y / \lambda \le f^{-1}(\mu),\ \mathcal{T}(\bar{1} - f^{-1}(\mu)) \ge r,$

$\mathcal{T}^*(\bar{1} - f^{-1}(\mu)) \le s\}$

$\ge f(C_{\tau,\tau^*}(\lambda, r, s))$

$\therefore f(C_{\tau,\tau^*}(\lambda, r, s)) \le C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s)$

(2) ⇒ (3)

Take $\mu \in I^X$, let $\lambda = f^{-1}(\mu)$.

By (2),

$f(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)) \le C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s))$ for every $\lambda \in I^Y$

$\therefore f(C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s)) \le C_{\mathcal{U},\mathcal{U}^*}(f(f^{-1}(\mu), r, s))$

$\Rightarrow f(C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s)) \le C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s)$

since $(f(f^{-1}(\mu)) \le \mu$ and by remark 3.1.4

$\Rightarrow f^{-1}(f(C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s)) \le f^{-1}(C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s))$

$\Rightarrow C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s)) \le f^{-1}(C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s))$ since $f^{-1}(f(\lambda)) \ge \lambda$

(3) ⇒ (1)

If follows from $C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s) = \mu$ implies $C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s) = f^{-1}(\mu)$.

Section 3.2

(r, s)-connectedness in intuitionistic fuzzy topological spaces

In this section we discuss the concept of (r, s)-connectedness in IFTS and obtain some characterization theorems. We also discuss some interesting properties of (r, s)-connected fuzzy sets and (r, s)-components which are analogous to the corresponding properties in general topological spaces.

Definition : 3.2.1

Let (X, $\mathcal{T}$, $\mathcal{T}^*$) be an IFTS. For λ, $\mu \in I^X$, λ and μ are called **(r, s)-separated** if for $r \in I_0$ and $s \in I_1$.

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \wedge \mu = C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) \wedge \lambda = \bar{0}$$

Definition : 3.2.2

A fuzzy set ρ is called **(r, s)-connected** if there do not exist (r, s)-separated fuzzy sets λ, $\mu \in I^X - \{\bar{0}\}$ such that $\rho = \lambda \vee \mu$.

A triplet (X, $\mathcal{T}$, $\mathcal{T}^*$) is called **(r, s)-connected** if $\bar{1}$ is (r, s)-connected.

Definition : 3.2.3

A fuzzy set ρ is called **connected** if it is (r, s)-connected for all $r \in I_0$ and $s \in I_1$.

Remark : 3.2.4

Let λ and μ be (r, s)-separated. For each $\rho \in I^X$ and $r_1 \leq r$, $s_1 \geq s$, since $C_{\mathcal{T},\mathcal{T}^*}(\rho, r_1, s_1) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s)$ we get λ and μ are (r_1, s_1)-separated.

Hence, from this fact, we get that if ρ is (r_1, s_1)-connected for $r_1 \leq r$, $s_1 \leq s$, then ρ is (r, s)-connected.

Let us now consider are example of a (r, s)-connected space.

Example : 3.2.5

Let X = {x, y} be a set. We define an IGO ($\mathcal{T}$, $\mathcal{T}^*$) on X as follows : For each $\lambda \in I^X$.

$$\mathcal{T}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ 1/3 & \text{if } \lambda = \chi_{\{x\}} \\ 1/2 & \text{if } \lambda = \chi_{\{y\}} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{T}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ 2/3 & \text{if } \lambda = \chi_{\{x\}} \\ 1/2 & \text{if } \lambda = \chi_{\{y\}} \\ 1 & \text{otherwise} \end{cases}$$

where $\chi_{\{x\}}$ denotes the characteristic function corresponding to the set {x}.

We can obtain

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \begin{cases} \bar{0} & \text{if } \lambda = \bar{0}, r \in I_0, s \in I_1 \\ \chi_{\{x\}} & \text{if } \bar{0} \neq \lambda \leq \chi_{\{x\}}, r \leq 1/2, s \geq 1/2 \\ \chi_{\{y\}} & \text{if } \bar{0} \neq \lambda \leq \chi_{\{y\}}, r \leq 1/3, s \geq 2/3 \\ \bar{1} & \text{otherwise} \end{cases}$$

If $r \leq 1/3$, $s \geq 2/3$, then $(\chi_{\{x\}} = C_{\mathcal{T},\mathcal{T}^*}(\chi_{\{x\}}, r, s)) \wedge \chi_{\{y\}} = \bar{0}$

and $\chi_{\{x\}} \wedge (\chi_{\{y\}} = C_{\mathcal{T},\mathcal{T}^*}(\chi_{\{y\}}, r, s)) = \bar{0}$

Thus, $\bar{1}_X = \chi_{\{x\}} \vee \chi_{\{y\}}$ is not (r, s)-connected for $r \leq 1/3$ and $s \leq 2/3$.

If $r > 1/3$ and $s < 2/3$, (X, $\mathcal{T}$, $\mathcal{T}^*$) is (r, s)-connected.

Characterization theorems

Theorem : 3.2.6

Let (X, $\mathcal{T}$, $\mathcal{T}^*$) be an IFTS. The following statements are equivalent

(1) (X, $\mathcal{T}$, $\mathcal{T}^*$) is (r, s)-connected.

(2) If $\lambda \vee \mu = \bar{1}$ and $\lambda \wedge \mu = \bar{0}$ for ($\mathcal{T}(\lambda) \geq r$, $\mathcal{T}^*(\lambda) \leq s$) and

$(\mathcal{T}(\mu) \geq r, \mathcal{T}^*(\mu) \leq s)$, then $\lambda = \bar{0}$ or $\mu = \bar{0}$

(3) If $\lambda \vee \mu = \bar{1}$ and $\lambda \wedge \mu = \bar{0}$ for $(\mathcal{T}(\bar{1} - \lambda) \geq r$, $\mathcal{T}^*(\bar{1} - \lambda) \geq s)$ and $(\mathcal{T}(\bar{1} - \mu) \geq r, \mathcal{T}^*(\bar{1} - \mu) \leq s)$, then $\lambda = \bar{0}$ or $\mu = \bar{0}$.

Proof : (1) ⇒ (2)

Suppose that there exist $\lambda, \mu \in I^X - \{\bar{0}\}$ such that for $(\mathcal{T}(\lambda) \geq r$, $\mathcal{T}^*(\lambda) \leq s)$ and $\mathcal{T}(\mu) \geq r, \mathcal{T}^*(\mu) \leq s)$, $\lambda \vee \mu = \bar{1}$, $\lambda \wedge \mu = \bar{0}$.

It implies

$(\bar{1} - \lambda) \wedge (\bar{1} - \mu) = \bar{0}$, $(\bar{1} - \lambda) \vee (\bar{1} - \mu) = \bar{1}$ (i)

Since $C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \lambda, r, s) = \bar{1} - \lambda$ and

$C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s) = \bar{1} - \mu$, from Theorem 3.1.5

We get from (i), $C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \lambda, r, s) \wedge (\bar{1} - \mu) = \bar{0}$ and

$C_{\mathcal{T},\mathcal{T}^*}(\bar{1} - \mu, r, s) \wedge (\bar{1} - \lambda) = \bar{0}$

∴ By definition, $\bar{1} - \lambda$ and $\bar{1} - \mu$ are (r, s)-separated.

Suppose $\lambda = \bar{1}$. Then $\mu = \lambda \wedge \mu = \bar{0}$

This is a contradiction.

∴ $\lambda \in I^X - \{\bar{1}\}$.

Thus, $\bar{1} - \lambda \in I^X - \{\bar{0}\}$.

Similarly $\bar{1} - \mu \in I^X - \{\bar{0}\}$.

Also, from (i), we have

$(\bar{1} - \lambda) \vee (\bar{1} - \mu) = \bar{1}$

Hence $\bar{1}$ is not (r, s)-connected.

i.e., $(X, \mathcal{T}, \mathcal{T}^*)$ is not (r, s)-connected.

This is a contradiction.

Hence there do not exist $\lambda, \mu \in I^X - \{\bar{0}\}$ satisfying the condition.

i.e., Either $\lambda = \bar{0}$ or $\mu = \bar{0}$.

(2) ⇒ (3)

By De Morgan's law, it is easily proved.

(3) ⇒ (1)

If (X, $\mathscr{T}$, $\mathscr{T}^*$) is not (r, s)-connected, then there exist (r, s)-separated fuzzy sets $\lambda, \mu \in I^X - \{\bar{0}\}$ such that $\lambda \vee \mu = \bar{1}$.

Since $\lambda \wedge \mu \leq C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) \wedge \mu = \bar{0}$, we have, $\lambda \wedge \mu = \bar{0}$

Since $\lambda \vee \mu = \bar{1}$, we get

$$(\bar{1} - \lambda) \wedge (\bar{1} - \mu) = \bar{0} \qquad \text{(ii)}$$

$\Rightarrow (\bar{1} - \mu)(x) = 0$ (or) $(\bar{1} - \lambda)(x) = 0$

If $(\bar{1} - \mu)(x) = 0$, then

$(\bar{1} - \mu)(x) \leq \lambda(x)$.

If $(\bar{1} - \mu)(x) \neq 0$, then

$(\bar{1} - \lambda)(x) = 0$

$\Rightarrow \lambda(x) = 1$

$\Rightarrow (\bar{1} - \mu)(x) \leq \lambda(x)$.

Hence $(\bar{1} - \mu) \leq \lambda$ (iii)

Now $C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) \wedge \mu = \bar{0}$

$\Rightarrow C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) \wedge \mu = (\bar{1} - \lambda) \wedge (\bar{1} - \mu)$ by (ii)

$\Rightarrow C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) \leq \bar{1} - \mu$

$\Rightarrow C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) \leq \lambda$ [by (iii)]

By Definition 3.1.2 (C2), $\lambda \leq C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s)$

$$\therefore\ C_{\mathscr{T},\mathscr{T}^*}(\lambda, r, s) = \lambda.$$

From Theorem 3.1.5, we have $\mathscr{T}(\bar{1} - \lambda) \geq r$ and $\mathscr{T}^*(\bar{1} - \lambda) \leq s$.

Similarly, we have $\mathcal{T}(\bar{1} - \mu) \geq r$ and $\mathcal{T}^*(\bar{1} - \mu) \leq s$.

$\therefore$ By (3), $\lambda = \bar{0}$ (or) $\mu = \bar{0}$

This is a contradiction.

$\therefore$ $(X, \mathcal{T}, \mathcal{T}^*)$ is (r, s)-connected.

Lemma : 3.2.7

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS and $\lambda, \mu, \rho \in I^X$. If μ and ρ are (r, s)-separated, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r, s)-separated.

Proof

Let μ and ρ be (r, s)-separated.

Then $C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) \wedge \rho = C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) \wedge \mu = \bar{0}$

Since $C_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \mu, r, s) \wedge (\lambda \wedge \rho) \leq C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) \wedge \rho = \bar{0}$,

we get $C_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \mu, r, s) \wedge (\lambda \wedge \rho) = \bar{0}$

Similarly $(\lambda \wedge \mu) \wedge C_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \rho, r, s) = \bar{0}$

Hence $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r, s)-separated.

Theorem : 3.2.8

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS and $\lambda \in I^X$. The following statements are equivalent :

(1) λ is (r, s)-connected.

(2) If μ and ρ are (r, s)-separated such that $\lambda \leq \mu \vee \rho$, then $\lambda \wedge \mu = \bar{0}$ or $\lambda \wedge \rho = \bar{0}$.

(3) If μ and ρ are (r, s)-separated such that $\lambda \leq \mu \vee \rho$, then $\lambda \leq \mu$ or $\lambda \leq \rho$.

Proof : (1) $\Rightarrow$ (2)

Let μ and ρ be (r, s)-separated such that $\lambda \leq \mu \vee \rho$.

By lemma 3.2.7, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r, s)-separated.

Since λ is (r, s)-connected and $\lambda = \lambda \wedge (\mu \vee \rho)$

$= (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$

We get either $\lambda \wedge \mu = \bar{0}$ or $\lambda \wedge \rho = \bar{0}$

Hence (2).

(2) ⇒ (3)

Let μ and ρ are (r, s)-separated such that $\lambda \leq \mu \vee \rho$

By (2), $\lambda \wedge \mu = \bar{0}$ or $\lambda \wedge \rho = \bar{0}$

Suppose $\lambda \wedge \mu = \bar{0}$

Consider $\lambda = \lambda \wedge (\mu \vee \rho)$ (since $\lambda \leq \mu \vee \rho$)

$= (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$

$= \bar{0} \vee (\lambda \wedge \rho)$

$\lambda = \lambda \wedge \rho$

Hence $\lambda \leq \rho$

Similarly, if $\lambda \wedge \rho = \bar{0}$, we get $\lambda \leq \mu$

Hence (3).

(3) ⇒ (1)

Let μ and ρ be (r, s)-separated such that $\lambda = \mu \vee \rho$

By (3), $\lambda \leq \mu$ or $\lambda \leq \rho$

Suppose $\lambda \leq \mu$

Since $\rho \wedge (\mu \vee \rho) = \rho$, we get $\rho = \rho \wedge \lambda$

$\therefore \rho \leq \rho \wedge \mu$ (since $\lambda \leq \mu$)

$\leq \rho \wedge C_{\mathcal{T}}(\mu, r, s)$

$= \bar{0}$ (since μ and ρ are (r, s)-connected).

Hence $\rho = \bar{0}$

Similarly, if $\lambda \leq \rho$, we can prove $\mu = \bar{0}$.

Properties of (r, s)-connected fuzzy sets

Theorem : 3.2.9

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS and $\lambda, \mu \in I^X$.

(1) If λ is (r, s)-connected and $\lambda \leq \mu \leq C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$ then μ is (r, s)-connected.

(2) If λ and μ are (r, s)-connected fuzzy sets which are not (r, s)-separated, then $\lambda \vee \mu$ is (r, s)-connected.

Proof

(1) Let ν and ρ be (r, s)-separated such that $\mu = \nu \vee \rho$. Put $\nu_1 = \lambda \wedge \nu$ and $\rho_1 = \lambda \wedge \rho$. Then ν_1 and ρ_1 are (r, s)-separated by Lemma 3.2.7

Consider $\nu_1 \vee \rho_1 = (\lambda \wedge \nu) \vee (\lambda \wedge \rho)$

$= \lambda \wedge (\nu \vee \rho)$

$= \lambda \wedge \mu$

$= \lambda$ as $\lambda \leq \mu$

$\therefore \lambda = \nu_1 \vee \rho_1$

Since λ is (r, s)-connected

either $\nu_1 = \bar{0}$ or $\rho_1 = \bar{0}$

If $\nu_1 = \bar{0}$, then $\lambda = \nu_1 \vee \rho_1$

$\Rightarrow \lambda = \bar{0} \vee \rho_1$

$\Rightarrow \lambda = \rho_1$

$\Rightarrow \lambda = \lambda \wedge \rho$

$\Rightarrow \lambda \leq \rho$

Given $\mu \leq C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$

$\leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s)$ as $\lambda \leq \rho$

Hence $\nu = \nu \wedge \mu \leq \nu \wedge C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) = \bar{0}$ since ν and ρ are (r, s)-separated.

Hence $\nu = \bar{0}$

Similarly, if $\rho_1 = \bar{0}$, we get $\rho = \bar{0}$, we get $\rho = \bar{0}$

Thus, μ is (r, s)-connected.

(2) Let ν and ρ be (r, s)-separated such that $\lambda \vee \mu = \nu \vee \rho$

Since λ is (r, s)-connected and $\lambda \leq \lambda \vee \mu$

$= \nu \vee \rho$,

by (1) $\Rightarrow$ (3) of Theorem 3.2.8, we get $\lambda \leq \nu$ or $\lambda \leq \rho$.

Similarly, we get $\mu \leq \nu$ or $\mu \leq \rho$

Suppose $\lambda \leq \nu$

Also, suppose that $\mu \leq \rho$

Since $(\lambda \wedge \mu) \vee \nu = \lambda$ and $(\lambda \vee \mu) \wedge \rho = \mu$ and ν and ρ are (r, s)-separated, by Lemma , we get λ and μ are (r, s)-separated.

This is a contradiction.

Hence $\mu \leq \nu$

$\therefore \lambda \vee \mu \leq \nu$

$\therefore \lambda \vee \mu$ is (r, s)-connected (by (3) $\Rightarrow$ (1) of Theorem 3.2.8).

Theorem : 3.2.10

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS. Let $\mathcal{A} = \{\lambda_i \,/\, i \in \Gamma\}$ be a family of (r, s)-connected fuzzy sets in $(X, \mathcal{T}, \mathcal{T}^*)$ such that no two members of $\mathcal{A}$ are (r, s)-separated. Then $\vee_{i\in\Gamma} \lambda_i$ is (r, s)-connected.

Proof

Given $\mathcal{A} = \{\lambda_i \,/\, i \in \Gamma\}$ is a family of (r, s)-connected fuzzy sets.

Let $\lambda = \vee_{i\in\Gamma} \lambda_I$

Suppose μ and ρ be (r, s)-separated such that $\lambda \leq \mu \vee \rho$

Since any two members $\lambda_i, \lambda_j \in \mathcal{A}$ are not (r, s)-separated, by Theorem 3.2.9 (2), $\lambda_i \vee \lambda_j$ is (r, s)-connected.

Since $\lambda_i \vee \lambda_j \leq \lambda$, and $\lambda = \mu \vee \rho$, we get $\lambda_i \vee \lambda_j \leq \mu \vee \rho$

From (1) $\Rightarrow$ (3) of Theorem 3.2.8, we get

$\lambda_i \vee \lambda_j \le \mu$ or $\lambda_i \vee \lambda_j \le \rho$

If $\lambda_i \vee \lambda_j \le \mu$, then $\lambda \le \mu$.

Hence λ is (r, s)-connected by (3) $\Rightarrow$ (1) of Theorem 3.2.8

Corollary : 3.2.11

Let (X, $\mathscr{T}$, $\mathscr{T}^*$) be an IFTS. Let $\{\lambda_i \,/\, i \in \Gamma\}$ be a family of (r, s)-connected fuzzy sets in (X, $\mathscr{T}$, $\mathscr{T}^*$) If $\wedge_{i\in\Gamma} \lambda_i \neq \bar{0}$, then $\vee_{i\in\Gamma} \lambda_i$ is (r, s)-connected.

Proof

Obvious from Theorem 3.2.10.

Theorem : 3.2.12

Let (X, $\mathscr{T}_1$, $\mathscr{T}_1^*$) and (Y, $\mathscr{T}_2$, $\mathscr{T}_2^*$) be IFTS's.

If f : (X, $\mathscr{T}_1$, $\mathscr{T}_1^*$) $\to$ (Y, $\mathscr{T}_2$, $\mathscr{T}_2^*$) is intuitionistic continuous and λ is (r, s)-connected, then f(λ) is (r, s)-connected.

Proof

Let μ and ρ be (r, s)-separated such that $f(\lambda) \le \mu \vee \rho$.

Consider $\lambda \le f^{-1}(f(\lambda))$

$\le f^{-1}(\mu \vee \rho)$

$= f^{-1}(\mu) \vee f^{-1}(\rho)$

Since f is intuitionstic continuous, by Theorem 3.1.7 (3), we get

$C_{\mathscr{T}_1,\mathscr{T}_1^*}(f^{-1}(\mu), r, s) \le f^{-1}(C_{\mathscr{T}_2,\mathscr{T}_2^*}(\mu, r, s))$.

$C_{\mathscr{T}_1,\mathscr{T}_1^*}(f^{-1}(\mu), r, s) \wedge f^{-1}(\rho) \le f^{-1}(C_{\mathscr{T}_2,\mathscr{T}_2^*}(\mu, r, s)) \wedge f^{-1}(\rho)$

$= f^{-1}(C_{\mathscr{T}_2,\mathscr{T}_2^*}(\mu, r, s)) \wedge \rho)$

$= f^{-1}(\bar{0})$, since μ and ρ are (r, s)-separated

$= \bar{0}$

Similarly, we have $(f^{-1}(\mu) \wedge C_{\mathcal{T}_1, \mathcal{T}_1^*}(f^{-1}(\rho), r, s) = \bar{0}$.

Hence $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are (r, s)-separated. Since λ is (r, s)-connected, by (1) $\Rightarrow$ (3) of Theorem 3.2.8

We get $\lambda \leq f^{-1}(\mu)$ or $\lambda \leq f^{-1}(\rho)$

If $\lambda \leq f^{-1}(\mu)$, then

$f(\lambda) \leq f(f^{-1}(\mu))$

$\leq \mu$

$\therefore$ $f(\lambda)$ is (r, s)-connected, by (3) $\Rightarrow$ (1) of Theorem 3.2.8.

(r, s)-components

Definition : 3.2.13

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS. A fuzzy set λ is a **(r, s)-component** in $(X, \mathcal{T}, \mathcal{T}^*)$ if λ is a maximal (r, s)-connected fuzzy set in $(X, \mathcal{T}, \mathcal{T}^*)$, (i.e.,) If $\mu \geq \lambda$ and μ is (r, s)-connected, then $\mu = \lambda$.

Theorem : 3.2.14

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS.

(1) If λ is a (r, s)-component, then $C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s) = \lambda$.

(2) If λ_1 and λ_2 are (r, s)-components in $(X, \mathcal{T}, \mathcal{T}^*)$ such that $\lambda_1 \wedge \lambda_2 = \bar{0}$, then λ_1 and λ_2 are (r, s)-separated.

(3) Each fuzzy point x_t is connected.

(4) Every (r, s)-component is a crisp set.

Proof

(1) Since λ is a (r, s)-component, λ is (r, s)-connected. Also, $\lambda \leq C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s)$.

By Theorem 3.2.9 (1), $C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s)$ is (r, s)-connected. Since λ is a (r, s)-component, by definition we get $C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s) = \lambda$.

(2) Let λ_1 and λ_2 be (r, s)-components such that $\lambda_1 \wedge \lambda_2 = \bar{0}$

By (1), $C_{\mathcal{J},\mathcal{J}^*}(\lambda_1, r, s) = \lambda_1$

and $C_{\mathcal{J},\mathcal{J}^*}(\lambda_2, r, s) = \lambda_2$.

Hence we get

$$C_{\mathcal{J},\mathcal{J}^*}(\lambda_1, r, s) \wedge \lambda_2 = \lambda_1 \wedge C_{\mathcal{J},\mathcal{J}^*}(\lambda_2, r, s)$$

$$= \lambda_1 \wedge \lambda_2$$

$$= \bar{0}$$

$\therefore$ λ_1 and λ_2 are (r, s)-separated.

(3) Let λ and μ be (r, s)-separated such that $x_t = \lambda \vee \mu$.

Then $x_t = \lambda$ or $x_t = \mu$

If $x_t = \lambda$, consider $\mu = \mu \wedge (\lambda \vee \mu)$

$$= \mu \wedge x_t$$

$$= \mu \wedge \lambda$$

$$\leq \mu \wedge C_{\mathcal{J},\mathcal{J}^*}(\lambda_1, r, s) \text{ since } \lambda \leq C_{\mathcal{J},\mathcal{J}^*}(\lambda_1, r, s)$$

$$= \bar{0} \text{ since } \lambda \text{ and } \mu \text{ are (r, s)-separated.}$$

$$\therefore \mu = \bar{0}$$

Similarly, if $x_t = \mu$, then $\lambda = \bar{0}$

Hence x_t is (r, s)-connected.

This is true for every $r \in I_0$ and $s \in I_1$

Hence x_t is connected.

(4) Let λ be a (r, s)-component with $x \in \text{supp}(\lambda) = \{x \in X / \lambda(x) > 0\}$ and μ is a (r, s)-component containing x_1.

Since $\lambda \wedge \mu \geq x_\lambda(x) \wedge x_1$

$$= x_\lambda(x)$$

$$\neq \bar{0} \text{ since } x \in \text{supp}(\lambda)$$

$$\therefore \lambda \wedge \mu \neq \bar{0}$$

By corollary 3.2.11, $\lambda \vee \mu$ is (r, s)-connected.

Also as $\lambda \leq \lambda \vee \mu$, $\mu \leq \lambda \vee \mu$ and λ and μ are (r, s)-components, by definition of components, we must have $\lambda = \lambda \vee \mu$

and $\mu = \lambda \vee \mu$

Thus $\lambda = \mu$

$\therefore\ \lambda(x) = \mu(x)$. Since $x_1 \leq \mu$, we get $\mu(x) = 1$

$\therefore\ \lambda(x) = 1$

$\therefore\ x \in \text{supp}(\lambda) \Rightarrow \lambda(x) = 1$.

Hence λ is a crisp set.

Section 3.3

(r, s)-components in stratification of intuitionistic fuzzy topological spaces

First we obtain stratification of an intuitionistic fuzzy topological space and then show that every (r, s)-component in an IFTS is a (r, s)-component in the stratification of it.

Theorem : 3.3.1

Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an IFTS. Define the functions $\mathcal{T}_{st}, \mathcal{T}^*_{st} : I^X \to I$ as follows : For each $\lambda \in I^X$

$$\mathcal{T}_{st}(\lambda) = \vee\left\{\bigwedge_{j\in J} \mathcal{T}(\lambda_j) / \lambda = \bigvee_{j\in J} (\lambda_j \wedge \overline{\alpha}_j)\right\}$$

where the first $\vee$ is taken over all families $\{\lambda_j / j \in J\}$ with $\lambda = \vee_{j\in J} (\lambda_j \wedge \overline{\alpha}_j)$

$$\mathcal{T}^*_{st}(\lambda) = \wedge\left\{\bigvee_{j\in J} \mathcal{T}^*(\lambda_j) / \lambda = \bigvee_{j\in J} (\lambda_j \wedge \overline{\alpha}_j)\right\}$$

where the first $\wedge$ is taken over all families $\{\lambda_j / j \in J\}$ with $\lambda = \vee_{j\in J} (\lambda_j \wedge \overline{\alpha}_j)$. Then $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is the coarsest stratified IGO on X which is finer than $(\mathcal{T}, \mathcal{T}^*)$.

Proof

First, we will show that $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is a stratified IGO as X.

(IGO1)

Suppose there exists $\lambda \in I^X$ such that

$\mathcal{T}_{st}(\lambda) + \mathcal{T}^*_{st}(\lambda) > 1$

There exists $r \in I$ and a family $\{\lambda_j / j \in J\}$

with $\lambda = \vee_{j\in J} (\lambda_j \wedge \bar{\alpha}_j)$ such that

$\mathcal{T}_{st}(\lambda) \geq \bigwedge_{j\in J} \mathcal{T}(\lambda_j) > r > 1 - \mathcal{T}^*_{st}(\lambda)$.

Since $\mathcal{T}(\lambda_j) > r$, for each $j \in J$, there exists r_j such that $\mathcal{T}(\lambda_j) \geq r_j > r$

Since $\mathcal{T}(\lambda_j) + \mathcal{T}^*(\lambda_j) \leq 1$, we get

$\mathcal{T}^*(\lambda_j) \leq 1 - \mathcal{T}(\lambda_j)$

$\leq 1 - r_j$

Hence there exists m_j such that

$\mathcal{T}^*(\lambda_j) \leq m_j \leq 1 - r_j$

Hence

$$\mathcal{T}^*_{st}(\lambda) \leq \bigvee_{j\in J} \mathcal{T}^*(\lambda_j) \leq \bigvee_{j\in J} m_j$$

$$\leq \bigvee_{j\in J} (1 - r_j)$$

$$\leq 1 - \bigwedge_{j\in J} r_j$$

$$\leq 1 - r$$

It is a contradiction since $\mathcal{T}^*_{st}(\lambda) > 1 - r$.

Hence $\mathcal{T}_{st}(\lambda) + \mathcal{T}^*_{st}(\lambda) \leq 1 \ \forall \lambda \in I^X$.

(IGO2) and (IS)

For each $\alpha \in I$ there exists a family $\{\bar{1}\}$ with $\bar{\alpha} = \bar{\alpha} \wedge \bar{1}$.

Hence $\lambda_j = \bar{1}$ for every j

Hence by definition we have

$\mathcal{T}_{st}(\bar{\alpha}) \geq \wedge \mathcal{T}(\lambda_j) = \wedge \mathcal{T}(\bar{1}) = \mathcal{T}(\bar{1}) = 1$

and

$\mathcal{T}_{st}^{*}(\bar{\alpha}) \leq \vee\, \mathcal{T}^{*}(\lambda_j) = \vee\, \mathcal{T}^{*}(\bar{1}) = \mathcal{T}^{*}(\bar{1}) = 0$

Hence $\mathcal{T}_{st}(\bar{\alpha}) = 1$ and $\mathcal{T}_{st}^{*}(\bar{\alpha}) = 0$.

(IGO3)

Suppose there exist $\mu, \nu \in I^X$ and $r \in I_0$, $m \in I_1$ with

$\mathcal{T}_{st}(\mu \wedge \nu) < r < \mathcal{T}_{st}(\mu) \wedge \mathcal{T}_{st}(\nu)$

and

$\mathcal{T}_{st}^{*}(\mu \wedge \nu) > m > \mathcal{T}_{st}^{*}(\mu) \vee \mathcal{T}_{st}^{*}(\nu)$

Since $\mathcal{T}_{st}(\mu) > r$ and $\mathcal{T}_{st}(\nu) > r$

and $\mathcal{T}_{st}^{*}(\mu) < m$ and $\mathcal{T}_{st}^{*}(\nu) < m$,

by definition of $(\mathcal{T}_{st}, \mathcal{T}_{st}^{*})$, there exists two families $\{\mu_j \,/\, j \in J\}$ with $\mu = \bigvee_{j\in J} (\mu_j \wedge \bar{\alpha}_j)$ and $\{r_k \,/\, k \in K\}$ with $\nu = \vee_{k\in K} (\nu_k \wedge \bar{\alpha}_k)$ such that

$$\mathcal{T}_{st}(\mu) \geq \bigwedge_{j\in J} \mathcal{T}(\mu_j), \quad \mathcal{T}_{st}(\nu) \geq \bigwedge_{k\in K} \mathcal{T}(\nu_k),$$

$$> r \qquad\qquad > r$$

and $$\mathcal{T}_{st}^{*}(\mu) \leq \bigvee_{j\in J} \mathcal{T}^{*}(\mu_j), \quad \mathcal{T}_{st}^{*}(\nu) \leq \bigvee_{k\in K} \mathcal{T}^{*}(\nu_k),$$

$$< m \qquad\qquad < m$$

Since I is a completely distributive lattice, we have

$$\mu \wedge \nu = \left(\bigvee_{j\in J} (\mu_j \wedge \bar{\alpha}_j)\right) \wedge \left(\bigvee_{k\in K} (\nu_k \wedge \bar{\alpha}_k)\right)$$

$$= \bigvee_{j,k} (\mu_j \wedge \nu_k) \wedge (\bar{\alpha}_j \wedge \bar{\alpha}_k)$$

$$= \bigvee_{j,k} (\mu_j \wedge \nu_k) \wedge \bar{\alpha}_{jk} \text{ where } \bar{\alpha}_{jk} = \bar{\alpha}_j \wedge \bar{\alpha}_k$$

Since $\mathcal{T}(\mu_j \wedge \nu_k) \geq \mathcal{T}(\mu_j) \wedge \mathcal{T}(\nu_k)$ and

$\mathcal{T}^{*}(\mu_j \wedge \nu_k) \leq \mathcal{T}^{*}(\mu_j) \wedge \mathcal{T}^{*}(\nu_k)$, We have

$$\mathcal{T}_{st}(\mu \wedge \nu) \geq \bigwedge_{j,k} \mathcal{T}(\mu_j \wedge \nu_k)$$

$$\geq \bigwedge_{j,k} (\mathcal{T}(\mu_j) \wedge \mathcal{T}(\nu_k))$$

$$= \left(\bigwedge_{j,k} \mathcal{T}(\mu_j)\right) \wedge \left(\bigwedge_{k \in K} \mathcal{T}(\nu_k)\right)$$

$$> r$$

$$\mathcal{T}^*_{st}(\mu \wedge \nu) \leq \bigvee_{j,k} \mathcal{T}^*(\mu_j \wedge \nu_k)$$

$$\geq \bigvee_{j,k} (\mathcal{T}^*(\mu_j) \vee \mathcal{T}^*(\nu_k))$$

$$= \left(\bigvee_{j,k} \mathcal{T}^*(\mu_j)\right) \vee \left(\bigvee_{k \in K} \mathcal{T}^*(\nu_k)\right)$$

$$< m$$

It is a contradiction.

Hence for all $\mu, \nu \in I^X$

$\mathcal{T}_{st}(\mu \wedge \nu) \geq \mathcal{T}_{st}(\mu) \wedge \mathcal{T}_{st}(\nu)$ and

$\mathcal{T}^*_{st}(\mu \wedge \nu) \leq \mathcal{T}^*_{st}(\mu) \vee \mathcal{T}^*_{st}(\nu)$

(IGO4)

Suppose there exists a family $\{\mu_i \in I^X / i \in \Gamma\}$ and $r \in I_0$ $m \in I_1$ with

$$\mathcal{T}_{st}\left(\bigvee_{i \in \Gamma} \mu_i\right) < r < \bigwedge_{i \in \Gamma} \mathcal{T}_{st}(\mu_i)$$

$$\mathcal{T}^*_{st}\left(\bigvee_{i \in \Gamma} \mu_i\right) > m > \bigvee_{i \in \Gamma} \mathcal{T}^*_{st}(\mu_i)$$

Since $\mathcal{T}_{st}(\mu_i) > r$ and $\mathcal{T}^*_{st}(\mu_i) < m$ for each $i \in \Gamma$, there exists a family $\{\mu_{ij} / j \in J_i\}$ with $\mu_i = \vee_{j \in J_i} (\mu_{ij} \wedge \overline{\alpha}_j)$ such that

$$\mathcal{T}_{st}(\mu_i) \geq \bigwedge_{i \in J_i} \mathcal{T}(\mu_{ij})$$

$$> r$$

$$\mathcal{T}^*_{st}(\mu_i) \leq \bigvee_{j \in J_i} \mathcal{T}^*(\mu_{ij})$$

$$< m$$

Since

$$\vee_{j\in\Gamma}\ \mu_i = \vee_{j\in\Gamma}\ (\bigvee_{j\in J_i} (\mu_{ij} \wedge \bar{\alpha}_j))$$

$$= \vee_{i,j}\ (\mu_{ij} \wedge \bar{\alpha}_j)\ .$$

We have

$$\mathcal{T}_{st}\Big(\bigvee_{i\in\Gamma}\mu_i\Big) \geq \bigwedge_{i,j}\ \mathcal{T}(\mu_{ij})$$

$$= \bigwedge_{i\in\Gamma}\left(\bigwedge_{j\in J_i}\mathcal{T}(\mu_{ij})\right)$$

$$\geq r$$

and $$\mathcal{T}^*_{st}\Big(\bigvee_{i\in\Gamma}\mu_i\Big) \leq \bigvee_{i,j}\ \mathcal{T}^*(\mu_{ij})$$

$$= \bigvee_{i\in\Gamma}\left(\bigvee_{j\in J_i}\mathcal{T}^*(\mu_{ij})\right)$$

$$\leq m$$

It is a contradiction.

Hence, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$

$\mathcal{T}_{st}\Big(\bigvee_{i\in\Gamma}\mu_i\Big) \geq \bigwedge_{i\in\Gamma}\ \mathcal{T}_{st}(\mu_i)$ and

$\mathcal{T}^*_{st}\Big(\bigvee_{i\in\Gamma}\mu_i\Big) \leq \bigvee_{i\in\Gamma}\ \mathcal{T}^*_{st}(\mu_i)$

For each $\lambda \in I^X$, there exists a family $\{\bar{1}\}$ with $\lambda = \bar{1} \wedge \lambda \ni \mathcal{T}_{st}(\lambda) \geq \mathcal{T}(\lambda)$ and $\mathcal{T}^*_{st}(\lambda) \leq \mathcal{T}^*(\lambda)$.

Hence $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is finer than $(\mathcal{T}, \mathcal{T}^*)$. (1)

Let $(\mathcal{U}, \mathcal{U}^*)$ be a stratified IGO which is finer than $(\mathcal{T}, \mathcal{T}^*)$.

We will show that $\mathcal{T}_{st}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*_{st}(\lambda) \geq \mathcal{U}^*(\lambda)$ for all $\lambda \in I^X$.

Suppose there exists $\mu \in I^X$ and $r \in I_0$, $m \in I_1$ such that

$\mathcal{T}_{st}(\mu) > r > \mathcal{U}(\mu)$ and $\mathcal{T}^*_{st}(\mu) < m < \mathcal{U}^*(\mu)$

Since $\mathcal{T}_{st}(\mu) > r$ and $\mathcal{T}^*_{st}(\mu) < m$, there exists a family $\{\mu_j / j \in J\}$ with

$\mu = \bigvee_{j\in J} (\mu_j \wedge \bar{\alpha}_j)$ such that $\mathcal{T}_{st}(\mu) \geq \bigwedge_{j\in J}\mathcal{T}(\mu_j)$

$$> r$$

and

$$\mathcal{T}^*_{st}(\mu) \leq \bigvee_{j\in J} \mathcal{T}^*(\mu_j)$$

$$< m$$

Since $(\mathcal{U}, \mathcal{U}^*)$ is finer than $(\mathcal{T}, \mathcal{T}^*)$, we get $\mathcal{U}(\mu_j) \geq \mathcal{T}(\mu_j)$ and $\mathcal{U}^*(\mu_j) \leq \mathcal{T}^*(\mu_j)$ for each $j \in J$.

$$\begin{aligned}
\text{Consider } \mathcal{U}(\mu) &= \mathcal{U}\left(\bigvee_{j\in J}(\mu_j \wedge \overline{\alpha}_j)\right) \\
&\geq \bigwedge_{j\in J} \mathcal{U}(\mu_j \wedge \overline{\alpha}_j) \\
&\geq \bigwedge_{j\in J} (\mathcal{U}(\mu_j) \wedge \mathcal{U}(\overline{\alpha}_j)) \\
&= \bigwedge_{j\in J} (\mathcal{U}(\mu_j) \wedge 1) \text{ as } (\mathcal{U}, \mathcal{U}^*) \text{ is stratified} \\
&= \bigwedge_{j\in J} \mathcal{U}(\mu_j) \\
&\geq \bigwedge_{j\in J} \mathcal{T}(\mu_j) \\
&> r.
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}^*(\mu) &= \mathcal{U}^*\left(\bigvee_{j\in J}(\mu_j \wedge \overline{\alpha}_j)\right) \\
&\leq \bigvee_{j\in J} \mathcal{U}^*(\mu_j \wedge \overline{\alpha}_j) \\
&\leq \bigvee_{j\in J} (\mathcal{U}^*(\mu_j) \vee \mathcal{U}^*(\overline{\alpha}_j)) \\
&= \bigvee_{j\in J} (\mathcal{U}^*(\mu_j) \vee 0) \text{ as } (\mathcal{U}, \mathcal{U}^*) \text{ is stratified} \\
&= \bigvee_{j\in J} \mathcal{U}^*(\mu_j) \\
&\leq \bigvee_{j\in J} \mathcal{T}^*(\mu_j) \\
&< m
\end{aligned}$$

It is a contradiction.

Hence $\mathcal{T}_{st}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*_{st}(\lambda) \geq \mathcal{U}^*(\lambda)$ for every $\lambda \in I^X$.

$\therefore$ $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is coarser than $(\mathcal{U}, \mathcal{U}^*)$. (2)

Hence from (1) and (2) we get that $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is the coarserst stratified IGO which is finer than $(\mathcal{T}, \mathcal{T}^*)$.

Definition : 3.3.2

In the above theorem $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is called the **stratification** of an IGO $(\mathcal{T}, \mathcal{T}^*)$ on X.

Example : 3.3.3

Let X = {x, y}. Define $\mu, \rho \in I^X$ as follows :

$\mu(x) = 0.5$, $\mu(y) = 0.5$ and $\rho(x) = 0.4$, $\rho(y) = 0.6$.

We define IGO on X as follows : For each $\lambda \in I^X$

$$\mathcal{T}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{1}, \bar{0} \\ 1/3 & \text{if } \lambda = \mu \\ 1/2 & \text{if } \lambda = \rho \\ 3/4 & \text{if } \lambda = \mu \vee \rho \\ 2/3 & \text{if } \lambda = \mu \wedge \rho \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{1}, \bar{0} \\ 2/3 & \text{if } \lambda = \mu \\ 1/2 & \text{if } \lambda = \rho \\ 1/4 & \text{if } \lambda = \mu \vee \rho \\ 1/3 & \text{if } \lambda = \mu \wedge \rho \\ 1 & \text{otherwise,} \end{cases}$$

Claim

If $\lambda(x) = \alpha$ for $0.5 < \alpha < 0.6$ and $\lambda(y) = 0.6$, for each $\beta \geq 0.6$, $\lambda = (\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge (\mu \vee \rho))$.

Consider

$(\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge (\mu \vee \rho))(x)$

$= \max\{(\bar{\alpha} \wedge \bar{1})(x), (\bar{\beta} \wedge (\mu \vee \rho))(x)\}$

$= \max\{\alpha, \min\{\beta, (\mu \vee \rho)(x)\}\}$

$= \max\{\alpha, \min\{\beta, 0.5\}\}$

$= \max\{\alpha, 0.5\}$

$= \alpha$

$= \lambda(x).$

$((\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge (\mu \vee \rho)))(y)$

$= \max\{(\bar{\alpha} \wedge \bar{1})(y), (\bar{\beta} \wedge (\mu \vee \rho))(y)\}$

$= \max\{\alpha, \min\{\beta, (\mu \vee \rho)(y)\}\}$

$= \max\{\alpha, \min\{\beta, 0.6\}\}$

$= \max\{\alpha, 0.6\}$

$= 0.6$

$= \lambda(y).$

$\therefore\ \lambda = (\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge (\mu \vee \rho))$

Hence the claim.

$\lambda = (\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge (\mu \vee \rho))$

$\lambda = (\bar{\alpha} \wedge \bar{1}) \vee (\bar{\beta} \wedge \rho)$

We have $\mathcal{T}_{st}(\lambda) = [\mathcal{T}(\bar{1}) \wedge \mathcal{T}(\mu \vee \rho)] \vee [\mathcal{T}(\bar{1}) \wedge \mathcal{T}(\rho)]$

$= [1 \wedge 3/4] \vee [1 \wedge 1/2]$

$= 3/4 \vee 1/2$

$= 3/4$

$\mathcal{T}^*_{st}(\lambda) = [\mathcal{T}^*(\bar{1}) \vee \mathcal{T}^*(\mu \vee \rho)] \wedge [\mathcal{T}^*(\bar{1}) \vee \tau^*(\rho)]$

$= [0 \vee 1/4] \wedge [0 \vee 1/2]$

$= 1/4 \vee 1/2$

$= 1/4$

If $\lambda(x) = \alpha$ for $0.5 < \alpha < 0.6$ and $\lambda(y) = \beta$ for $0.5 < \alpha, \beta < 0.6$ and $\alpha < \beta$ we have $\mathcal{T}_{st}(\lambda) = 3/4$ and $\mathcal{T}^*_{st}(\lambda) = 1/4$.

If $\lambda(x) = 0.5$ and $\lambda(y) = 0.6$, since for $\alpha \geq 0.5$ and $\beta \geq 0.6$,

$$\lambda = \bar{\beta} \wedge (\mu \vee \rho)$$
$$= (\bar{\beta} \wedge \mu) \vee (\bar{\beta} \wedge \rho)$$

We have $\mathcal{T}_{st}(\lambda) = 3/4$ and $\mathcal{T}^*_{st}(\lambda) = 1/4$.

If $\lambda(x) = 0.5$ and $\lambda(y) = \beta$ for $0.5 < \beta < 0.6$,

$$\lambda = \bar{\beta} \wedge (\mu \vee \rho)$$
$$= (\bar{\beta} \wedge \mu) \vee (\bar{\beta} \wedge \rho).$$

We have $\mathcal{T}_{st}(\lambda) = 3/4$ and $\mathcal{T}^*_{st}(\lambda) = 1/4$.

If $\lambda(x) = \alpha$ and $\lambda(y) = \beta$ for $0.4 < \alpha, \beta < 0.5$ and $\alpha < \beta$, since, for

$\lambda_1 \in \{\bar{1}, \mu, \mu \vee \rho\}$,

$\lambda_2 = \{\rho, \mu \wedge \rho\}$,

$\lambda = (\bar{\alpha} \wedge \lambda_1) \vee (\bar{\beta} \wedge \lambda_2)$

We have $\mathcal{T}^*_{st}(\lambda) = 2/3$ and $\mathcal{T}^*_{st}(\lambda) = 1/3$. By a similar method as the above cases, we can obtain the following :

$$\mathcal{T}_{st}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{\alpha}, \ \forall \alpha \in I, \\ 3/4 & \text{if } \lambda(x) = \alpha \text{ and } \lambda(y) = \beta \text{ for } 0.5 \leq \alpha, \beta \leq 0.6, \alpha < \beta, \\ 1/2 & \text{if } \lambda(x) = \alpha \text{ for } 0.4 \leq \alpha < 0.5, \lambda(y) = \beta \text{ for } 0.5 < \beta \leq 0.6 \\ 2/3 & \text{if } \lambda(x) = \alpha, \ \lambda(y) = \beta \text{ for } 0.4 \leq \alpha, \beta \leq 0.5, \alpha < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{T}^*_{st}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{\alpha}, \ \forall \alpha \in I, \\ 1/4 & \text{if } \lambda(x) = \alpha \text{ and } \lambda(y) = \beta \text{ for } 0.5 \leq \alpha, \beta \leq 0.6, \alpha < \beta, \\ 1/2 & \text{if } \lambda(x) = \alpha \text{ for } 0.4 \leq \alpha < 0.5, \lambda(y) = \beta \text{ for } 0.5 < \beta \leq 0.6 \\ 1/3 & \text{if } \lambda(x) = \alpha, \ \lambda(y) = \beta \text{ for } 0.4 \leq \alpha, \beta \leq 0.5, \alpha < \beta, \\ 1 & \text{otherwise} \end{cases}$$

Theorem : 3.3.4

Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(X, \mathcal{U}, \mathcal{U}^*)$ be IFTS. Let $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ and $(\mathcal{U}_{st}\ \mathcal{U}^*_{st})$ be stratification for $(\mathcal{T}, \mathcal{T}^*)$ and $(\mathcal{U}, \mathcal{U}^*)$ respectively.

If $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is intuitionistic continuous, then

$f : (X, \mathcal{T}_{st}, \mathcal{T}^*_{st}) \to (Y, \mathcal{U}_{st}\ \mathcal{U}^*_{st})$ is intuitionistic continuous.

Proof

Suppose there exist $\nu \in I^Y$ and $r \in I_0$, $m \in I_1$ such that

$\mathcal{U}_{st}(\nu) > r > \mathcal{T}_{st}(f^{-1}(\nu))$

$\mathcal{U}^*_{st}(\nu) < m < \mathcal{T}^*_{st}(f^{-1}(\nu))$

Since $\mathcal{U}_{st}(\nu) > r$ and $\mathcal{U}^*_{st}(\nu) < m$ by the definition of $(\mathcal{U}_{st}\ \mathcal{U}^*_{st})$, there exists a family $\{\nu_j / j \in J\}$ with $\nu = \bigvee_{j\in J} (\nu_j \wedge \overline{\alpha}_j)$ such that

$\mathcal{U}_{st}(\nu) \geq \bigwedge_{j\in J} \mathcal{U}(\nu_j)$

$> r$

and $\mathcal{U}^*_{st}(\nu) \leq \bigvee_{j\in J} \mathcal{U}^*(\nu_j)$

$< m$

Consider

$f^{-1}(\nu) = f^{-1}(\bigvee_{j\in J} (\nu_j \wedge \overline{\alpha}_j))$

$= \bigvee_{j\in J} (f^{-1}(\nu_j) \wedge \overline{\alpha}_j)$

By the definition of $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ we have

$\mathcal{T}_{st}(f^{-1}(\nu)) \geq \bigwedge_{j\in J} \mathcal{T}(f^{-1}(\nu_j))$ and

$\mathcal{T}^*_{st}(f^{-1}(\nu)) \leq \bigvee_{j\in J} \mathcal{T}^*(f^{-1}(\nu_j))$

Since $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is intuitionstic continuous, we get

$\mathcal{T}(f^{-1}(\nu_j)) \geq \mathcal{U}(\nu_j)$ and

$\mathcal{T}^*(f^{-1}(v_j)) \leq \mathcal{U}^*(v_j)$ for each $j \in J$.

i.e.,

$$\mathcal{T}_{st}(f^{-1}(v)) \geq \bigwedge_{j\in J} \mathcal{T}(f^{-1}(v_j))$$

$$\geq \bigwedge_{j\in J} \mathcal{U}(v_j) > r,$$

$$\mathcal{T}^*_{st}(f^{-1}(v)) \leq \bigvee_{j\in J} \mathcal{T}^*(f^{-1}(v_j))$$

$$\leq \bigvee_{j\in J} \mathcal{U}^*(v_j)$$

$$< m$$

It is a contradiction.

Hence $f : (X, \mathcal{T}_{st}, \mathcal{T}^*_{st}) \to (Y, \mathcal{U}_{st}\ \mathcal{U}^*_{st})$ is intuitionistic continuous.

The converse of the previous theorem is not true as is seen from the following example :

Example : 3.3.5

Let X be a nonempty set. Define IGO's $(\mathcal{T}, \mathcal{T}^*)$ and $(\mathcal{U}, \mathcal{U}^*)$ on X as follows :

For each $\lambda \in I^X$

$$\mathcal{T}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{1}, \bar{0} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{T}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{1}, \bar{0} \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{U}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{1}, \bar{0} \\ 1/3 & \text{if } \lambda = \overline{0.5} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{U}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{1}, \bar{0} \\ 2/3 & \text{if } \lambda = \overline{0.5} \\ 1 & \text{otherwise} \end{cases}$$

Since 0 = $\mathcal{T}(\overline{0.5}) < \mathcal{U}(\overline{0.5})$ = 1/3 and 1 = $\mathcal{T}^*(\overline{0.5}) > \mathcal{U}^*(\overline{0.5})$ = 2/3, the identify function

id_X : $(X, \mathcal{T}, \mathcal{T}^*) \to (X, \mathcal{U}, \mathcal{U}^*)$ is not intuitionistic continuous.

For a family $\{\bar{1}\}$ with $(\overline{0.5}) = (\overline{0.5}) \wedge \bar{1}$

We have, $\mathcal{U}_{st}(\overline{0.5}) \geq \mathcal{U}(\bar{1}) = 1$ and

$$\mathcal{U}^*_{st}(\overline{0.5}) \leq \mathcal{U}^*(\bar{1}) = 0$$

Hence $\mathcal{U}_{st}(\overline{0.5}) = 1$ and $\mathcal{U}^*_{st}(\overline{0.5}) = 0$. Thus,

$$\mathcal{T}_{st}(\lambda) = \mathcal{U}_{st}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{\alpha}, \forall \alpha \in L \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{T}^*_{st}(\lambda) = \mathcal{U}^*_{st}(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{\alpha}, \forall \alpha \in I \\ 1 & \text{otherwise} \end{cases}$$

id_X : $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st}) \to (X, \mathcal{U}_{st}, \mathcal{U}^*_{st})$ is intuitionistic continuous.

Theorem : 3.3.6

Let $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$ be a stratification of an IFTS $(X, \mathcal{T}, \mathcal{T}^*)$. A fuzzy set λ is a (r, s)-component in $(X, \mathcal{T}, \mathcal{T}^*)$ iff λ is a (r, s)-component in $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$.

Proof

(I) Let λ be a (r, s)-component in $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$. Suppose that λ is not (r, s)-connected in $(X, \mathcal{T}, \mathcal{T}^*)$. Then there exist $\mu \neq 0$ and $\rho = \bar{0}$ which are (r, s)-separated in $(X, \mathcal{T}, \mathcal{T}^*)$ such that $\lambda = \mu \vee \rho$.

Since $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is finer than $(\mathcal{T}, \mathcal{T}^*)$ by Theorem ...., we get $\mathcal{T} \leq \mathcal{T}_{st}$ and $\mathcal{T}^* \geq \mathcal{T}^*_{st}$

$\therefore \mathcal{T}(\lambda) \geq r \Rightarrow \mathcal{T}_{st}(\lambda) \geq r$

and $\mathcal{T}^*(\lambda) \leq r \Rightarrow \mathcal{T}^*_{st}(\lambda) \leq r$

Hence the collection corresponding to $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$ is larger than the collection corresponding to $\mathcal{T}$, $\mathcal{T}^*$

$\therefore$ $C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\mu, r, s) \le C_{\mathcal{T}, \mathcal{T}^*}(\mu, r, s)$

and $C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\rho, r, s) \le C_{\mathcal{T}, \mathcal{T}^*}(\rho, r, s)$ (1)

Since, μ and ρ are (r, s)-separated,

$C_{\mathcal{T}, \mathcal{T}^*}(\mu, r, s) \wedge \rho = C_{\mathcal{T}, \mathcal{T}^*}(\rho, r, s) \wedge \mu = \bar{0}$.

Using this in (1), we get

$C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\mu, r, s) \wedge \rho = C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\rho, r, s) \wedge \mu = \bar{0}$.

Hence μ and ρ are (r, s)-separated in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$), Thus, λ is not a (r, s)-component in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$).

This is a contradiction.

Hence λ is a (r, s)-connected in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$).

(II) We will show that if λ is a (r, s)-component in (X, $\mathcal{T}$, $\mathcal{T}^*$), then λ is (r, s)-connected in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$).

Let λ be a (r, s)-component in (X, $\mathcal{T}$, $\mathcal{T}^*$).

Then $C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s) = \lambda$ from Theorem 3.2.14 (1), suppose that λ is not (r, s)-connected in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$), then $\mu \neq \bar{0}$ and $\rho \neq \bar{0}$ are (r, s)-separated in (X, $\mathcal{T}_{st}$, $\mathcal{T}_{st}^*$) such that $\lambda = \mu \vee \rho$

Since $\mathcal{T} \le \mathcal{T}_{st}$ and $\mathcal{T}^* \ge \mathcal{T}_{st}^*$

$C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\lambda, r, s) \le C_{\mathcal{T}, \mathcal{T}^*}(\lambda, r, s)$

$= \lambda$.

Always $\lambda \le C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\lambda, r, s)$

So, $C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\lambda, r, s) = \lambda$

Since $\mu \le \lambda$, we have $C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\mu, r, s) \le \lambda$.

It implies $\lambda = C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s) \vee \rho$

Put $C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s) = \omega$

If $x \in supp(\omega)$, then $x \in supp(\lambda)$.

Since λ is (r, s)-component in (X, τ, τ^*), by Theorem 3.2.14 (4)

$x_1 \in \lambda = \omega \vee \rho$

i.e., $\omega(x) \vee \rho(x) = 1$.

Since $\omega \wedge \rho = \bar{0}$, $\omega(x) \wedge \rho(x) = 0$.

Since $x \in supp(\omega)$, $\omega(x) \neq 0$.

$\therefore \rho(x) = 0$.

So, $\omega(x) = 1$.

Hence ω is a crisp set.

Since $\mathcal{T}_{st}(\bar{1} - \omega) \geq r$ and $\mathcal{T}^*_{st}(\bar{1} - \omega) \leq s$ from Theorem 3.1.3 and Theorem 3.1.5

for any family $\{\bar{\alpha}_i \wedge n_i \,/\, \bar{1} - \omega = \vee_{i\in J} \bar{\alpha}_i \wedge n_i\}$,

$$\mathcal{T}_{st}(\bar{1} - \omega) = \vee\{\bigwedge_{i\in J} \mathcal{T}(n_i)\} \geq r$$

$$\mathcal{T}^*_{st}(\bar{1} - \omega) = \wedge\{\bigvee_{i\in J} \mathcal{T}^*(n_i)\} \leq s$$

Without loss of generality, we assume that $\bar{\alpha}_i \neq \bar{0}$. Since $\omega(x) = 1$ for $x \in supp(\omega)$.

$$(\bar{1} - \omega)(x) = \bigvee_{i\in J} (\bar{\alpha}_i \wedge n_i)(x)$$

$$\Rightarrow 1 = \omega(x) = \bigwedge_{i\in J} \{(\bar{1} - \bar{\alpha}_i)(x) \vee (\bar{1} - n_i)(x)\}$$

$$\Rightarrow (\bar{1} - \bar{\alpha}_i)(x) \vee (\bar{1} - n_i)(x) = 1 \ \forall\, i \in J \qquad (2)$$

Since $\bar{\alpha}_i \neq \bar{0}$, $\bar{\alpha}_i(x) = \alpha_i \neq 0$

$\therefore (\bar{1} - \bar{\alpha}_i)(x) = 1 - \alpha_I < 1$.

Hence if $x \in supp(\omega)$, from (2), we get

$(\bar{1} - n_i)(x) = 1 \ \forall\, i \in J$

$\Rightarrow (\bar{1} - n_i)(x) > (\bar{1} - \bar{\alpha}_i)(x)$

$\Rightarrow n_i(x) < \overline{\alpha}_i(x)$

$\Rightarrow (\overline{\alpha}_i \wedge n_i)(x) = n_i(x)$

Thus for $x \in supp(\omega)$, $(\bar{1} - \omega)(x) = \bigvee_{i \in J} (\overline{\alpha}_i \wedge n_i)(x)$

$$= \bigvee_{i \in J} n_i(x) \qquad (3)$$

If $y \notin supp(\omega)$, then

$1 = (\bar{1} - \omega)(y) = \bigvee_{i \in J} (\overline{\alpha}_i \wedge n_i)(y) \leq \bigvee_{i \in J} n_i(y)$

$$\therefore 1 = (\bar{1} - \omega)(y) = \bigvee_{i \in J} n_i(y) \text{ for } y \notin supp(\omega) \qquad (4)$$

Hence from (3) and (4) for any family

$\{\overline{\alpha}_i \wedge n_i \,/\, \bar{1} - \omega = \bigvee_{i \in J} \overline{\alpha}_i \wedge n_i\}$, we have

$\bar{1} - \omega = \bigvee_{i \in J} n_i$

It implies

$\mathcal{T}_{st}(\bar{1} - \omega) = \mathcal{T}(\bar{1} - \omega) = \bigwedge_{i \in J} \mathcal{T}(n_i) \geq r.$

$\mathcal{T}^*_{st}(\bar{1} - \omega) = \mathcal{T}^*(\bar{1} - \omega) = \bigwedge_{i \in J} \mathcal{T}^*(n_j) \leq s.$

So, $C_{\mathcal{T}, \mathcal{T}^*}(\omega, r, s) = \omega$

Substituting for ω, we get

$$C_{\mathcal{T}, \mathcal{T}^*}(C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s), r, s) = C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s) \qquad (5)$$

Similarly, we have

$$C_{\mathcal{T}, \mathcal{T}^*}(C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\rho, r, s), r, s) = C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\rho, r, s) \qquad (6)$$

Consider

$C_{\mathcal{T}, \mathcal{T}^*}(\mu, r, s) \wedge \rho \leq C_{\mathcal{T}, \mathcal{T}^*}(C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s), r, s) \wedge \rho$

$= C_{\mathcal{T}_{st}, \mathcal{T}^*_{st}}(\mu, r, s) \wedge \rho$ from (5)

$= \bar{0}$ as μ and ρ are (r, s)-separated in $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$.

$\mu \wedge C_{\mathcal{T}, \mathcal{T}^*}(\rho, r, s) \leq \mu \wedge C_{\mathcal{T}, \mathcal{T}^*}(C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\rho, r, s), r, s)$

$= \mu \wedge C_{\mathcal{T}_{st}, \mathcal{T}_{st}^*}(\rho, r, s)$ from (6)

$= \bar{0}$ as μ and ρ are (r, s)-separated in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$.

Hence μ and ρ are (r, s)-separated in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$.

Thus λ is not a (r, s)-component in $(X, \mathcal{T}, \mathcal{T}^*)$

It is contradiction.

Hence λ is (r, s)-connected in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$.

(III) Let λ be a (r, s)-component in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$. From (I), λ is (r, s)-connected in $(X, \mathcal{T}, \mathcal{T}^*)$. There exists a (r, s)-component μ in $(X, \mathcal{T}, \mathcal{T}^*)$ containing λ.

From (II), μ is (r, s)-connected in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$ and also it contains λ.

Thus by the maximality of λ, we must have $\lambda = \mu$.

Thus λ is a (r, s)-component in $(X, \mathcal{T}, \mathcal{T}^*)$. Similarly, if ρ is a (r, s)-component in $(X, \mathcal{T}, \mathcal{T}^*)$, from (II), we get ρ is (r, s)-connected in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$.

$\therefore$ There exists a (r, s)-component, ν in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$ containing ρ.

By (I) we get ν is (r, s)-connected in $(X, \mathcal{T}, \mathcal{T}^*)$ and it contains ρ.

$\therefore$ By the maximality of ρ, we must have $\rho = \nu$

$\therefore$ ρ is a (r, s)-component in $(X, \mathcal{T}_{st}, \mathcal{T}_{st}^*)$.