CHAPTER. 3

CHAPTER 3

SOME NEW TYPES OF GENERALIZED CLOSED SETS, CONTINUOUS AND IRRESOLUTE MAPS IN MINIMAL STRUCTURES.

This chapter deals with study of various generalizations of closed sets namely g-closed sets, gs-closed sets, sg-closed sets, π-closed sets, πg-closed sets, πgs-closed sets, gp-closed sets, πgp-closed sets, $\hat{g}$-closed sets, $\tilde{g}$s-closed sets, λ-closed sets and generalization of continuous and irresolute maps in minimal structures. Properties and characterizations and inter relationships between these concepts are analyzed.

SECTION: 3.1
PRELIMINARIES

Definition: 3.1.1

Let (X,m_X) be an m-space. A subset A of X is an **m_X-semiopen set** if there exists $U \in m_X$ such that $U \subseteq A \subseteq m_X\text{-cl}(U)$. A subset A of X is **$m_X$-semiclosed** if $X - A$ is m_X-semiopen.

Definition: 3.1.2

Let (X,m_X) be an m-space. A subset A of X is an **m_X-preopen** set if $A \subseteq m_X\text{-int}(m_X\text{-cl}(A))$. A subset A of X is **$m_X$-preclosed** if $X - A$ is m_X-preopen.

Definition: 3.1.3

Let (X,m_X) be an m-space. A sub set A of X is an m_X-regular open set if A = m_X-int (m_X-cl(A)). A sub set A of X is an **m_X-regular closed** set if X – A is an **m_X-regular open set**.

Definition: 3.1.4

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-π-open set** if A is the finite union of m_X-regular open sets.

Preposition: 3.1.5

Let A be an m-space (X,m_X).

(i) If A is m_X-closed, then it is m_X-semiclosed.

(ii) If A is m_X-closed, then it is m_X-preclosed.

Remark: 3.1.6

Converse of the above preposition [3.1.5] need not be true.

Example: 3.1.7

(i) Let X = {a, b, c, d}, m_X = { ϕ, X, {a}, {b}, {a, c}}. Then the set {b} is m_X-semiclosed but not m_X-closed.

(ii) Let X = {a, b, c, d}, m_X = { ϕ, X, {a, b, c}, {c, d}}. Then the set {b} is m_X-preclosed but not m_X-closed.

Definition: 3.1.8

Let (X,m_X) be an m-space and $B \subseteq X$. The **m_X-semiclosure** of B denoted by m_X-scl(B) is defined to be the intersection of all m_X-semiclosed sets of (X,m_X) containing B.

Properties of m_X-semiclosure: 3.1.9

(i) m_X-scl(ϕ) = ϕ.

(ii) m_X-scl(X) = X.

(iii) If $A \subseteq B$ then m_X-scl(A) $\subseteq m_X$-scl(B).

(iv) If $\phi \neq B \neq X$. Then m_X-scl(B) is not necessarily an m_X-semiclosed set.

(v) m_X-scl(X – A) = X – m_X-sint(A).

(vi) m_X-sint(X – A) = X – m_X-scl(A).

Definition: 3.1.10

The **m_X-preclosure** of B denoted by m_X-pcl(B) is defined to be the intersection of all m_X-preclosed sets of (X,m_X) containing B.

Properties of m_X-preclosure: 3.1.11

(i) m_X-pcl(ϕ) = ϕ.

(ii) m_X-pcl(X) = X.

(iii) If $A \subseteq B$ then m_X-pcl(A) $\subseteq m_X$-pcl(B).

(iv) If $\phi \neq B \neq X$. Then m_X-pcl(B) is not necessarily an m_X-preclosed set.

(v) m_X-pcl$(X - A) = X - m_X$-pint(A).

(vi) m_X-pint$(X - A) = X - m_X$-pcl(A).

Lemma: 3.1.12

Let m_X be an m-structure on X which satisfies the property of Maki. If $A_i \in SO(X,m_X)$ for each $i \in I$, then $\bigcup_{i\in I} A_i \in SO(X,m_X)$.

Proof:

Let $A_i \in SO(X,m_X)$ for each $i \in I$, then there exists a set $U_i \in m_X$ such that $U_i \subseteq A_i \subseteq m_X\text{-cl}(U_i)$, in consequence, $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} m_X\text{-cl}(U_i)$.

Since m_X-cl is a monotone operator, then $\bigcup_{i\in I} m_X\text{-cl}(U_i) \subseteq m_X\text{-cl}(\bigcup_{i\in I} U_i)$ and $\bigcup_{i\in I} U_i \in m_X$, threfore $\bigcup_{i\in I} U_i \in m_X$ and $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq m_X\text{-cl}(\bigcup_{i\in I} U_i)$ and hence $\bigcup_{i\in I} A_i \in SO(X,m_X)$.

Theorem: 3.1.13

(i) $x \in m_X$-scl(A) if and only if $U \cap A \neq \phi$ for every m_X-semi open set U containing x.

In the case that m_X satisfy the property of Maki.

(ii) A is an m_X- semiclosed set if and only if $A = m_X$-scl(A).

Theorem: 3.1.14

Let (X,m_X) be an m-space and $A \subseteq X$. If m_X satisfy the property of Maki. Then $m_X\text{-scl}(A) = A \cup m_X\text{-int}(m_X\text{-cl}(A))$.

Proof:

Since m_X satisfies the property of Maki, m_X-scl(A) is an

m_X-semiclosed set, using Definition [3.1.1], we obtain that m_X-int(m_X-cl(m_X-sclA))$\subseteq m_X$-scl(A). Therefore m_X-int(m_X-cl(A))$\subseteq m_X$-scl(A) and follows that $A\cup m_X$-int(m_X-cl(A)) $\subseteq$ m_X-scl(A). The opposite inclusion, we observe that m_X-int(m_X-cl($A\cup m_X$-int(m_X-cl(A)))) = m_X-int(m_X-cl(A)$\cup m_X$-cl(m_X-int(m_X-cl(m_X-cl(A))))$\subseteq$(m_X-cl(A))$\cup m_X$-int(m_X-cl(m_X-int(m_X-cl(A))))=$m_X$cl(A)$\cup m_X$-int(m_X-cl(A))=m_X-cl(A). Thus m_X-int(m_X-cl($A\cup m_X$-int(m_X-cl(A))))$\subseteq m_X$-Int(m_X-cl(A))$\subseteq A\cup m_X$-int(m_X-cl(A)). Follows that m_X-int(m_X-cl($A\cup m_X$-int(m_X-cl(A))))$\subseteq A\cup m_X$-int(m_X-cl(A)). Therefore by Definition [3.1.1], $A\cup m_X$-int (m_X-cl(A)) is an m_X-semiclosed set and so m_X-scl(A)$\subseteq$ $A\cup m_X$-int(m_X-cl(A)).

Remark: 3.1.15

The property of Maki condition is removed in the above theorem [3.1.14] the equality is not true.

Example: 3.1.16

Let X = N. Also, define the m-structure on X as follows:
$m_X = \{\phi, N, P(\{2n : n\in N\}), \{1\}\}$. Then, the m_X-closed sets ϕ, N, P ({2n: n∈N})^C and N – {1}. Also, SO(X,m_X) = {ϕ,N, P({2n : n∈N), {1}, F},
Where F $\cap$ {2n: n∈N} $\neq \phi$. If we take A = {3}, then m_X-scl (A) = {3}, m_X-cl(A) = {2n + 1 : n∈N} and m_X-int({2n + 1 : n∈N}) = {1}. It is clear that $A\cup m_X$-int(m_X-cl(A)) = {1, 3} $\supseteq$ {3} = m_X-scl(A). In consequence, m_X-scl(A)$\subseteq A\cup m_X$-int(m_X-cl(A)).

Theorem: 3.1.17

Let (X,m_X) be an m-space and $A \subseteq X$. If m_X satisfy the property of Maki. Then

(i) m_X-sint(A) = $A \cap m_X$-cl (m_X-int(A)).

(ii) m_X-pcl(A) = $A \cup m_X$-cl (m_X-int(A)).

(iii) m_X-pint(A) = $A \cap m_X$-int (m_X-cl(A)).

Proof:

(i) Follows from Theorems [3.1.13] and [3.1.14].

(ii) The proof is similar to the proof of Theorem [3.1.14].

(iii) Follows from (ii).

SECTION: 3.2
NEW GENERALIZED CLOSED SETS UNDER MINIMAL STRUCTURES

Definition: 3.2.1

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-πg-closed set** if the m_X-cl(A)$\subseteq$U whenever $A \subseteq U$ and U is m_X-π-open in X.

Definition: 3.2.2

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-gp-closed set** if the m_X-pcl(A)$\subseteq$U whenever $A \subseteq U$ and U belongs to m_X.

Definition: 3.2.3

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-πgp-closed set** if the m_X-pcl(A)$\subseteq$U whenever A$\subseteq$U and U m_X-π-open in X.

Definition: 3.2.4

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-gs-closed set** if the m_X-scl(A)$\subseteq$U whenever A$\subseteq$U and U belongs to m_X.

Definition: 3.2.5

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-sg-closed set** if the m_X-scl(A)$\subseteq$U whenever A$\subseteq$U and U is an m_X-semiopen set in X.

Definition: 3.2.6

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-πgs-closed set** if the m_X-scl(A)$\subseteq$U whenever A$\subseteq$U and U is m_X-π-open in X.

Example: 3.2.7

Let X = {a, b, c, d}, m_X = {ϕ, X, {a}, {b}, {a, c}}. Then SO(X,m_X) = {ϕ, X, {a}, {b},{a, c}, {a, d}, {b, d}, {a, c, d}}, PO(X,m_X) = {ϕ, X, {a}, {b}, {c}, {a, c},{a, b, c}}, RO(X,mX) = {ϕ, X, {b}, {a, c}}.

Example: 3.2.8

Let X = {a, b, c}, m_X = {ϕ, X, {a}, {b}}. Then SO(X,m_X)={ϕ, X, {a}, {b}, {b, c}, {a, c}}, PO(X,m_X) = {ϕ, X, {a} ,{b}, {a, b}}, RO(X,m_X) = {ϕ, X, {a}, {b}}.

Example: 3.2.9

Let X = {a, b, c, d}, m_X = { ϕ, X, {a}, {b}, {a, b, d}, {a, b, c}}. Then SO(X,m_X)= { ϕ, X, {a}, {b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b,d}, {a, c, d},{b, c, d}}, RO(X,m_X) = { ϕ, X, {a}, {b}} and PO(X,m_X) = { ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}.

Example: 3.2.10

Let X = {a, b, c, d}, m_X = { ϕ, X, {a}, {b}, {a, b}, {a, b, d}, {a, b,c}}. Then SO(X,m_X) = { ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d},{a, c, d}, {b, c, d}, {a, b, c}, {a, b, d}}, RO(X,m_X) = { ϕ, X, {a}, {b}} and PO(X,m_X) = { ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}.

Example: 3.2.11

Let X = {a, b, c, d}, m_X = { ϕ, X, {a, b, c}, {c, d}}. Then SO(X,m_X) = { ϕ,X, {c, d}, {a, b, c},{a, c, d}, {b, c, d}}, PO(X,m_X) = { ϕ, X, {c}, {a, c}, {a, d},{b, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}, {a, b, d}}, RO(X,m_X) = { ϕ,X}.

Example: 3.2.12

Let X = {a, b, c, d}, m_X = { ϕ,X, {a, b}, {a, c}, {a}, {b}, {c}}. Then SO(X,m_X)= { ϕ, X, {a, b}, {a, c}, {a}, {b}, {c}}, PO(X,m_X) = { ϕ, X, {a}, {b}, {c},{a, b}, {a, c}, {b, c}, {a, b, c}}, RO(X,m_X) = { ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}.

Remark: 3.2.13

(i) In Example [3.2.7], the set {a, c} is m_X-gs-closed but not m_X-sg-closed.

(ii) In Example [3.2.9], the set {a, b} is m_X-sg-closed but not m_X-gs-closed.

Preposition: 3.2.14

Let A be an m-space (X,m_X). If A is m_X-semiclosed, then it is m_X-gs-closed.

Remark: 3.2.15

Converse of the above preposition [3.2.14] need not be true.

Example: 3.2.16

In Example [3.2.7], the set {c} is m_X-gs-closed but not m_X-semiclosed.

Preposition: 3.2.17

Let A be an m-space (X,m_X).

(i) If A is m_X-g-closed, then it is m_X-gs-closed.

(ii) If A is m_X-g-closed, then it is m_X-gp-closed.

Remark: 3.2.18

Converse of the above preposition [3.2.17] need not be true.

Example: 3.2.19

(i) In Example [3.2.7], the set {b} is m_X-gs-closed but not m_X-g-closed.

(ii) In Example [3.2.7], the set {a, d} is m_X-gp-closed but not m_X-gclosed.

Preposition: 3.2.20

Let A be an m-space (X,m_X).

(i) If A is m_X-πg-closed, then it is m_X-πgs-closed.

(ii) If A is m_X-πg-closed, then it is m_X-πgp-closed.

Remark: 3.2.21

Converse of the above preposition [3.2.20] need not be true.

Example: 3.2.22

(i) In Example [3.2.7], the set {a} is m_X-πgs-closed but not m_X-πg-closed.

(ii) In Example [3.2.12], the set {a} is m_X-gp-closed but not m_X-πg-closed.

Theorem: 3.2.23

Let (X,m_X) be an m-space where m_X satisfy the property of Maki and $A \subseteq X$. The following properties hold:

(i) If A is m_X-gs-closed, then A is m_X-πgs-closed.

(ii) If A is m_X-gp-closed, then A is m_X-πgp-closed.

(iii) If A is m_X-g-closed, then A is m_X-πg-closed.

(iv) If A is m_X-sg-closed, then A is m_X-πsg-closed.

Remark: 3.2.24

If the condition of Maki is dropped in Theorem [3.2.23], the result is not necessarily true as we can see as follows:

(i) The set {a, b} in Example [3.2.8] is m_X-gs-closed but not m_X-πgs- closed.

(ii) The set {a, b, c} in Example [3.2.7] is m_X-gp-closed but not m_X-πgp- closed.

(iii) The set {a, b} in Example [3.2.8] is m_X-g-closed but not m_X-πg- closed.

(iv) The set {a, b} in Example [3.2.8] is is m_X-sg-closed, but not m_X-πsg- closed.

Remark: 3.2.25

In Theorem [3.2.23], none of the implications are reversible.

(i) In Example [3.2.10], the set {a, b, c} is m_X-πgs closed but not m_X-gs-closed.

(ii) In Example [3.2.11], the set {a, b, c} is m_X-πgp-closed but not m_X-gp-closed.

(iii) In Example [3.2.10], the set {a, c} is m_X-πg-closed but not m_X-g-closed.

(iv) In Example [3.2.11], the set {c} is m_X-π-sg-closed but not m_X-sg-closed.

Remark: 3.2.26

The notions m_X-gs-closedness and m_X-gp-closedness are independent.

(i) In Example [3.2.10], the set {a} is m_X-gs-closed but not m_X-gp-closed.

(ii)In Example [3.2.11], the set {c} is m_X-gp-closed but not m_X-gs-closed.

Remark: 3.2.27

The notions m_X-πgs-closed is different from the notion of m_X-πgp-closed set

(i) In Example [3.2.10], the set {b} is m_X-πgs-closed but not m_X-πgp-closed.

(ii) In Example [3.2.12], the set {a} is m_X-πgp closed but not m_X-πgs-closed.

Definition: 3.2.28

An m-space (X,m_X) is said to be **m_X-$T_{1/2}$ space** if every m_X-g-closed set is m_X-closed.

Definition: 3.2.29

An m-space (X,m_X) is said to be **m_X-s$T_{1/2}$ space** if every m_X-gs-closed set is m_X-semiclosed.

Definition: 3.2.30

An m-space (X,m_X) is said to be **m_X-πgp-$T_{1/2}$ space** if every m_X-πgp-closed set is m_X-preclosed.

Theorem: 3.2.31

The following results are true for a subset A of an m-space (X,m_X).

(i) A is m_X-sg-closed if and only if m_X-cl(A)$\subseteq m_X$-sKer(A).

(ii) A is m_X-g-closed if and only if m_X- cl(A) $\subseteq$ m_X-Ker(A).

(iii) A is m_X-gs-closed if and only if m_X-scl(A)$\subseteq m_X$-sKer(A).

(iv) A is m_X-gp-closed if and only if m_X-pcl(A)$\subseteq m_X$-pKer(A).

(v) A is m_X-πg-closed if and only if m_X-cl(A)$\subseteq m_X$-π-Ker(A).

(vi) A is m_X-πgs-closed if and only if m_X-scl(A)$\subseteq m_X$-π-Ker(A).

(vii) A is m_X-πgp-closed if and only if m_X-pcl(A)$\subseteq m_X$-π-Ker(A).

Where m_X-Ker(A) (resp. m_X-sKer(A), m_X-pKer(A), m_X-π-Ker(A)) is defined as the intersection of all m_X-open sets of (resp. m_X-semi-open, m_X-pre-open, m_X-π-open) containing A.

Proof:

Let A $\subseteq$ X be an m_X-sg-closed set. Let D = {S: S $\subseteq$ X, A $\subseteq$ S, S$\in$SO(X,m_X)}. Then m_X-sKer(A) = $\bigcap_{S\in D}$ S. For all S$\in$D, m_X-scl(A)$\subseteq$S as A is m_X-sg-closed. Hence m_X-scl(A)$\subseteq$ $\bigcap_{S\in D}$ S = m_X-sKer(A).

Conversely assume that m_X-scl(A)$\subseteq m_X$-sKer(A), take S$\in$SO(X,m_X) such that A$\subseteq$S. Then by hypothesis m_X-scl(A)$\subseteq m_X$-sKer(A)$\subseteq$S. Therefore A is m_X-sg-closed.

The proof of (ii), (iii), (iv), (v), (vi) and (vii) are similar.

Theorem: 3.2.32

Let m_X be an m-structure on X satisfying the property of Maki and A$\subseteq$X. Then:

(i) A is m_X-gs-closed if and only if there does not exists a nonempty m_X-closed set F such that F$\subseteq m_X$-scl(A) – A.

(ii) A is an m_X-sg- closed if and only if there does not exists a nonempty m_X-semiclosed set F such that F$\subseteq m_X$-scl(A) – A.

(iii) A is an m_X-g-closed if and only if there does not exists a nonempty m_X -closed set F such that F$\subseteq m_X$-cl(A) – A

(iv) A is an m_X-πg-closed if and only if there does not exists a nonempty

m_X-π-closed set F such that $F \subseteq m_X\text{-cl}(A) - A$.

(v) A is an m_X-gp-closed if and only if there does not exists a nonempty m_X-closed set F such that $F \subseteq m_X\text{-pcl}(A) - A$.

(vi) A is an m_X-πgp-closed if and only if there does not exists a nonempty m_X-π-closed set F such that $F \subseteq m_X\text{-pcl}(A) - A$.

(vii) A is an m_X-πgs-closed if and only if there does not exists a nonempty m_X-π-closed set F such that $F \subseteq m_X\text{-scl}(A) - A$.

Proof:

(i) Let A be an m_X-gs-closed set and let $F \subseteq X$ be an m_X-closed set such that $F \subseteq m_X\text{-scl}(A) - A$. Then $A \subseteq X - F$ and $X - F$ is an m_X- open set. Since A is an m_X-gs-closed, we have that $m_X\text{-scl}(A) \subseteq X - F$ and $F \subseteq X - m_X\text{-scl}(A)$. Hence $F \subseteq m_X\text{-scl}(A) \cap (X - m_X\text{-scl}(A)) = \phi$. Therefore $F = \phi$.

Conversely let $A \subseteq U$ and U be an m_X-open set. Then

$m_X\text{-scl}(A) \cap (X - U) \subseteq m_X\text{-scl}(A) \cap (X - A) = m_X\text{-scl}(A) - A$.

Since $m_X\text{-scl}(A) - A$ does not containing any nonempty m_X-closed.

We obtain that $m_X\text{-scl}(A) \cap (X - U) = \phi$.Hence $m_X\text{-scl}(A) \subseteq U$. Therefore A is m_X-gs-closed in (X, m_X).

(ii)Let A be an m_X-sg-closed set and let F be an m_X-semiclosed set of (X, m_X) such that $F \subseteq m_X\text{-scl}(A) - A$. Then $A \subseteq X - F$ and $X - F$ is an m_X-semiopen set, since A is an m_X-sg-closed, we have that $m_X\text{-scl}(A) \subseteq X - F$ and $F \subseteq X - m_X\text{-scl}(A)$. Hence $F \subseteq m_X\text{-scl}(A) \cap (X - m_X\text{-scl}(A)) = \phi$. Therefore $F = \phi$.

Conversely, let $A \subseteq U$ and U be an m_X-semiopen set, then

$m_X\text{-scl}(A) \cap (X - U) \subseteq m_X\text{-scl}(A) \cap (X - A) = m_X\text{-scl}(A) - A$.

Since $m_X\text{-scl}(A) - A$ does not contain any nonempty m_X-semiclosed sets, we obtain that $m_X\text{-scl}(A) \cap (X - U) = \phi$. Hence $m_X\text{-scl}(A) \subseteq U$.

Therefore A is an m_X-gs-closed.

(iv) Let A be an m_X-πg-closed set and let $F \subseteq X$ be an m_X-π-closed set such that $F \subseteq m_X\text{-cl}(A) - A$. Then $A \subseteq X - F$ and $X - F$ is an m_X-π- open set. Since A is an m_X-πg-closed, we have that $m_X\text{-cl}(A) \subseteq X - F$ and $F \subseteq X - m_X\text{-cl}(A)$. Hence $F \subseteq m_X\text{-cl}(A) \cap (X - m_X\text{-cl}(A)) = \phi$. Therefore $F = \phi$.

Conversely, let $A \subseteq U$ and U be an m_X-π- open set, then $m_X\text{-cl}(A) \cap (X - U) \subseteq m_X\text{-cl}(A) \cap (X - A) = m_X\text{-cl}(A) - A$. Since $m_X\text{-cl}(A) - A$ does not contain any nonempty m_X-π-closed sets, we obtain that $m_X\text{-cl}(A) \cap (X - U) = \phi$. Therefore $m_X\text{-cl}(A) \subseteq U$. Hence A is an m_X-πg-closed.

The proof of (iii), (v), (vi) and (vii) are similar.

Example: 3.2.33

(i) In Example [3.2.9], the set {a, b} is sg-closed, and {d} is an m_X-semi-closed such that $\{d\} \subseteq m_X\text{-scl}(\{a, b\}) - \{a, b\})$.

(ii) In Example [3.2.9], the set {a} is not m_X-g-closed and there does not exist m_X-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-cl}(A) - A$.

(iii) In Example [3.2.7], the set {a} is not m_X-gs-closed, and there does not exist m_X-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-scl}(\{a\}) - \{a\}$.

(iv) In Example [3.2.12] the set {c} is not m_X-gp-closed, and there does not exists a m_X-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-pcl}(\{c\}) - \{c\}$.

Theorem: 3.2.34

Let (X,m_X) be an m-space satisfying the property of Maki and $A \subseteq X$. Then the following properties are equivalent:

(i) A is m_X-π-open and m_X-π-gs-closed.

(ii) A is m_X-regular open.

Proof:

(i) ⇒ (ii)

Let A be m_X-π-open and m_X-π-gs-closed. Then $m_X\text{-scl}(A) \subseteq A$. Using Theorem [3.1.14], $m_X\text{-scl}(A) = A \cup m_X\text{-int}(m_X\text{-cl}(A))$.

We obtain that $m_X\text{-int}(m_X\text{-cl}(A)) \subseteq A$. Since A is m_X-π-open, A is m_X-open and A is m_X-preopen. Hence $A \subseteq m_X\text{-int}(m_X\text{-cl}(A))$.

Therefore $m_X\text{-int}(m_X\text{-cl}(A)) \subseteq A \subseteq m_X\text{-int}(m_X\text{-cl}(A))$.

Hence A is m_X-regular open.

(ii) ⇒ (i)

Let A is m_X-regular open. Every m_X-regular open set is m_X-π-open and m_X-open. Therefore m_X-π-open and m_X-semiopen. By Theorem [3.1.14], $m_X\text{-scl}(A) = A$. Hence A is m_X-π-gs-closed.

Theorem: 3.2.35

Let (X,m_X) be an m-space satisfying the property of Maki and $A \subseteq X$. If A is m_X-π-open and m_X-π-gs-closed then A is m_X-semiclosed and hence m_X-gs-closed.

Proof:

Let A be an m_X-π-open and m_X-π-gs-closed. By Theorem [3.2.32], A is m_X-regular open. Using Theorem [3.1.14], $m_X\text{-scl}(A) = A$. Hence A is m_X-semi-closed and hence A is m_X-gs-closed.

Example: 3.2.36

In Example [3.2.7], the set {a, b, c} is m_X-π-open and m_X-gs-closed but not m_X-π-gs-closed.

Definition: 3.2.37

A subset A of an m-space (X,m_X) is called **m_X-clopen** if m_X-int (m_X-cl(A)) = A.

Theorem: 3.2.38

Let (X,m_X) be an m-space satisfying the property of Maki and $A \subseteq X$. Then the following properties are equivalent:

(i) A is m_X-π-clopen, that is m_X-π-open and m_X-π-closed.

(ii) A is m_X-π-open, m_X-clopen and m_X-π-gs-closed.

Proof:

(i) $\Rightarrow$ (ii)

Let A be an m_X-π-clopen. Then A is m_X-π-open and m_X-π-closed.
Hence A is m_X-open and m_X-closed and m_X-scl(A) =A.
Therefore A is m_X-π-open, m_X-clopen and m_X-π-gs-closed.

(ii) $\Rightarrow$ (i)

Let A be an m_X-π-open and m_X-π-gs-closed. By Theorem [3.2.31], A is m_X-regular open. Now if A is m_X-clopen then m_X-int(m_X-cl(A)) = A. Hence A is m_X-π-clopen.

Theorem: 3.2.39

Let (X,m_X) be an m-space satisfying the property of Maki and A,B be subsets of X. Then the following properties hold:

(i) If A is m_X-g-closed and $A \subseteq B \subseteq m_X$-cl(A), then B is m_X-g-closed.

(ii) If A is m_X-π-g-closed and $A \subseteq B \subseteq m_X$-cl(A), then B is m_X-π-g-closed.

(iii) If A is m_X-gp-closed and $A \subseteq B \subseteq m_X$-pcl(A), then B is m_X-gp-closed.

(iv) If A is m_X-π-gp-closed and $A \subseteq B \subseteq m_X$-pcl(A), then B is m_X-π-gpclosed.

(v) If A is m_X-gs-closed and $A \subseteq B \subseteq m_X$-scl(A), then B is m_X-gs-closed.

(vi) If A is m_X-π-gs-closed and $A \subseteq B \subseteq m_X$-scl(A), then B is m_X-π-gsclosed.

Proof:

(i) Let A be an m_X-g-closed subset and $B \subseteq U$.

Since $B \subseteq m_X$-cl(A) then m_X-cl(B) $\subseteq m_X$-cl(A). As A is m_X-g-closed, m_X-cl(A) $\subseteq U$ and m_X-cl(B) $\subseteq U$. Therefore B is m_X-g-closed.

The proof of (ii), (iii), (iv), (v) and (vi) are similar.

Theorem: 3.2.40

Let (X,m_X) be an m-space satisfying the property of Maki and $A \subseteq X$. Then the following properties hold:

(i) A is m_X-g-open if and only if $F \subseteq m_X$-int (A) whenever F is m_X-closed and $F \subseteq A$.

(ii) A is m_X-π-g-open if and only if $F \subseteq m_X$-int (A) whenever F is m_X-π-closed and $F \subseteq A$.

(iii) A is m_X-gp-open if and only if $F \subseteq m_X$-pint (A) whenever F is m_X-closed and $F \subseteq A$.

(iv) A is m_X-π-gp-open if and only if $F \subseteq m_X$-pint (A) whenever F is m_X-π-closed and $F \subseteq A$.

(v) A is m_X-gs-open if and only if $F \subseteq m_X$-sint (A) whenever F is m_X-closed and $F \subseteq A$.

(vi) A is m_X-π-gs-open if and only if $F \subseteq m_X$-sint (A) whenever F is m_X-π-closed and $F \subseteq A$.

Proof:

(i) Let A be an m_X-g-open set in X. Then A^C is m_X-g-closed set in X. Let F be m_X-closed and $F \subseteq A$. Then F^C is m_X -open and $A^C \subseteq F^C$. So m_X-cl(A^C) $\subseteq A^C$, $[m_X\text{-cl}(A)]^C \subseteq F^C$. Thus $F \subseteq m_X$-int(A).

Conversely assume that $F \subseteq m_X$-int(A)whenever F is m_X-closed and $F \subseteq A$. Let G be a m_X-open set and $A^C \subseteq G$. Now G^C is m_X-closed. $G^C \subseteq (A^C)^C = A$. By hypothesis $G \subseteq [m_X\text{-int}(A)]^C$, $G \supseteq [m_X\text{-cl}(A)]^C$ and hence A^C is m_X-g-closed. Thus A is m_X-g-open.

The proof of (ii), (iii), (iv), (v) and (vi) are similar.

Definition: 3.2.41

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-$\hat{g}$-closed set** if the m_X-cl(A)$\subseteq$U whenever $A \subseteq U$ and $U \in SO(X,m_X)$.

The complement of m_X-$\hat{g}$-closed set is called **m_X-$\hat{g}$-open**.

Definition: 3.2.42

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-$^{\#}$gs-closed set** if the m_X-scl(A)$\subseteq$U whenever $A \subseteq U$ and U is m_Xg-open.

The complement of m_X-$^{\#}$gs-closed set is called **m_X-$^{\#}$gs-open**.

Definition: 3.2.43

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-$\widetilde{g}$s-closed set** if the m_X-scl(A)$\subseteq$U whenever A$\subseteq$U and U is m_X-$^{\#}$gs-open.

The complement of m_X-$\widetilde{g}$s-closed set is called **m_X-$\widetilde{g}$s-open.**

Definition: 3.2.44

Let (X,m_X) be an m-space. A sub set A of X is an **m_X-λ-closed set** if the m_X-cl(A)$\subseteq$U whenever A$\subseteq$U and U is m_X-$\widetilde{g}$s-open.

The complement of m_X-λ-closed set is called **m_X-λ-open**.

Remark: 3.2.45

The family of all m_X-λ-open sets of (X,m_X) is denoted by $\lambda O(X,m_X)$.

Definition: 3.2.46

Let (X,m_X) be an m-space and A be a subset of X. The intersection of all m_X-λ-closed sets containing A is called the **m_X-λ-closure** of A and is denoted by m_X-λ-cl(A).

Example: 3.2.47

Let $X = \{a, b, c, d\}$, $m_X = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$. Then $SO(X,m_X) = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, c, d\}\}$, $PO(X,m_X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b, c\}\}$, $RO(X,m_X) = \{\phi, X, \{b\}, \{a, c\}\}$ and $\lambda O(X,m_X) = \{\phi, X, \{d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Example: 3.2.48

Let X = {a, b, c}, m_X = {ϕ, X, {a}, {b}}. Then SO(X,m_X)={ϕ, X, {a}, {b}, {b, c}, {a, c}}, PO(X,m_X) = {ϕ, X, {a} ,{b}, {a, b}}, RO(X,m_X) = {ϕ, X, {a}, {b}} and λO(X,m_X) = {ϕ, X, {c}, {a, c}, {b, c}}.

Example: 3.2.49

Let X = {a, b, c, d}, m_X = {ϕ, X, {a}, {b}, {a, b, d}, {a, b, c}}. Then SO(X,m_X)= {ϕ, X, {a}, {b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}, RO(X,m_X)={ ϕ, X, {a}, {b}}, PO(X,m_X) = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}} and λO(X,m_X) = {ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}}.

Example: 3.2.50

Let X = {a, b, c, d}, m_X = {ϕ, X, {a}, {b}, {a, b}, {a, b, d}, {a, b, c}}. Then SO(X,m_X) = {ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, c, d}, {b, c, d}, {a, b, c}, {a, b, d}}, RO(X,m_X) = {ϕ, X, {a}, {b}} PO(X,m_X) = {ϕ, X, {a}, {b}, {a, b}, {a, b, c},{a, b, d}} and λO(X,m_X) = {ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}}.

Example: 3.2.51

Let X = {a, b, c, d}, m_X = {ϕ, X, {a, b, c}, {c, d}}. Then SO(X,m_X) = {ϕ,X, {c, d}, {a, b, c},{a, c, d}, {b, c, d}}, PO(X,m_X) = {ϕ, X, {c}, {a, c}, {a, d},{b, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}, {a, b, d}},

$RO(X,m_X) = \{\phi, X\}$ and $\lambda O(X,m_X) = \{\phi, X, \{d\}, \{a, b\}, \{a, b, d\}\}$.

Example: 3.2.52

Let $X = \{a, b, c, d\}$, $m_X = \{\phi, X, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$. Then $SO(X,m_X) = \{\phi, X, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$, $PO(X,m_X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $RO(X,m_X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ $\lambda O(X,m_X) = \{\phi, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Theorem: 3.2.53

Let (X,m_X) be any m-space with m_X satisfing the property of Maki and A be any subset of X. Then

(i) If A is m_X-$\tilde{g}$-closed, then A is m_X-g-closed.

(ii) If A is m_X-$^{\#}$gsclosed then A is m_X-$\tilde{g}$s closed.

Remark: 3.2.54

If the property of Maki is dropped in theorem [3.2.53], the result is not necessarily true.

(i) Take $A = \{a, b, c\}$ in Example [3.2.47].

(ii) Take $A = \{a, b, d\}$ in Example [3.2.48].

Definition: 3.2.55

A m-space (X,m_X) is said to be **m_X-$sT_{1/2}$** if every m_X-$^{\#}$gs-closed set is m_X-semiclosed.

Example: 3.2.56

The Example [3.2.50], shows that (X,m_X) is an m_X-$T_{1/2}$ space.

Theorem: 3.2.57

Let m_X be an m-structure on X.

(i) $A\subseteq X$ is an m_X-$\hat{g}$-closed set if and only if m_X-cl(A)$\subseteq m_X$-sKer(A).

(ii) $A\subseteq X$ is an m_X-g-closed set if and only if m_X-cl(A)$\subseteq m_X$-$\hat{g}$ Ker(A).

(iii) $A\subseteq X$ is an m_X-$^{\#}$gs-closed set if and only if m_X-scl(A)$\subseteq m_X$-gKer(A).

(iv) $A\subseteq X$ is an m_X-$\tilde{g}$ s-closed set if and only if m_X-scl(A) $\subseteq m_X$-$^{\#}$gsKer(A).

(v) $A\subseteq X$ is an m_X-λ-closed set if and only if m_X-cl(A)$\subseteq m_X$-$\tilde{g}$ sKer(A).

Where m_X-gKer(A) (resp. $\hat{g}$ Ker(A), $^{\#}$gsKer(A), $\tilde{g}$ sKer(A)) is defined as the intersection of all m_X-g-open sets of (resp. m_X-$\hat{g}$-open, m_X-$^{\#}$gs-open, m_X-$\tilde{g}$ s-open) containing A.

Proof:

(i) Let A be an m_X-$\hat{g}$-closed and Let D = {S: $S\subseteq X$, $A\subseteq S$, $S\in SO(X,m_X)$}. Then m_X-sKer(A) = $\bigcap_{S\in D} S$, for all $S\in D$, m_X-cl(A)$\subseteq S$ then $A\subseteq X$. Hence, m_X-cl(A) $\subseteq m_X$-sKer(A).

Conversely let m_X-cl(A)$\subseteq m_X$-sKer(A), take $S\in SO(X,m_X)$ such that $A\subseteq S$, then by hypothesis m_X-cl(A) $\subseteq m_X$-Ker(A) $\subseteq S$. Hence A is m_X-$\hat{g}$-closed.

The other proofs are similar.

Theorem: 3.2.58

Let m_X be an m-structure on X.

(i) The set $A\subseteq X$ is an m_X-$\hat{g}$-closed set if and only if there does not exist an m_X-semiclosed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-cl}(A) - A$.

(ii) The set $A \subseteq X$ is an m_X-g-closed set if and only if there does not exist an m_X-$\hat{g}$-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-cl}(A) - A$.

(iii) The set $A \subseteq X$ is an m_X-$^{\#}$gs-closed set if and only if there does not exist an m_X-g-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-scl}(A) - A$.

(iv) The set $A \subseteq X$ is an m_X-$\tilde{g}$s-closed set if and only if there does not exist an m_X-$^{\#}$gs-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-scl}(A) - A$.

(v) The set $A \subseteq X$ is an m_X-λ-closed set if and only if there does not exist an m_X-$\tilde{g}$s-closed set F such that $F \neq \phi$ and $F \subseteq m_X\text{-cl}(A) - A$.

Proof:

(i) Let A be an m_X-$\hat{g}$-closed and let $F \subseteq X$ be an m_X-semi closed set such that $F \subseteq m_X\text{-cl}(A) - A$. Then $A \subseteq X - F$ and $X - F$ is an m_X-semiopen set, since A is an m_X-$\hat{g}$-closed, $m_X\text{-cl}(A) \subseteq X - F$ and $F \subseteq X - m_X\text{-cl}(A)$. Therefore $F \subseteq m_X\text{-cl}(A) \cap (X - m_X\text{-cl}(A)) = \phi$. Hence $F = \phi$.

Conversely assume that $A \subseteq U$ and U be an m_X-semiopen set, then $m_X\text{-cl}(A) \cap (X - U) \subseteq m_X\text{-cl}(A) \cap (X - A) = m_X\text{-cl}(A) - A$. Since $m_X\text{-cl}(A) - A$ does not contain subsets m_X-semiclosed different from the empty set, we obtain that $m_X\text{-cl}(A) \cap (X - U) = \phi$, and therefore $m_X\text{-cl}(A) \subseteq U$. Hence A is an m_X-$\hat{g}$-closed.

The other proofs are similar.

Theorem: 3.2.59

Let (X, m_X) be an m-space and A,B subsets of X. The following properties hold:

(i) If A is m_X-$\hat{g}$-closed and $A \subseteq B \subseteq m_X$-cl(A) then B is m_X-$\hat{g}$-closed.

(ii) If A is m_X-$^{\#}$gs-closed and $A \subseteq B \subseteq m_X$-scl(A) then B is m_X-$^{\#}$gsclosed.

(iii) If A is m_X-$\tilde{g}$s-closed and $A \subseteq B \subseteq m_X$-scl(A) then B is m_X-$\tilde{g}$s-closed.

(iv) If A is m_X-λ-closed and $A \subseteq B \subseteq m_X$-cl(A) then B is m_X-λ-closed.

Proof:

(i) Let A be an m_X-$\hat{g}$-closed set, $B \subseteq U$ and U is m_X-semi open. Then $A \subseteq B \subseteq U$. Since A is m_X-$\hat{g}$-closed, m_X-cl(A)$\subseteq$U. As $B \subseteq$ m_X-cl(A), m_X-cl(B) $\subseteq$ m_X-cl(A) $\subseteq$U. Therefore B is m_X-$\hat{g}$-closed.

The other proofs are similar.

Theorem: 3.2.60

Let (X, m_X) be an m-space and $A \subseteq X$. The following properties hold:

(i) A is m_X-$\hat{g}$-open if and only if $F \subseteq m_X$-int(A) whenever F is m_X-semi closed and $F \subseteq A$.

(ii) A is m_X-g-open if and only if $F \subseteq m_X$-int(A) whenever F is m_X-$\hat{g}$-closed and $F \subseteq A$.

(iii) A is m_X-$^{\#}$gs-open if and only if $F \subseteq m_X$-sint(A) whenever F is m_X-g-closed and $F \subseteq A$.

(iv) A is m_X-$\tilde{g}$s-open if and only if $F \subseteq m_X$-int(A) whenever F is m_X-$^{\#}$gs-closed and $F \subseteq A$.

(v) A is m_X-λ-open if and only if $F \subseteq m_X$-int(A) whenever F is m_X-$\tilde{g}$s-closed and $F \subseteq A$.

Proof:

(i) Let A be m_X-$\hat{g}$-open and F be m_X-semiclosed such that $F \subseteq A$. Then $X - A \subseteq X - F$ and $X - F$ is m_X-semiopen. Then $X - A$ is m_X-$\hat{g}$-closed and $X - F$ is m_X-semiopen. Therefore $m_X\text{-cl}(X - A) \subseteq X - F$. Therefore $X - m_X\text{-int}(A) \subseteq X - F$. Hence $F \subseteq m_X\text{-int}(A)$.

Conversely let F be m_X-semiclosed, $F \subseteq A$ and $F \subseteq m_X\text{-int}(A)$. Let $X - A \subseteq U$ where U is m_X-semiopen. Then $X - U \subseteq A$ and $X - U$ is m_X-semiclosed. By hypothesis $X - U \subseteq m_X\text{-int}(A)$. Follows that $X - m_X\text{-int}(A) \subseteq U$. Hence $m_X\text{-cl}(X - A) \subseteq U$. Therefore $X - A$ is m_X-$\hat{g}$-closed and hence A is m_X-$\hat{g}$-open.

In a similar form, we can prove (ii), (iii) (iv) and (v).

Theorem: 3.2.61

An arbitrary intersection of m_X-λ-closed sets is m_X-λ-closed.

Proof:

Let $F = \{A_i : i \in I\}$ be a family of m_X-λ-closed sets and let $A = \bigcap A_i$. Since $A \subseteq A_i$ for all $i \in I$, $X_I \cap m_X\text{-cl}(A) \subseteq X_I \cap m_X\text{-cl}(A_i) \subseteq A_i$ for all $i \in I$. It follows that $X_I \cap m_X\text{-cl}(A) \subseteq X_I \cap m_X\text{-cl}(A_i) \subseteq A$. Hence A is m_X-λ-closed.

SECTION: 3.3

(m_x,m_y)-CONTINUOUS MAPS, (m_x,m_y) -λ-IRRESOLUTE MAPS AND (m_x,m_y)- CONTRA λ-CONTINUOUS MAPS.

Definition: 3.3.1

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-closed in X for all m_Y-closed set $O \in Y$.

Definition: 3.3.2

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **π-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-π-closed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.3

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **π-gs-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-π-gs-closed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.4

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **πg-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-πg-closed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.5

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **πgp-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-πgp-closed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.6

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **s-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-semiclosed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.7

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **gs-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-gs-closed in X for every m_Y-closed set O of (Y,m_Y).

Definition: 3.3.8

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **gp-(m_X,m_Y)-continuous** if $f^{-1}(O)$ is m_X-gp-closed in (X,m_X) for every m_Y-closed set O of (Y,m_Y).

Example: 3.3.9

In the Example [3.2.7], take X = Y = {a, b, c}, $m_X = m_Y = \{\phi, X, \{a\}, \{b\}\}$ and f: $(X,m_X) \rightarrow (Y,m_Y)$ defined as f(a) = f(c) = c and f(b) = a. Then the function f satisfies all different notions of continuity.

Theorem: 3.3.10

Let f: $(X,m_X) \rightarrow (Y,m_Y)$, then:

(i) If f is (m_X,m_Y)-continuous, then it is g-(m_X,m_Y)-continuous.

(ii) If f is (m_X,m_Y)-continuous, then it is s-(m_X,m_Y)-continuous.

Proof:

(i) Let f be (m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y).Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. Since every m_X-closed set is m_X-g-closed, $f^{-1}(O)$ is m_X-g-closed in (X,m_X).

Therefore f is g-(m_X,m_Y)-continuous.

(ii) Let f be (m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y). Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. Since every m_X-closed set is m_X-s-closed, $f^{-1}(O)$ is m_X-s-closed in (X,m_X).

Therefore f is s-(m_X,m_Y)-continuous.

Remark: 3.3.11

The converse of the above theorem [3.3.10] need not be true.

Theorem: 3.3.12

Let f: (X,m_X) $\rightarrow$ (Y,m_Y) then:

If f is s-(m_X,m_Y)-continuous, then it is gs-(m_X,m_Y)-continuous.

Proof:

Let f be s-(m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y).Since f is s-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-s-closed. Since every m_X-s-closed set is m_X-gs-closed, f^{-1} (O) is m_X-gs-closed in (X,m_X). Therefore f is gs-(m_X,m_Y)-continuous.

Remark: 3.3.13

The converse of the above theorem [3.3.12] need not be true.

Theorem: 3.3.14

Let f: (X,m_X) $\rightarrow$ (Y,m_Y) then

(i) If f is g-(m_X,m_Y)-continuous, then it is gp-(m_X,m_Y)-continuous.

(ii) If f is g-(m_X,m_Y)-continuous, then it is gs-(m_X,m_Y)-continuous.

Proof:

(i) Let f be g-(m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y). Since f is g-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-g-closed. Since every m_X-g-closed set is m_X-gp-closed, $f^{-1}(O)$ is m_X-gp-closed in (X,m_X). Therefore f is gp-(m_X,m_Y)-continuous.

(ii) Let f be g-(m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y). Since f is g-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-g-closed. Since every m_X-g-closed set is m_X-gs-closed, $f^{-1}(O)$ is m_X-gs-closed in (X,m_X). Therefore f is gs-(m_X,m_Y)-continuous.

Remark: 3.3.15

The converse of the above theorem [3.3.14] need not be true.

Theorem: 3.3.16

Let f: $(X,m_X) \rightarrow (Y,m_Y)$, where m_X satisfy the property of Maki, then If f is π-(m_X,m_Y)-continuous then f is (m_X,m_Y)-continuous.

Proof:

Let f be π - (m_X,m_Y)-continuous and let O an m_Y-closed set in (Y,m_Y). Since f is π - (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X- π-closed.

Since every m_X-π-closed set is m_X-closed, $f^{-1}(O)$ is m_X-closed in (X,m_X). Therefore f is (m_X,m_Y)-continuous.

Example: 3.3.17

In the Example [3.2.8], $X = Y = \{a, b, c\}$, $m_X = m_Y = \{\phi, X, \{a\}, \{b\}\}$ and f: $(X,m_X) \rightarrow (Y,m_Y)$ defined as $f(a) = f(b) = a$ and $f(c) = c$. Then f is π-(m_X,m_Y)- continuous but does not is (m_X,m_Y)- continuous.

Theorem: 3.3.18

Let f: $(X,m_X) \rightarrow (Y,m_Y)$, where m_X satisfies the condition of Maki then:

(i) If f is g-(m_X,m_Y)-continuous, then f if πg-(m_X,m_Y)-continuous.

(ii) If f is gs-(m_X,m_Y)-continuous, then f if π-gs-(m_X,m_Y)-continuous.

(iii) If f is gp-(m_X,m_Y)-continuous, then f if π-gp-(m_X,m_Y)-continuous.

Proof:

(i) Let f be g-(m_X,m_Y)-continuous and let O be an m_Y-closed set in (Y,m_Y). Since f is g-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-g-closed. Since every m_X-g-closed set is m_X-πg-closed, $f^{-1}(O)$ is m_X-πg-closed in (X,m_X). Therefore f is πg-(m_X,m_Y)-continuous.

(ii) Let f be gs-(m_X,m_Y)-continuous and let O be an m_Y-closed set in (Y,m_Y). Since f is gs-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-gs-closed. Since every m_X-gs-closed set is m_X-πgs-closed, $f^{-1}(O)$ is m_X-πgs-closed in (X,m_X). Therefore f is πgs-(m_X,m_Y)-continuous.

(iii) Let f be gp-(m_X,m_Y)-continuous and let O be an m_Y-closed set in (Y,m_Y). Since f is gp-(m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-gp-closed. Since every m_X-gp-closed set is m_X-πgp-closed, $f^{-1}(O)$ is m_X-πgp-closed in (X,m_X). Therefore f is πg-(m_X,m_Y)-continuous.

Remark: 3.3.19

The converse of the above Theorem [3.3.18] need be not true.

Example: 3.3.20

In the Example [3.2.8], take X = Y = {a, b, c}, $m_X = m_Y$ = {ϕ, X, {a}, {b}} and f: $(X,m_X) \rightarrow (Y,m_Y)$ defined as: f(a) = c, f(b) = a , f(c) = b. Then:

(i) f is g-(m_X,m_Y)-continuous but not π-g-(m_X,m_Y)-continuous.

(ii) f is gs-(m_X,m_Y)-continuous but not πgs-(m_X,m_Y)-continuous.

Example: 3.3.21

In the Example [3.2.8], take X = {a, b, c}, m_X = {ϕ, X, {a}, {b}} and Y = {x, y}, and m_Y = {ϕ, Y, {x}}. Let f: $(X,m_X) \rightarrow (Y,m_Y)$ be defined as: f(a) = f(c) = x and f(b) = y. Then f is π-gs-(m_X,m_Y)-continuous but not π-g-(m_X,m_Y)-continuous and π-gp-(m_X,m_Y)-continuous.

Example: 3.3.22

In the Example [3.2.12], take X = {a, b, c, d}, m_X = {ϕ, X, {a, b}, {a, c}, {a}, {b}, {c}} and Y = {x, y}, and m_Y = {ϕ, Y, {x}}.

Let f : $(X,m_X) \rightarrow (Y,m_Y)$ be defined as: f(b) = f(c) = f(d) = x and f(a) = y. Then f is π-gp-(m_X,m_Y)-continuous but not π-gs-(m_X,m_Y)-continuous and π-g-(m_X,m_Y)-continuous.

Example: 3.3.23

In the Example [3.2.7], X = {a, b, c, d}, m_X = {ϕ, X, {a}, {b}, {a, c}} and Y = {x, y}, and m_Y = {ϕ, Y, {x}}, and f: $(X,m_X) \rightarrow (Y,m_Y)$,

$f(a) = f(b) = f(c) = y$ and $f(d) = y$. Then f is gp-(m_X,m_Y)-continuous but none of π-gp-(m_X,m_Y)-continuous, π-gs-(m_X,m_Y)-continuous, π-g(m_X,m_Y)-continuous and π-(m_X,m_Y)-continuous.

Definition: 3.3.24

A map $f: (X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-$\hat{g}$-continuous** if $f^{-1}(O)$ is m_X-$\hat{g}$-closed in X for all m_Y-closed set $O \in Y$.

Definition: 3.3.25

A map $f: (X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-$^{\#}$gs-continuous** if $f^{-1}(O)$ is m_X-$^{\#}$gs-closed in X for all m_Y-closed set $O \in Y$.

Definition: 3.3.26

A map $f: (X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-$\tilde{g}$s-continuous** if $f^{-1}(O)$ is m_X-$\tilde{g}$s-closed in X for all m_Y-closed set $O \in Y$.

Definition: 3.3.27

A map $f: (X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-λ-continuous** if $f^{-1}(O)$ is m_X-λ-closed in X for all m_Y-closed set $O \in Y$.

Example: 3.3.28

In the Example [3.2.48], take $X = Y = \{a, b, c\}$, $m_X = m_Y = \{\phi, X, \{a\}, \{b\}\}$ and $f: X \rightarrow Y$ defined as: $f(a) = f(c) = c$ and $f(b) = a$. Then f satisfies all different notions of continuity described the above definitions

Theorem: 3.3.29

Let f: $(X,m_X) \rightarrow (Y,m_Y)$, then:

(i) If f is (m_X,m_Y)-continuous, then f is (m_X,m_Y)-$\hat{g}$-continuous.

(ii) If f is (m_X,m_Y)-continuous, then f is (m_X,m_Y)-$^{\#}$gs-continuous.

(iii)If f is (m_X,m_Y)-continuous, then f is (m_X,m_Y)-$\tilde{g}$ s-continuous.

(iv) If f is (m_X,m_Y)-continuous, then f is (m_X,m_Y)-λ-continuous

Proof:

(i) Let f be (m_X,m_Y)-continuous and let O be an m_Y-closed set. Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-closed is m_X-$\hat{g}$-closed, $f^{-1}(O)$ is m_X-$\hat{g}$-closed in (X,m_X). Therefore f is (m_X,m_Y)-$\hat{g}$-continuous.

(ii) Let f be (m_X,m_Y)-continuous and let O be an m_Y-closed set. Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-closed is m_X-$^{\#}$gs-closed, $f^{-1}(O)$ is m_X-$^{\#}$gs-closed in (X,m_X). Therefore f is (m_X,m_Y)-$^{\#}$gs-continuous.

(iii) Let f be (m_X,m_Y)-continuous and let O be an m_Y-closed set. Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-closed is m_X-$\tilde{g}$ s-closed, $f^{-1}(O)$ is m_X-$\tilde{g}$ s -closed in (X,m_X). Therefore f is (m_X,m_Y)-$\tilde{g}$ s-continuous.

(iv) Let f be (m_X,m_Y)-continuous and let O be an m_Y-closed set. Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-closed is m_X-λ-closed, $f^{-1}(O)$ is m_X-λ-closed in (X,m_X). Therefore f is (m_X,m_Y)-λ-continuous.

Remark: 3.3.30

Converse of the above theorem [3.3.29] is not true.

Example: 3.3.31

In the Example [3.2.48], X = Y = {a, b, c}, $m_X = m_Y = \{\phi, X, \{a\}, \{b\}\}$ and f: X→Y be defined as: f (a) = f (b) = a and f(c) = c. Then f is (m_X,m_Y)-$\hat{g}$-continuous, (m_X,m_Y)-g-continuous, (m_X,m_Y)-$^{\#}$gs-continuous, (m_X,m_Y)-$\tilde{g}$ s-continuous and (m_X,m_Y)-λ-continuous but not (m_X,m_Y)-continuous.

Theorem: 3.3.32

Let f: $(X,m_X) \rightarrow (Y,m_Y)$, where m_X satisfies the condition of Maki *then:*

(i) If f is (m_X,m_Y)-$\hat{g}$-continuous, then it is (m_X,m_Y)-g-continuous.

(ii) If f is (m_X,m_Y)-$^{\#}$gs-continuous, then it is (m_X,m_Y)-$\tilde{g}$ s-continuous.

Proof:

(i) Let f be (m_X,m_Y)-$\hat{g}$-continuous and let O be an m_Y-closed set. Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-$\hat{g}$-closed is m_X-g-closed, $f^{-1}(O)$ is m_X-g-closed in (X,m_X). Therefore f is (m_X,m_Y)-g-continuous.

(ii) Let f be (m_X,m_Y)-$^{\#}$gs -continuous and let O be an m_Y-closed set.Since f is (m_X,m_Y)-continuous, $f^{-1}(O)$ is m_X-closed. As every m_X-$^{\#}$gs-closed is m_X-$\tilde{g}$ s-closed, $f^{-1}(O)$ is m_X-$\tilde{g}$ s-closed in (X,m_X). Therefore f is (m_X,m_Y)-$\tilde{g}$ s-continuous.

Remark: 3.3.33

Converse of the above theorem [3.3.32] need not be true.

Example: 3.3.34

In the Example [3.2.48], X = Y = {a, b, c}, $m_X = m_Y = \{\phi, X, \{a\}, \{b\}\}$ and f: X → Y be defined as: f (a) = c, f (b) = a and f(c) = b. Then

(i) f is (m_X,m_Y)-$\tilde{g}$ s-continuous but not (m_X,m_Y)-g-continuous.

(ii) f is (m_X,m_Y)-$\tilde{g}$ s-continuous but not (m_X,m_Y)-λ-continuous.

Example: 3.3.35

In the Example [3.2.47], take X = {a, b, c, d}, $m_X = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$ and Y = {x, y}, and $m_Y = \{\phi, Y, \{x\}\}$, and f: X → Y be defined as: f (a) = f (b) = f (d) = y and f(c) = x. Then f is (m_X,m_Y)-$^{\#}$gs-continuous but not (m_X,m_Y)-$\tilde{g}$ s-continuous and (m_X,m_Y)-λ-continuous.

Example: 3.3.36

In the Example [33.2.47], X = {a, b, c, d}, $m_X = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$ and Y = {x, y}, and $m_Y = \{\phi, Y, \{x\}\}$ and f: X → Y be defined as: f (a) = f (b) = f(c) = y and f (d) = y. Then f is (m_X,m_Y)-$\hat{g}$-continuous, but not (m_X,m_Y)-g-continuous,(m_X,m_Y)-$^{\#}$gs-continuous,(m_X,m_Y)-$\tilde{g}$ s-continuous, (m_X,m_Y)-λ-continuous.

Definition: 3.3.37

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-semiclosed in X for every m_Y-semiclosed set O of (Y,m_Y).

Definition: 3.3.38

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **π-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-π-closed in (X,m_X) for every m_Y-π-closed set O of (Y,m_Y).

Definition: 3.3.39

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **g-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-g-closed in (X,m_X) for every m_Y-g-closed set O of (Y,m_Y).

Definition: 3.3.40

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **π-gp-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-π-gp-closed in (X,m_X) for every m_Y-π-gp-closed set O of (Y,m_Y).

Definition: 3.3.41

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **π-gs-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-π-gs-closed in (X,m_X) for every m_Y-π-gs-closed set O of (Y,m_Y).

Definition: 3.3.42

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **gs-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-gs-closed in (X,m_X) for every m_Y-gs-closed set O of (Y,m_Y).

Definition: 3.3.43

A map f : $(X,m_X) \rightarrow (Y,m_Y)$ is called **gp-(m_X,m_Y)-irresolute** if $f^{-1}(O)$ is m_X-gp-closed in (X,m_X) for every m_Y-gp-closed set O of (Y,m_Y).

Example: 3.3.44

In the Example [3.2.12], $X = \{a, b, c, d\}$, $m_X = \{\phi, X, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$. $Y = \{x, y\}$, and $m_Y = \{\phi, Y, \{x\}\}$ and $f: (X,m_X) \to (Y,m_Y)$ be defined as: $f(b) = f(c) = f(d) = x$ and $f(a) = y$. Then f is π-gp-(m_X,m_Y)-irresolute but none of π-gs-(m_X,m_Y)-irresolute and π-g-(m_X,m_Y)-irresolute.

Example: 3.3.45

In the Example [3.2.9], $X = \{a, b, c, d\}$, $m_X = \{\phi, X, \{a, b, d\},$

$\{a, b, c\}, \{a\}, \{b\}\}$. $Y = \{x, y\}$, and $m_Y = \{\phi, Y, \{x\}\}$ and

$f : (X,m_X) \to (Y,m_Y)$ defined as: $f(b) = f(c) = f(d) = x$ and $f(a) = y$. Then f is π- gs-(m_X,m_Y)-irresolute but none of π-gp-(m_X,m_Y)-irresolute,

π-g-(m_X,m_Y)-irresolute, g-(m_X,m_Y)-irresolute and gp-(m_X,m_Y)-irresolute.

Example: 3.3.46

In the Example [3.2.7], $X = \{a, b, c, d\}$, $m_X = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$. $Y = \{x, y\}$, and $m_Y = \{\phi, Y, \{x\}\}$, $f: (X,m_X) \to (Y,m_Y)$ defined as: $f(a) = f(b) = f(c) = y$ and $f(d) = x$. Then f is gp-(m_X,m_Y)-irresolute but none of π-gp-(m_X,m_Y)-irresolute, π-gs-(m_X,m_Y)-irresolute, π-g-(m_X,m_Y)-irresolute and π--(m_X,m_Y)-irresolute.

Definition: 3.3.47

A map $f: (X,m_X) \to (Y,m_Y)$ is called **(m_X,m_Y)-pre semi closed** if f(O) is m_Y-semi closed in (Y,m_Y) for all $m_X$semiclosed set O of (X,m_X).

Definition: 3.3.48

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-pre semi open** if f(O) is m_Y-semi open in (Y,m_X) for all m_X-semi open set O of (X,m_X).

Definition: 3.3.49

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-regular open** if f(O) is m_Y-regular open in (Y,m_Y) for every m_X-open set O of (X,m_X).

Lemma: 3.3.50

Let (X,m_X) and (Y,m_Y) be two m-spaces where m_X satisfies the property of Maki. If f: $(X,m_X) \rightarrow (Y,m_Y)$ is π-(m_X,m_Y)-irresolute function and (m_X,m_Y)-pre semiclosed, then f(A) is m_Y-π-gs-closed for every m_X-π-gs-closed set A in X.

Proof:

Let A be any m_X-π-gs-closed set in X and U be an m_Y-π-open set in y such that $f(A) \subseteq U$. By hypothesis $f^{-1}(U)$ is m_X-open set in X and $A \subseteq f^{-1}(U)$. Then m_X-scl(A)$\subseteq f^{-1}(U)$ and hence f(m_X-scl(A))$\subseteq$U. Since A$\subseteq m_X$-scl(A), then f(A)$\subseteq$f(m_X-scl(A)), hence m_Y-scl(f(A))$\subseteq m_Y$-scl(f(m_X-scl(A))). Since f is (m_X,m_Y)-pre semiclosed, m_Y-scl(f(m_X-scl(A)))=f(m_X-scl(A)). Then m_Y-sclf((A))$\subseteq$f(m_X-scl(A))$\subseteq$U. Hence f(A) is m_Y-πgs-closed set in Y.

Lemma: 3.3.51

Let (X,m_X) and (Y,m_Y) be two m-spaces, where m_Y satisfies the property of Maki. If f: $(X,m_X) \rightarrow (Y,m_Y)$ is (m_X,m_Y)-irresolute, (m_X,m_Y)-regular open and bijective, then f is π-gs-(m_X,m_Y)-irresolute.

Proof:

Let F be any m_Y-π-gs-closed set in Y and U any m_X-π-open set in X such that $f^{-1}(F) \subseteq U$. Then $F \subseteq f(U)$ and since f(U) is m_Y-π-open, m_Y-scl(F) $\subseteq$ f(U). Therefore $f^{-1}(m_Y\text{-scl}(F)) \subseteq U$. Since f is (m_X,m_Y)-irresolute, $f^{-1}(m_Y\text{-scl}(F))$ is m_X-semiclosed. Hence $m_X\text{-scl}(f^{-1}(F)) \subseteq m_X\text{-scl}(f^{-1}(m_Y\text{-scl}(F)) = (f^{-1}(m_Y\text{-scl}(F)) \subseteq U$. Thus $f^{-1}(F)$ is m_X- π-gs-closed in X.

Lemma: 3.3.52

Let (X,m_X) and (Y,m_Y) be two m-spaces where m_Y satisfies the property of Maki. The following conditions are equivalent:

(i) f: $(X,m_X) \rightarrow (Y,m_Y)$ is a (m_X,m_Y)-irresolute function.

(ii) For each subset $A \subseteq X$, $f(m_X\text{-scl}(A)) \subseteq m_Y\text{-scl}(f(A))$.

(iii) For each m_Y semiclosed subset $V \subseteq Y$, the inverse image $f^{-1}(V)$ is an m_X-semiclosed in X.

(iv) For all $B \subseteq Y$, $m_X\text{-scl}(f^{-1}(B)) \subseteq f^{-1}(m_Y\text{-scl}(B))$.

Proof:

(iii) $\Rightarrow$ (ii)

Let A be a subset of X and assume that $y \notin m_Y\text{-scl}(f(A))$, then there exists a m_Y-semi open set G in Y, such that $y \in G$ and $f(A) \cap G = \phi$, therefore $f^{-1}(f(A) \cap G) = \phi$, $A \cap f^{-1}(G) = \phi$. Hence $m_X\text{-scl}(A) \subseteq (f^{-1}(G))^C$, then $f(m_X\text{-scl}(A)) \cap G = \phi$ and therefore $y \notin f(m_X\text{-scl}(A))$.

Hence $f(m_X\text{-scl}(A)) \subseteq m_Y\text{-scl}(f(A))$ for all subset A of X.

(ii) $\Rightarrow$ (iii)

Let V be any m_Y-semiclosed subset in Y, then $f^{-1}(V) \subseteq X$.

By hypothesis $f(m_X\text{-}scl(f^{-1}(V)))\subseteq m_Y\text{-}scl(f(f^{-1}(V)))$,

$f(m_Xscl(f^{-1}(V)))\subseteq m_Y\text{-}scl(V)$. Hence $f(m_X\text{-}scl(f^{-1}(V)))\subseteq V$, then

$m_X\text{-}scl(f^{-1}(V)\subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is an m_X-semiclosed set.

(ii) $\Rightarrow$ (iv)

Let B be a subset of Y. Then $f^{-1}(B)\subseteq X$. By the hypothesis

$f(m_X\text{-}scl(f^{-1}(B)))\subseteq m_Y\text{-}scl(f(f^{-1}(B)))\subseteq m_Y\text{-}scl(B)$.

Therefore $m_X\text{-}scl(f^{-1}(B))\subseteq f^{-1}(m_Y\text{-}scl(B))$.

(iv) $\Rightarrow$ (iii)

Let V be any m_Y -semiclosed set in Y. Then $f^{-1}(V)\subseteq X$. By hypothesis, $m_X\text{-}scl(f^{-1}(V))\subseteq f^{-1}(m_Y\text{-}scl(V))$. But V is a m_Y-semiclosed set, and hence $m_Y\text{-}scl(V) = V$. Hence $m_X\text{-}scl(f^{-1}(V))\subseteq f^{-1}(V)$.

Therefore $f^{-1}(V)$ is an m_X-semiclosed set in X.

The others implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), follow from the definition of (m_X,m_Y)-irresolute function and the complement of a set.

Theorem: 3.3.53

Let $f: (X,m_X) \rightarrow (Y,m_Y)$, then:

(i) If f is π-g-(m_X,m_Y)-irresolute, then f is π-g-(m_X,m_Y)-continuous.

(ii) If f is π-gs-(m_X,m_Y)-irresolute, then f is π-gs-(m_X,m_Y)-continuous.

(iii) If f is π-gp-(m_X,m_Y)-irresolute, then f is π-gp-(m_X,m_Y)-continuous.

Remark: 3.3.54

The converse of the above theorem [3.3.53] need not be true.

Example: 3.3.55

Let X = Y = {a, b, c, d}, m_X = { ϕ, X, {a}, {b}, {c},{a, b}, {c, d}}, and m_Y = { ϕ, Y , {a}, {b}, {a, c}}. Then the m_X-regular open sets of (X,m_X) are { ϕ, X, {a}, {b}, {a, b}, {b, c}, {c, d}, {a, c, d}, {b, c, d} and the m_X-π-open sets are { ϕ, X, {a}, {b}, {a, b}, {b, c}, {c,d}, {a, c, d}, {b, c, d}and {a, b, c}}. Let f: (X,m_X) $\rightarrow$ (Y,m_Y) be defined as: f (a) = f (d) = d, f (b) = a and f(c) = c. Then f is π-(m_X,m_Y)-irresolute but not π-(m_X,m_Y)-continuous.

Definition: 3.3.56

A map f: (X,m_X) $\rightarrow$ (Y,m_Y) is called **(m_X,m_Y)-$\hat{g}$-irresolute** if f^{-1}(O) is m_X-$\hat{g}$-closed in X for every m_Y-$\hat{g}$-closed set O$\in$Y.

Definition: 3.3.57

A map f: (X,m_X) $\rightarrow$ (Y,m_Y) is called **(m_X,m_Y)-$^{\#}$gs-irresolute** if f^{-1}(O) is m_X-$^{\#}$gs-closed in X for every m_Y -$^{\#}$gs-closed set O$\in$Y.

Definition: 3.3.58

A map f: (X,m_X) $\rightarrow$ (Y,m_Y) is called **(m_X,m_Y)-$\tilde{g}$s-irresolute** if f^{-1}(O) is m_X-$\tilde{g}$s-closed in X for every m_Y-$\tilde{g}$s-closed set O$\in$Y.

Definition: 3.3.59

A map f: (X,m_X) $\rightarrow$ (Y,m_Y) is called **(m_X,m_Y)-λ-irresolute** if f^{-1}(O) is m_X-λ-closed in X for every m_Y-λ-closed set O$\in$Y.

Example: 3.3.60

In the Example [3.2.52], X = {a, b, c, d}, m_X = { ϕ, X, {a, b}, {a, c}, {a}, {b}, {c}}. Y = {x, y}, and m_Y = { ϕ, Y, {x}}, and the function f: X → Y defined by f (b) = f(c) = f (d) = x and f (a) = y. f is (m_X,m_Y)-irresolute but not (m_X,m_Y)- $\hat{g}$ -irresolute, (m_X,m_Y)-g-irresolute, (m_X,m_Y)-$^{\#}$gs-irresolute, (m_X,m_Y)- $\tilde{g}$ s-irresolute and (m_X,m_Y)-λ-irresolute.

Example: 3.3.61

In the Example [3.2.49], X = {a, b, c, d}, m_X = { ϕ, X, {a, b, d}. {a, b, c}, {a}, {b}}. Y = {x, y}, and m_Y = { ϕ, Y, {x}} and f: X → Y is defined as f (b) = f(c) = f(d) = x and f(a) = y. Then f is (m_X,m_Y)-$^{\#}$gs-irresolute but not (m_X,m_Y)- $\hat{g}$ -irresolute, (m_X,m_Y)-g-irresolute and (m_X,m_Y)-λ-irresolute.

Example: 3.3.62

In the Example [3.2.47], X = {a, b, c, d}, m_X = { ϕ, X, {a}, {b}, {a, c}}. Y = {x, y}, and m_Y = { ϕ, Y, {x}}, f: X → Y is defined as f (a) = f (b) = f(c) = y and f (d) = x. Then f is (m_X,m_Y)- $\hat{g}$ -irresolute but not (m_X,m_Y)-irresolute, (m_X,m_Y)-g-irresolute, (m_X,m_Y)-$^{\#}$gs-irresolute, (m_X,m_Y)- $\tilde{g}$ s-irresolute and (m_X,m_Y)-λ-irresolute.

Definition: 3.3.63

A map f: (X,m_X) → (Y,m_Y) is called **(m_X,m_Y)-contra-continuous** if f^{-1} (O) is m_X-closed in X for every m_Y-open set O ∈ Y.

Definition: 3.3.64

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-contra g-continuous** if $f^{-1}(O)$ is m_X-g-closed in X for every m_Y-open set $O \in Y$.

Definition: 3.3.65

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-contra $\hat{g}$-continuous** if $f^{-1}(O)$ is m_X-$\hat{g}$-closed in X for every m_Y-open set $O \in Y$.

Definition: 3.3.66

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-contra $^{\#}$gs-continuous** if $f^{-1}(O)$ is m_X-$^{\#}$gs-closed in X for every m_Y-open set $O \in Y$.

Definition: 3.3.67

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-contra $\tilde{g}$s-continuous** if $f^{-1}(O)$ is m_X-$\tilde{g}$s-closed in X for every m_Y-open set $O \in Y$.

Definition: 3.3.68

A map f: $(X,m_X) \rightarrow (Y,m_Y)$ is called **(m_X,m_Y)-contra-λ-continuous** if $f^{-1}(O)$ is m_X-λ-closed in X for every m_Y-open set $O \in Y$.

Lemma: 3.3.69

Let (X,m_X) and (Y,m_Y) be two m-spaces where m_X satisfies the property of Maki. The following conditions are equivalent:

(i) f: $(X,m_X) \rightarrow (Y,m_Y)$ is (m_X,m_Y)-contra λ-continuous function.

(ii) For each m_X-closed set $A \subseteq Y$, $f^{-1}(A)$ is m_X-λ-open.

(iii) For each $x \in X$ and each m_Y -closed set $A \subseteq Y$ with $f(x) \in A$, there exists an m_X-λ-open $U \in X$ such that $f(U) \subseteq A$.

(iv) For all $A \subseteq X$, $f(m_X\text{-}\lambda\text{-cl}(A)) \subseteq m_Y\text{-Ker}(f(A))$.

(v) For all $B \subseteq Y$, $m_X\text{-}\lambda\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(m_Y\text{-Ker}(B))$.

Proof:

(iii) $\Rightarrow$ (ii)

Let A be any m_Y -closed set of Y and $x \in f^{-1}(A)$. Then $f(x) \in A$ and there exists an m_X-λ-open set U_X of X such that $x \in A$ and $f(U_X) \subseteq A$.

Hence $f^{-1}(A) = \cup \{U_X / x \in f^{-1}(A)\}$ is an m_X-λ-open.

(ii) $\Rightarrow$ (iv)

Let A be any subset of X. Let $y \notin \text{Ker}(f(A)))$. Then there exists an m_Y-closed set F in Y such that $y \in F$ and $f(A) \cap F = \phi$. Thus $A \cap f^{-1}(F) = \phi$ and $m_X\text{-}\lambda\text{-cl}(A) \cap f^{-1}(F) = \phi$. Therefore $f(m_X\text{-}\lambda\text{-cl}(A)) \cap F = \phi$ and $y \notin f(m_X\text{-}\lambda\text{-cl}(A))$. Hence $f(m_X\text{-}\lambda\text{-cl}(A)) \subseteq m_Y\text{-Ker}(f(A))$.

(iv) $\Rightarrow$ (v)

Let B any subset of Y.

Then $f(m_X\text{-}\lambda\text{-cl}(f^{-1}(B)) \subseteq m_Y\text{-Ker}(f(f^{-1}(B))) \subseteq m_Y\text{-Ker}(B)$.

Hence $m_X\text{-}\lambda\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(m_Y\text{-Ker}(B))$.

(v) $\Rightarrow$ (i)

Let V any m_X-open set in Y.

Then $m_X\text{-}\lambda\text{-cl}(f^{-1}(V)) \subseteq f^{-1}(m_Y\text{-Ker}(V)) = f^{-1}(V)$.

Hence $f^{-1}(V)$ is m_X-λ-closed.

Lemma: 3.3.70

Let (X,m_X) and (Y,m_Y) be two m-spaces and $f: (X,m_X) \rightarrow (Y,m_Y)$ be a function. Then the following statements are equivalent:

(i) f is (m_X,m_Y)-λ-continuous.

(ii) For each point x in X and each m_Y-open set V with $f(x) \in V$, there is m_X-λ-open set U_X such that $x \in U_X$, and $f(U_X) \in V$.

Proof:

(i) $\Rightarrow$ (ii)

Let $f(x) \in V$. Since f is (m_X,m_Y)-λ-continuous, we have $x \in f^{-1}(V)$ and $f^{-1}(V)$is m_X-λ-open. Let $U = f^{-1}(V)$. We have $x \in U$ and $f(U) \subseteq V$.

(ii) $\Rightarrow$ (i)

Let V an m_Y-open set and let $x \in f^{-1}(V)$. Then $f(x) \in V$ follows that there exists a m_X-λ-open set U_X such that $x \in U_X$, $f(U_X) \subseteq V$. We have $x \in U_X$, and $f^{-1}(V) = \bigcup_{X \in f^{-1}(V)} U_X$. by using Theorem [3.2.38], $f^{-1}(V)$ is m_X-λ-open.