CHAPTER II

CHAPTER - II

$(1, 2)^*$-SEMI GENERALIZED SEPARATIONS IN BITOPOLOGICAL SPACES

In this chapter the notion of $(1,2)^*$-semi-generalized separation axioms, $(1,2)^*$- ψ -separation axioms due to Lellis Thivagar and Nirmala Mariappan [32] and $(1,2)^*$-semi-g-regular spaces and $(1,2)^*$-semi-g-normal spaces due to Lellis Thivagar, Hatir and Nirmala Mariappan [39] are discussed .Properties ,characterizations and applications are analyzed.

SECTION-2.1

PRELIMINARIES

Definition: 2.1.1

A subset S of a bitopological space is said to be (X, τ_1, τ_2) is said to be $\tau_{1,2}$**-open** if $S=A\cup B$ where $A\in \tau_1$ and $B\in \tau_2$. A subset S of X is $\tau_{1,2}$**-closed** ,if the complement of S is $\tau_{1,2}$-open.

Definition: 2.1.2

Let S be a subset of X. Then

(i) The $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int(S) is defined by

$$\cup \{G/G \subseteq S \text{ and } G \text{ is } \tau_{1,2}\text{-open}\}.$$

(ii) The $\tau_{1,2}$-closure of S denoted by $\tau_{1,2}$-cl(S) is defined by

$$\cap \{F/S\subseteq F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\}.$$

Remark: 2.1.3

(i) $\tau_{1,2}$-int(S) is $\tau_{1,2}$-open for each $S\subseteq X$ and $\tau_{1,2}$-cl(S) is $\tau_{1,2}$-closed for each $S\subseteq X$.

(ii) A set $S\subseteq X$ is $\tau_{1,2}$-open iff $S = \tau_{1,2}$-int(S) and is $\tau_{1,2}$-closed iff $S = \tau_{1,2}$-cl(S).

(iii) $\tau_{1,2}$-int(S) = τ_1 -int(S) $\cup$ τ_2-int(S) and $\tau_{1,2}$-cl(S) = τ_1 - cl(S) $\cap$ cl(S) for any set S $\subseteq$ X.

(iv)For any family $\{S_i / i \in I\}$ of subsets of X we have

a) $$\bigcup_i \tau_{1,2} - \mathrm{int}(s_i) \subseteq \tau_{1,2} - \mathrm{int}(\bigcup_i s_i)$$

b) $$\bigcup_i \tau_{1,2} - \mathrm{cl}(s_i) \subseteq \tau_{1,2} - \mathrm{cl}(\bigcup_i s_i)$$

c) $$\tau_{1,2}\text{-}\mathrm{int}(\bigcap_i s_i) \subseteq \bigcap_i \tau_{1,2} - \mathrm{int}(s_i)$$

d) $$\tau_{1,2}\text{-}\mathrm{cl}(\bigcap_i s_i) \subseteq \bigcap_i \tau_{1,2} - \mathrm{cl}(s_i)$$

(v) $\tau_{1,2}$-open sets need not form a topology.

Definition: 2.1.4

A subset A of a bitopological space (X, τ_1, τ_2) is called

(i) **(1,2)* -semi-open** if A $\subseteq$ $\tau_{1,2}$-cl($\tau_{1,2}$-int(A))

(ii) **(1,2)* -preopen** if A $\subseteq$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(A))

(iii) **(1,2)* -*α*-open** if A $\subseteq$ $\tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A)))

(iv) **(1,2)*-semi-closed** if A^c is (1,2)* - semi-open.

(v) **(1,2)*-generalised closed** if $\tau_{1,2}$-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\tau_{1,2}$-open in *X*.

Definition: 2.1.5

(i) The **(1,2)*-semi-closure** of a subset A of X, denoted by (1,2)*-scl(A) is defined to be the intersection of all (1,2)* -semi-closed sets containing A.

(ii)The **(1,2)* -semi-interior** of a subset A of X, denoted by (1,2)*-sint(A) is defined to be the union of all (1,2)* -semi-open)sets contained in A.

Definition: 2.1.6

A subset A of a bitopological space (X, τ_1, τ_2) is called

(i) **(1, 2)*-semi-generalized closed** if (1, 2)*-scl(A) $\subseteq$ U whenever A$\subseteq$ U and U is (1,2)* -semi-open in X.

(ii) **(1, 2)* -sg-open** if A^C is (1, 2)* -sg-closed.

Remark: 2.1.7

(i) Since arbitrary union (resp. intersection) of (1,2)* -semi-open (resp.(1,2)*-semi-closed) sets is (1,2)* -semi-open (resp. (1,2)*-semi-closed), (1,2)*-sintA (resp. (1,2)*-cl(A)) is (1,2)*-semi-open (resp. (1,2)*-semi-closed).

(ii) For a bitopological space (X, τ_1, τ_2) , a subset A of X is (1,2)* -semi-open (resp. (1,2)* -semi-closed) if and only if (1,2)* -sint(A) (resp. (1,2)* -scl(A)) = A.

Definition 2.1.8

A bitopological space (X, τ_1, τ_2) is called a

(i) **(1, 2)* - $T_{1/2}$-space** if every (1,2)*- g-closed set is $\tau_{1,2}$- closed.

(ii) **(1, 2)* - semi- $T_{1/2}$-space** if every (1,2)* -sg-closed set is (1,2)* -semi-closed.

(iii) **(1, 2)* - semi-T_0- space** if to each pair of distinct points x, y of X, there exists a (1, 2)* - semi-open set containing one but not the other.

(iv) **(1, 2)* -semi-T_1- space** if to each pair of distinct points x, y of X, there exists a pair of (1, 2)* -semi-open sets, one containing x but not y and the other containing y but not x.

(v) **(1, 2)* - semi-T_2- space** if to each pair of distinct points x, y of X, there exists a pair of disjoint (1, 2)* - semi-open sets U and V such that $x \in U$ and $y \in V$.

Theorem: 2.1.9

Let (X, τ_1, τ_2) be a bitopological space. Then

(1, 2)*-scl(A) = $A \cup \tau_{1,2}$-int ($\tau_{1,2}$-cl(A)) and (1,2)*-sint(A) = $A \cap \tau_{1,2}$-cl($\tau_{1,2}$-int(A))

Proof:

By definition (1, 2)*-scl (A) contains A.

Let F be any $(1, 2)^*$ -semi-closed set containing A.

Then $F \supseteq \tau_{1,2}$-int $(\tau_{1,2}$-cl (F)) $\supseteq \tau_{1,2}$-int $(\tau_{1,2}$-cl (A)). This is true for every $(1, 2)^*$-semi-closed set containing A.

Hence $(1, 2)^*$-scl(A) contains $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)).

Thus $\tau_{1,2}$-scl(A) $\supseteq A \cup \tau_{1,2}$-(int($\tau_{1,2}$-cl(A)).

$$\begin{aligned}\text{Now } \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A\cup \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A)))) &\subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A\cup \tau_{1,2}\text{-cl}(A))) \\ &\subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-cl}(A))) \\ &\subseteq A\cup \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A)).\end{aligned}$$

Therefore $A \cup \tau_{1,2}$-int $(\tau_{1,2}$-cl (A)) is a $(1, 2)^*$-semi-closed set containing A which implies $A \cup \tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $\supseteq (1,2)^*$-scl(A).

Hence $(1, 2)^*$-scl (A) = $A \cup \tau_{1,2}$-int $(\tau_{1,2}$-cl (A)).

By definition $(1, 2)^*$ -sint(A) is contained in A. Let S be any $(1, 2)^*$ -semi-open set contained in A.

Then $S \subseteq \tau_{1,2}$-cl $(\tau_{1,2}$-int(S)) $\subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int(A)). This is true for every $(1, 2)^*$-semi-open set contained in A.

Hence $(1, 2)^*$-sint (A) is contained in $\tau_{1,2}$- cl $(\tau_{1,2}$-int(A)).

Thus $(1,2)^*$-sint(A)$\subseteq$A $\tau_{1,2}$-cl($\tau_{1,2}$-int(A)).

$$\begin{aligned}\text{Now } \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A\cap \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)))) &\supseteq \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A\cap \tau_{1,2}\text{-int}(A))) \\ &\supseteq \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(\tau_{1,2}\text{-int}(A))) \\ &\supseteq A \cap \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)).\end{aligned}$$

Therefore $A \cap \tau_{1,2}$-cl($\tau_{1,2}$-int(A)) is a $(1,2)^*$-semi-open set contained in A which implies $A \cap \tau_{1,2}$-cl($\tau_{1,2}$-int(A)) $\supseteq (1,2)^*$-sint(A).

Hence $(1, 2)^*$ -sint (A) = $A \cap \tau_{1,2}$-cl $(\tau_{1,2}$-int(A)).

Theorem: 2.1.10

A bitopological space (X, τ_1, τ_2) is $(1,2)^*$-semi-$T_{1/2}$ if and only if every singleton in X is $(1,2)^*$-semi-open or $(1,2)^*$-semi-closed.

Proof:

Let (X, τ_1, τ_2) be $(1, 2)^*$-semi-$T_{1/2}$ and $x \in X$.

If {x} is not $(1, 2)^*$-semi-closed then X – {x} is not $(1, 2)^*$-semi-open.

Therefore X is the only $(1, 2)^*$-semi-open set containing X - {x} and hence X – {x} is $(1, 2)^*$-sg-closed.

Since (X, τ_1, τ_2) is $(1, 2)^*$-semi-$T_{1/2}$, X –{x} is $(1,2)^*$-semi-closed or {x} is $(1,2)^*$-semi-open.

Conversely,

Let every singleton in X be $(1, 2)^*$-semi-open or $(1, 2)^*$-semi-closed.

Let $A \subseteq X$ be $(1, 2)^*$-sg-closed.

Suppose A is not $(1, 2)^*$-semi-closed.

Then $(1, 2)^*$-scl(A) $\neq$ A.

Choose $x \in (1,2)^*$-scl(A) - A.

Case (i):

{x} is $(1,2)^*$-semi-open.

Then X – {x} is a $(1,2)^*$-semi-closed set containing A.

Therefore $(1, 2)^*$ -scl(A) $\subseteq$ X-{x}.

Case (ii):

{x} is $(1,2)^*$-semi-closed.

Then X – {x} is a $(1, 2)^*$ -semi-open set containing A and A is $(1, 2)^*$ -sg-closed.

Therefore $(1, 2)^*$-scl (A) $\subseteq$ X – {x}.

Thus in any case x $\notin (1, 2)^*$ -scl(A) which is a contradiction

SECTION-2.2

(1, 2)*-SEMI GENERALIZED SEPARATIONS AXIOMS

In this section some new separation axioms using $(1,2)^*$ - sg-open sets which are weaker than $(1,2)^*$ - semi-separation axioms are analyzed.

Definition :2.2.1

A bitopological space (X, τ_1, τ_2) is called a **$(1,2)^*$ -semi-generalised-T_0** (briefly $(1,2)^*$ -sg-T_0) space iff to each pair of distinct points x,y of X, there exists a $(1,2)^*$ -sg-open set containing one but not the other.

Example :2.2.2

Let $X = \{a, b, c\}$; $\tau_1 = \{\varphi, \{a, c\}, X\}$ $\tau_2 = \{\varphi, \{b, c\}, X\}$; $\tau_{1,2}$-open sets = $\{\varphi, \{a, c\}, \{b, c\}, X\}$ = $(1,2)^*$- semi-open sets; $(1,2)^*$ -sg-open sets = $\{\varphi, \{a, c\}, \{b, c\}, \{c\}, X\}$.(X, τ_1, τ_2) is $(1,2)^*$ -sg- T_0 but not $(1,2)^*$ -semi- T_0.

Theorem : 2.2.3

If in any bitopological space, (X, τ_1, τ_2), $(1,2)^*$ -semi-generalized closures of distinct points are distinct, then (X, τ_1, τ_2) is$(1,2)^*$-sg- T_0.

Proof

Let $x, y \in X$ and $x \neq y$.

By hypothesis, $(1,2)^*$ - sgcl($\{x\}$) $\neq$ $(1,2)^*$ -sgcl($\{y\}$).

Then there exists a point $z \in X$ such that z belongs to exactly one of the two sets , say $(1,2)^*$ -sgcl($\{y\}$) but not to $(1,2)^*$ -sgcl($\{x\}$).

If $y \in (1,2)^*$ -sgcl($\{x\}$), then $(1,2)^*$ -sgcl($\{y\}$)$\subseteq(1,2)^*$-sgcl($\{x\}$) which implies $z \in (1,2)^*$ -sgcl($\{x\}$), a contradiction.

So $y \in X - (1,2)^*$ -sgcl($\{x\}$), a $(1,2)^*$ -sg-open set which does not contain x. This shows that (X, τ_1, τ_2) is $(1,2)^*$-sg- T_0.

Theorem :2.2.4

In any bitopological space (X, τ_1, τ_2), $(1,2)^*$ - semi-generalized closures of distinct points are distinct.

Proof

Let $x, y \in X$ and $x \neq y$.

Case (i)

$\{x\}$ is $(1,2)^*$ - semi-closed. Then $\{x\}$ is $(1,2)^*$ - sg-closed. Now $y \neq x$ implies

$y \notin \{x\} = (1,2)^*$ - sgcl($\{x\}$).

Hence $(1,2)^*$ - sgcl($\{y\}$) $\neq$ $(1,2)^*$ - sgcl($\{x\}$).

Case (ii)

$\{x\}$ is not $(1,2)^*$ - semi-closed. Then $X-\{x\}$ not $(1,2)^*$ - semi-open and therefore X is the only $(1,2)^*$- semi-open set containing $X-\{x\}$.

Hence $X-\{x\}$ is $(1,2)^*$ - sg-closed.

Now $y \in X - \{x\}$ implies $(1, 2)^*$ - sgcl($\{y\}$) $\subseteq X - \{x\}$.

Hence $x \notin (1, 2)^*$ - sgcl($\{y\}$) and $(1,2)^*$ - sgcl($\{y\}$) $\neq$ $(1,2)^*$ - sgcl($\{x\}$).

Remark: 2.2.5

Every bitopological space (X, τ_1, τ_2) is $(1,2)^*$ - sg- T_0.

Definition :2.2.6

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-semi-C_0** (resp. $(1, 2)^*$-αC_0) space iff to each pair of distinct points x,y of X, there exists a $(1, 2)^*$-semi-open (resp. $(1, 2)^*$-α-open) set such that $(1, 2)^*$-scl(G) (resp. $(1, 2)^*$-αcl(G))contains one of x and y, but not the other.

Theorem :2.2.7

If a bitopological space (X, τ_1, τ_2) is

(i) $(1, 2)^*$-α C_0 then it is $(1, 2)^*$ -semi- C_0.

(ii) $(1, 2)^*$-semi- C_0 then it is $(1, 2)^*$-semi- T_0.

Proof :

(i) Let (X, τ_1, τ_2) be $(1, 2)^*$ -α C_0 and x, y ∈X with x≠ y.

Then there exists a $(1, 2)^*$-α-open set G of X such that x ∈ $(1, 2)^*$-αcl(G) and y∉$(1, 2)^*$-αcl(G).

Since G is $(1, 2)^*$-α-open, $G_1 = (1, 2)^*$-αcl(G) = $\tau_{1,2}$-cl($\tau_{1,2}$-int(G) .

Since $(1, 2)^*$-scl(G_1) ⊆ $(1, 2)^*$-αcl((G_1) =$(1, 2)^*$-αcl(G), y∉$(1, 2)^*$-scl(G_1).

But x∈ $(1, 2)^*$-αcl(G) = G_1 ⊆ $(1, 2)^*$-scl(G_1).

Hence (X, τ_1, τ_2) is $(1, 2)^*$-semi- C_0.

(ii) Let (X, τ_1, τ_2) be $(1, 2)^*$-semi- C_0 and x, y ∈X with x≠y.

Then there exists a $(1, 2)^*$-semi-open set G of X such that x ∈$(1, 2)^*$-scl(G) and y∉$(1, 2)^*$-scl(G).

Since G is $(1, 2)^*$-semi-open, $(1, 2)^*$-scl(G) is $(1, 2)^*$-semiopen. Also x ∈ $(1, 2)^*$-scl(G) but y∉$(1, 2)^*$-scl(G).Hence (X, τ_1, τ_2) is $(1, 2)^*$-semi- T_0.

Definition :2.2.8

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -semi-generalised-T_1** (briefly written as $(1, 2)^*$ -sg- T_1) space if to each pair of distinct points x,y of X, there exists a pair of $(1, 2)^*$-sg-open sets, one containing x but not y and the other containing y but not x.

Remark 2.2.9

A $(1, 2)^*$ -semi-T_1 bitopological space is $(1, 2)^*$-sg-T_1 since every $(1, 2)^*$-semi-open set is $(1, 2)^*$ -sg-open..

Definition 2.2.10

A subset A of a bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-semi-generalized neighbourhood** (briefly $(1, 2)^*$-sgnbd.) of a point x of X if there exists a $(1, 2)^*$-sg-open set U containing x such that U⊆A.

Definition :2.2.11

The union of all $(1, 2)^*$ sg-open sets of a bitopological space (X, τ_1, τ_2) which are contained in a subset A of X is called the **$(1, 2)^*$ -semi generalized interior of A** and is denoted by $(1, 2)^*$ -sgint(A).

Lemma :2.2.12

Every singleton set {x} of a bitopological space (X, τ_1, τ_2) is either $(1, 2)^*$-nowhere dense or $(1, 2)^*$ - preopen.

Proof

Let $x \in X$. If {x} is not $(1, 2)^*$ -nowhere dense then G = $\tau_{1,2}$-int($\tau_{1,2}$-cl{x}) $\neq \varphi$.

Suppose x is not in G. Then G^C contains x.

Since G^C is $\tau_{1,2}$-closed, $G^C \supseteq \tau_{1,2}$-cl{x} $\supseteq G$ which implies $G = \varphi$, a contradiction.

Hence {x} $\subseteq \tau_{1,2}$- int($\tau_{1,2}$-cl{x}) or {x} is $(1, 2)^*$-preopen.

Remark:2.2.13

Lemma 2.2.12 provides a decomposition $X = X_1 \cup X_2$ of (X, τ_1, τ_2) where $X_1 = \{x \in X / \{x\}$ is $(1, 2)^*$-nowhere dense $\}$ and $X_2 = \{x \in X / \{x\}$ is $(1, 2)^*$ -preopen$\}$.

Lemma :2.2.14

Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X.Then

(i) A is $(1, 2)^*$-sg -closed if and only if $X_1 \cap (1, 2)^*$-scl(A) $\subseteq$ A.

(ii) $(1, 2)^*$ -pcl(A) $\subseteq X_1 \cup A$.

Proof:

(i) Let $A \subseteq X$ be $(1, 2)^*$ -sg-closed and let $x \in X_1 \cap (1, 2)^*$-scl(A).

Now $\tau_{1,2}$-int($\tau_{1,2}$-cl{x}) = φ implies {x} is $(1, 2)^*$ - semi-closed.

If x is not in A then $A \subseteq \{x\}^c$, a $(1, 2)^*$-semi-open set.

Since A is $(1, 2)^*$-sg -closed, $(1, 2)^*$ -scl(A) $\subseteq \{x\}^c$ which implies $x \notin (1, 2)^*$-scl(A), a contradiction.

Hence $x \in A$ and $X_1 \cap (1, 2)^*$-scl(A) $\subseteq$ A.

Conversely ,

Let $X_1 \cap (1, 2)^*$-scl(A)$\subseteq$A.

Let U be any $(1, 2)^*$-semi-open set containing A.

It is enough to prove that $X_2 \cap (1, 2)^*$-scl(A) $\subseteq$ U.

Let $x \in X_2 \cap (1, 2)^*$-scl(A).

$\{x\}$ is $(1, 2)^*$-preopen implies $\{x\} \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl$\{x\}$) = G.

Suppose x is not in U.

Then x is in U^C.

Therefore G = $\tau_{1,2}$-int($\tau_{1,2}$-cl$\{x\}$) $\subseteq$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(U^C) $\subseteq U^C$ since U^C is $(1, 2)^*$-semi-closed.

Then $G \cap U = \varphi$ which implies $G \cap A = \varphi$.

This is a contradiction since, $x \in G$, a $(1, 2)^*$-semi-open set and $x \in (1, 2)^*$-scl(A) imply $G \cap A \neq \varphi$.

(ii) Let $x \in (1, 2)^*$-pcl(A).

Suppose $x \notin X_1$.

Then $\{x\}$ is $(1, 2)^*$-preopen and thus $\{x\} \cap A \neq \varphi$.

This implies $x \in A$.

Theorem :2.2.15

Arbitrary intersection of $(1, 2)^*$-sg-closed sets in a bitopological space is $(1, 2)^*$-sg-closed.

Proof:

Let $(A_i)_{i\in I}$ $(A_i)_{i\in I}$ be any collection of $(1, 2)^*$-sg-closed subsets of the bitopological space (X, τ_1, τ_2) and let $A = \cap_{i\in I} A_i$.

It is enough if we prove that $X_1 \cap (1, 2)^*$-scl(A) $\subseteq$ A. Let $x \in X_1 \cap (1, 2)^*$-scl(A).

Then $\{x\}$ is $(1, 2)^*$-no where dense.

Then $\{x\}$ is $(1, 2)^*$-semi-closed.

Suppose $x \notin A$. Then $x \notin A_i$ for some $i \in I$.

Then $A_i \subseteq X - \{x\}$, a $(1, 2)^*$ -semi-open set.

Since A_i is $(1, 2)^*$-sg-closed, $(1, 2)^*$-scl$(A_i) \subseteq X - \{x\}$.

This implies $x \notin (1, 2)^*$-scl(A_i), a contradiction, since $x \in (1, 2)^*$-scl(A) implies $x \in (1, 2)^*$-scl(A_i) for every i.

Remark :2.2.16

From Theorem 2.2.15 it follows that arbitrary union of $(1, 2)^*$- sg-open sets in a bitopological space is $(1, 2)^*$-sg-open. Hence $(1, 2)^*$-sgint(A) is $(1, 2)^*$-sg-open.

Lemma :2.2.17

A subset of a bitopological space is $(1, 2)^*$ -sg-open iff it is a $(1, 2)^*$-sgnbd of each of its points.

Definition :2.2.18

A point x of X is called a **$(1, 2)^*$ -semi-generalised interior point** (briefly $(1, 2)^*$ -sg- interior point) of A iff $x \in (1, 2)^*$-sgint(A).

Lemma :2.2.19

Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then any point $x \in X$ is a $(1, 2)^*$-sg-interior point of A iff A is $(1, 2)^*$-sgnbd of x.

Theorem :2.2.20

For a bitopological space (X, τ_1, τ_2) the following are equivalent.

(i) (X, τ_1, τ_2) is $(1, 2)^*$-sg-T_1.

(ii) Each one point set is $(1, 2)^*$ -sg-closed in X.

Proof

(i)$\Rightarrow$(ii)

Let (X, τ_1, τ_2) be $(1, 2)^*$-sg-T_1 and $x \in X$.

Suppose $(1, 2)^*$-sgcl$(\{x\}) \neq \{x\}$.

Then we can find an element $y \in (1, 2)^*$-sgcl$(\{x\})$ with $y \neq x$.

Since X is $(1, 2)^*$-sg-T_1, there exist $(1, 2)^*$-sg-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Now $x \in V^C$ and V^C is $(1, 2)^*$-sg-closed.

Therefore $(1, 2)^*$-sgcl$(\{x\}) \subseteq V^C$ which implies $y \in V^C$, contradiction.

Hence$(1, 2)^*$-sgcl$(\{x\}) = \{x\}$ or $\{x\}$ is $(1, 2)^*$-sg-closed.

(ii) $\Rightarrow$ (i)

Let $x, y \in X$ with $x \neq y$.

Then $\{x\}$ and $\{y\}$ are $(1, 2)^*$-sg-closed.

Therefore $U = (\{x\})^c$ and $V = (\{y\})^c$ are $(1, 2)^*$-sg-open and $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Hence (X, τ_1, τ_2) is $(1, 2)^*$-sg-T_1

Definition :2.2.21

A point x of X is called a **$(1, 2)^*$-sg-limit point** of a subset A of X if and only if $A \cap (G - \{x\}) \neq \varphi$ for every $(1, 2)^*$-sg-open set G containing x.

Remark:2.2.22

The set of all $(1, 2)^*$-sg-limit points of a set A of X is denoted by $(1, 2)^*$-sgd(A).

Lemma :2.2.23

If A is a subset of a space X, then $(1, 2)^*$ -sgcl(A) = $A \cup$ $(1, 2)^*$-sgd((A)).

Proof :

Let $x \in (1, 2)^*$-sgcl(A) and suppose $x \notin A$.

Let G be any $(1, 2)^*$-sg-open set containing x. If $G \cap A = \varphi$ then $A \subseteq G^C$, a $(1, 2)^*$-sg-closed set and therefore $(1, 2)^*$-sgcl(A)$\subseteq G^C$.

This implies $x \notin (1, 2)^*$-sgcl(A), a contradiction.

Hence $G \cap A \neq \varphi$, in particular $A \cap (G - \{x\}) \neq \varphi$ and therefore $x \in (1, 2)^*$ -sgdA.

Remark :2.2.24

(i) A point $x \in (1, 2)^*$-sgcl(A) if and only if every $(1, 2)^*$-sg-open set containing x contains a point of A.

(ii) If X is $(1, 2)^*$-sg-T_1 and if $p \in (1, 2)^*$ -sgd(A) for some subset A of X, then it is not necessary that every $(1, 2)^*$ -sgnbd of p contains infinitely many points of A.

Example :2.2.25

Let X = {a, b, c, d} ; τ_1 = {φ, {a}, {a, c}, {a, d}{a, c, d},X} τ_2 = {φ, {c, d},X}; $\tau_{1,2}$-open sets = {φ, {a}, {a, c}, {a, d}, {a, c, d}, {c, d},X}, $(1, 2)^*$ -semi-open sets ={φ, {a}, {a, c}, {a, d}, {a, c, d}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, b},X} = $(1, 2)^*$ -sg-open sets .(X, τ_1, τ_2) is $(1, 2)^*$ -sg-T_1. For the set A = {a, c, d}, b is a $(1, 2)^*$-sg-limit point but the $(1, 2)^*$ -sgnbd {b, c, d}of b contains only a finite number of points from A.

Definition :2.2.26

A mapping f : X → Y is said to be a **$(1, 2)^*$-sg-irresolute mapping** if $f^{-1}(G)$ is $(1, 2)^*$ -sg-open in X whenever G is $(1, 2)^*$ -sg-open in Y .

Theorem :2.2.27

Let f : X → Y be an injective and $(1, 2)^*$ -sg-irresolute mapping. If Y is $(1, 2)^*$ -sg-T_1 then so also is X.

Definition :2.2.28

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -semi-generalised-T_2** (briefly $(1, 2)^*$-sg-T_2) space if to each pair of distinct points x,y of X, there exists a pair of disjoint $(1, 2)^*$-sg-open sets U and V such that $x \in U$ and $y \in V$.

Remark :2.2.29

A $(1, 2)^*$ -semi-T_2 bitopological space is $(1, 2)^*$-sg-T_2 since every $(1, 2)^*$-semi-open set is $(1, 2)^*$-sg-open.

Theorem :2.2.30

Let f : X → Y be an injective and $(1, 2)^*$-sg-irresolute mapping. If Y is $(1, 2)^*$ -sg-T_2, then X is also $(1, 2)^*$ -sg-T_2.

Definition :2.2.31

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -semi-generalised-R_0** (briefly $(1, 2)^*$-sg-R_0) space if for each $(1, 2)^*$ -sg-open set $G, x \in G$, implies $(1, 2)^*$-sgcl$(x) \subseteq G$.

Definition :2.2.32

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -semi-generalised-R_1** (briefly $(1, 2)^*$-sg-R_1) space if for $x, y \in X$ with $(1, 2)^*$-sgcl$(x) \neq (1, 2)^*$ -sgcl(y), there exists a pair of disjoint $(1, 2)^*$-sg-open sets U and V such that $(1, 2)^*$ -sgcl$(x) \subseteq U$ and $(1, 2)^*$ -sgcl$(y) \subseteq V$.

Theorem :2.2.33

If a bitopological space (X, τ_1, τ_2) is $(1, 2)^*$ -sg-R_1 then every singleton set is $(1, 2)^*$ -sg-closed.

Proof

Suppose $\{x\}$ is not $(1, 2)^*$ -sg-closed. Then $x \neq y \in (1, 2)^*$-sgcl$(\{x\})$.
Then $(1, 2)^*$ -sgcl$(\{y\}) \subseteq (1, 2)^*$ -sgcl$(\{x\})$ and $(1, 2)^*$-sgcl$(\{y\}) \neq (1, 2)^*$-sgcl$(\{x\})$.
Since X is $(1, 2)^*$ -sg-R_1, $(1, 2)^*$ -sgcl$(\{y\})$ and $(1, 2)^*$ -sgcl$(\{x\})$ cannot be separated by disjoint $(1, 2)^*$ -sg-open sets.

Corollary :2.2.34

If a bitopological space (X, τ_1, τ_2) is $(1, 2)^*$ -sg-R_1 then it is $(1, 2)^*$ -sg-R_0.

Theorem :2.2.35

The following are equivalent in a bitopological space (X, τ_1, τ_2)

(i) X is $(1, 2)^*$ -sg-T_2.
(ii) X is $(1, 2)^*$-sg-R_1 and $(1, 2)^*$ -sg-T_1.
(iii) X is $(1, 2)^*$-sg-R_1 and $(1, 2)^*$-sg-T_0.

Proof :

(i)$\Rightarrow$ (ii)

X is $(1, 2)^*$ -sg-T_2 implies X is $(1, 2)^*$-sg-T_1 and therefore by Theorem 2.2.20, every singleton set in X is $(1, 2)^*$ -sg-closed.

Let x, y$\in$ X and x $\neq$ y.

Since X is $(1, 2)^*$ -sg-T_2, there exist two disjoint $(1, 2)^*$ -sg-open sets U and V containing x and y respectively.

Since {x} and {y} are $(1, 2)^*$ -sg-closed,X is $(1, 2)^*$ -sg-R_1.

(ii) $\Rightarrow$ (iii)

Since X is $(1, 2)^*$ -sg-T_1 implies X is $(1, 2)^*$ -sg-T_0, X is $(1, 2)^*$-sg-R_1 and $(1, 2)^*$-sg-T_0.

(iii) $\Rightarrow$ (i)

X is $(1, 2)^*$ -sg-R_1 implies {x} is $(1, 2)^*$ -sg-closed for every x$\in$X.

Also x $\neq$ y implies $(1, 2)^*$ -sgcl({x})$\neq$ $(1, 2)^*$ -sgcl({y}).

Hence X is $(1, 2)^*$ -sg-R_1 implies X is $(1, 2)^*$ -sg-T_2.

SECTION- 2.3

(1, 2)*-ψ SEPARATION AXIOMS

In this section, $(1, 2)^*$-ψ-open sets which are stronger than $(1, 2)^*$-semi-generalized separation axioms are analyzed.

Definition :2.3.1

A subset A of a bitopological space (X, τ_1, τ_2) is called **$(1, 2)^*$-ψ-closed** if $(1, 2)^*$-scl(A) $\subseteq$ U whenever A$\subseteq$U and U is $(1, 2)^*$-sg-open.A set **A** is called **$(1, 2)^*$ -ψ-open** if A^C is $(1, 2)^*$ -ψ-closed.

Remark :2.3.2

If A is $(1, 2)^*$-ψ-closed and U is $(1, 2)^*$ -sg-open with A $\subseteq$U, then $(1, 2)^*$ -scl(A) $\subseteq$$(1, 2)^*$-sint(U). This follows from the definitions of$(1, 2)^*$ -ψ-closed set and $(1, 2)^*$-sg-open set.

Remark :2.3.3

(i) Every $(1, 2)^*$ -semi-closed set, and thus every $\tau_{1,2}$ -closed set and every $(1, 2)^*$ -α-closed set is $(1, 2)^*$-ψ-closed.

(ii) Every $(1, 2)^*$-ψ-closed set is $(1, 2)^*$-sg-closed, and thus $(1, 2)^*$-semi-preclosed and also $(1, 2)^*$ -gs-closed.

.

Example :2.3.4

Let X = {a, b, c, d} ; τ_1 = {φ, {a, b},X} τ_2 = {φ, {a, c},X}; $\tau_{1,2}$ -open sets = {φ, {a, b}, {a, c}, {a, b, c},X} A = {b, c, d} is $(1, 2)^*$ -ψ-closed but not $(1, 2)^*$ -semi-closed.

Example :2.3.5

Let X = {a, b, c, d, e} ; τ_1 = {φ, {a, d, e}, {b, c},X} τ_2 = {φ, {b, c, d},X}; $\tau_{1,2}$ -open sets = {φ, {a, d, e}, {b, c}, {b, c, d},X} A = {b} is $(1, 2)^*$ -sg-open and $(1, 2)^*$ -sg-closed. Since $(1, 2)^*$ -scl(A) = {b, c},A is not $(1, 2)^*$-ψ-closed.

Thus the class of $(1, 2)^*$ -ψ-closed sets properly contains the class of $(1, 2)^*$ - semi-closed sets, and thus properly contains the class of $(1, 2)^*$ - α-closed sets and also properly contains the class of $\tau_{1,2}$ -closed sets. Also the class of $(1, 2)^*$ -ψ-closed sets is properly contained in the class of $(1, 2)^*$ -sg-closed sets and hence it is properly contained in the class of $(1, 2)^*$ -semi-pre closed sets and contained in the class of $(1, 2)^*$ -gs-closed sets.

Definition :2.3.6

A subset A of a bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -ψ- neighbourhood** of a point x of X if there exists a $(1, 2)^*$ -ψ-open set U containing x such that $U \subseteq A$.

Definition :2.3.7

The union of all $(1, 2)^*$-ψ-open sets of a bitopological space (X, τ_1, τ_2) which are contained in a subset A of X is called the **$(1, 2)^*$ -ψ-interior** of A and is denoted by $(1, 2)^*$-ψ-int(A).

Lemma :2.3.8

A subset of a bitopological space is $(1, 2)^*$ -ψ-open iff it is a $(1, 2)^*$-ψ-nbd of each of its points.

Definition :2.3.9

A point x of X is called a **$(1, 2)^*$ -ψ-interior point** of A iff $x \in (1, 2)^*$-ψ-int(A).

Lemma :2.3.10

Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then any point $x \in X$ is a $(1, 2)^*$-ψ-interior point of A iff A is a $(1, 2)^*$ -ψ-nbd of x.

Definition :2.3.11

The **$(1, 2)^*$ -ψ-closure** of a subset A of X is the intersection of all $(1, 2)^*$-ψ-closed sets that contains A and is denoted by $(1, 2)^*$-ψ-cl(A).

Definition :2.3.12

A point x of X is called a **$(1, 2)^*$ -ψ-limit point** of a subset A of X if and only if $A \cap (G - \{x\}) \neq \varphi$ for every $(1, 2)^*$-ψ-open set G containing x.

Definition :2.3.13

The set of all $(1, 2)^*$-ψ-limit points of a set A of X is denoted by $(1, 2)^*$ -ψ-d(A), is called **$(1, 2)^*$-ψ-derived set** of A.

Definition:2.3.14

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-ψ-T_0 space** iff to each pair of distinct points x,y of X, there exists a $(1, 2)^*$-ψ-open set containing one but not the other.

Theorem :2.3.15

If in any bitopological space, (X, τ_1, τ_2), $(1, 2)^*$-ψ-closures of distinct points are distinct, then (X, τ_1, τ_2) is $(1, 2)^*$-ψ-T_0.

Proof

Let $x, y \in X$ and $x \neq y$.
By hypothesis, $(1, 2)^*$-$\psi cl(x) \neq (1, 2)^*$-$\psi cl(y)$.
Then there exists a point $z \in X$ such that z belongs to exactly one of the two sets , say $(1, 2)^*$-$\psi cl(y)$ but not to $(1, 2)^*$-$\psi cl(x)$.
If $y \in (1, 2)^*$-$\psi cl(x)$, then $(1, 2)^*$-$\psi cl(y) \subseteq (1, 2)^*$-$\psi cl(x)$ which implies $z \in (1, 2)^*$-$\psi cl(x)$, a contradiction.
So $y \in X - (1, 2)^*$-$\psi cl(x)$, a $(1, 2)^*$-ψ-open set which does not contain x.
Hence (X, τ_1, τ_2) is $(1, 2)^*$-ψ-T_0.

Definition :2.3.16

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-ψ-T_1 space** if to each pair of distinct points x,y of X, there exists a pair of $(1, 2)^*$-ψ-open sets, one containing x but not y and the other containing y but not x.

Remark :2.3.17

Every $(1, 2)^*$-semi-T_1 bitopological space is $(1, 2)^*$-ψ-T_1 and every $(1, 2)^*$-ψ-T_1 bitopological space is $(1, 2)^*$-sg-T_1 since every $(1, 2)^*$-semi-open set is $(1, 2)^*$-ψ-open and every $(1, 2)^*$-ψ-open set is $(1, 2)^*$-sg-open.

Theorem :2.3.18

For a bitopological space (X, τ_1, τ_2) the following are equivalent.

(i) (X, τ_1, τ_2) is $(1, 2)^*$-ψ-T_1.

(ii) Each one point set is $(1, 2)^*$-ψ-closed in X.

Proof

(i) $\Rightarrow$ (ii)

Let (X, τ_1, τ_2) be $(1, 2)^*$-ψ-T_1 and $x \in X$.

Suppose $(1, 2)^*$-sgcl($\{x\}$)$\neq$ $\{x\}$.

Then $y\in$ $(1, 2)^*$-ψcl($\{x\}$) with $y \neq x$.

Since X is $(1, 2)^*$-ψ-T_1, there exist $(1, 2)^*$ -sg-open sets U and V such that $x\in U$, $y\notin U$ and $y\in V$, $x \notin V$.

Now $x \in V^C$and V^C is $(1, 2)^*$-ψ-closed.

Therefore $(1, 2)^*$-ψcl($\{x\}$)$\subseteq V^C$ which implies $y\in V^C$, a contradiction.

Hence $(1, 2)^*$ -ψcl($\{x\}$) = $\{x\}$ or $\{x\}$ is $(1, 2)^*$-ψ-closed.

(ii) $\Rightarrow$ (i)

Let $x, y\in X$ with $x \neq y$.

Then $\{x\}$ and $\{y\}$ are $(1, 2)^*$-ψ-closed.

Therefore $U = (\{y\})^c$ and $V = (\{x\})^c$ are $(1, 2)^*$-ψ-open and $x \in U$, $y\notin U$ and $y\in V$, $x\notin V$.

Hence (X, τ_1, τ_2) is $(1, 2)^*$ -ψ-T_1.

Definition :2.3.19

A mapping $f : X \rightarrow Y$ is said to be a **$(1, 2)^*$ -ψ-irresolute mapping** if $f^{-1}(G)$ is $(1, 2)^*$ -ψ-open in X whenever G is $(1, 2)^*$ -ψ-open in Y .

Theorem :2.3.20

Let $f : X \rightarrow Y$ be an injective and $(1, 2)^*$-ψ-irresolute mapping.If Y is $(1,2)^*$-ψ -T_1then X is also $(1, 2)^*$-ψ-T_1.

Definition :2.3.21

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -ψ-T_2 space** if to each pair of distinct points x,y of X, there exists a pair of disjoint $(1, 2)^*$-ψ-open sets U and V such that $x\in U$ and $y\in V$.

Remark :2.3.22

Every $(1, 2)^*$-semi-T_2 bitopological space is $(1, 2)^*$ -ψ-T_2 and every $(1, 2)^*$ -ψ-T_2 bitopological space is $(1, 2)^*$-sg-T_2 since every $(1, 2)^*$-semi-open set is $(1, 2)^*$ -ψ-open and every $(1, 2)^*$ -ψ-open set is $(1, 2)^*$-sg-open.

Theorem :2.3.23

Let $f : X \rightarrow Y$ be an injective and $(1, 2)^*$-ψ-irresolute mapping. If Y is $(1, 2)^*$-ψ-T_2 then X is also $(1, 2)^*$-ψ-T_2.

Definition :2.3.24

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-ψ-R_0 space** if for each $(1, 2)^*$-ψ-open set G, $x \in G$, implies $(1, 2)^*$-$\psi cl(x) \subseteq G$.

Definition :2.3.25

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$-ψ-R_1 space** if for x, $y \in X$ with $(1, 2)^*$-$\psi cl(\{x\}) \neq (1, 2)^*$-$\psi cl(\{y\})$, there exists a pair of disjoint $(1, 2)^*$-ψ-open sets U and V such that $(1, 2)^*$-$\psi cl(\{x\}) \subseteq U$ and $(1, 2)^*$-$\psi cl(\{y\}) \subseteq V$.

Theorem :2.3.26

Every $(1, 2)^*$-ψ-R_1 bitopological space is $(1, 2)^*$-ψ-R_0.

Proof

Let (X, τ_1, τ_2) be $(1, 2)^*\psi$-R_1 and let G be a $(1, 2)^*\psi$-open set containing x.

If $(1, 2)^*\psi cl(x) \not\subset G$ then there exists an element $y \in ((1, 2)^*\psi cl(x) \cap G^c$.

Since G^c is $(1, 2)^*\psi$-closed, $(1, 2)^*\psi cl(y) \subseteq G^c$.

Now $(1, 2)^*$-$\psi cl(x) \neq (1, 2)^*$ - $\psi cl(y)$ and X is $(1, 2)^*\psi$-R_1.

Hence there exist disjoint $(1, 2)^*$-ψ-open sets containing $(1, 2)^*$-$\psi cl(x)$ and $(1, 2)^*$-$\psi cl(y)$ respectively.

This is not possible since $y \in (1, 2)^*$- $\psi cl(x) \cap (1, 2)^*\psi cl(y)$.

Theorem :2.3.27

The following are equivalent in a bitopological space (X, τ_1, τ_2)

(i) X is $(1, 2)^*$-ψ-T_2.

(ii) X is $(1, 2)^*$-ψ-R_1 and $(1, 2)^*$-ψ-T_1.

(iii) X is $(1, 2)^*$-ψ-R_1 and $(1, 2)^*$-ψ-T_0.

Proof

(i) $\Rightarrow$ (ii)

X is $(1, 2)^*$ -ψ-T_2 implies X is $(1, 2)^*$ -ψ-T_1 and therefore by Theorem 4.17, every singleton set in X is $(1, 2)^*$ -ψ-closed.

Let $x, y \in X$ and $x \neq y$.

Since X is $(1, 2)^*$-ψ-T_2, there exist two disjoint $(1, 2)^*$ -ψ-open sets U and V containing x and y respectively.

Since $\{x\}$ and $\{y\}$ are $(1, 2)^*$ -ψ-closed, X is $(1, 2)^*$ -ψ-R_1.

(ii) ⇒ (iii)

Since X is $(1, 2)^*$ -ψ-T_1 implies X is $(1, 2)^*$-ψ-T_0 , X is $(1, 2)^*$-ψ-R_1 and $(1, 2)^*$-ψ-T_0.

(iii) ⇒ (i)

Let $x, y \in X$ and $x \neq y$.

Case(i):

$(1, 2)^*$ -ψcl$(\{x\}) \neq (1, 2)^*$ -ψcl$(\{y\})$.

Since X is $(1, 2)^*$ -ψ-R_1, there exist two disjoint $(1, 2)^*$ -ψ-open sets U and V such that $U \supseteq (1, 2)^*$ -ψcl$(\{x\})$ and $V \supseteq (1, 2)^*$ -ψcl$(\{y\})$.

Then $x \in U$ and $y \in V$.

Case(ii):

$(1, 2)^*$ -ψcl$(\{x\}) = (1, 2)^*$ -ψcl$(\{y\})$.

Since $x \neq y$ and X is $(1, 2)^*$-ψ-T_0, there exists a $(1, 2)^*$ -ψ-open set U containing x but not y.

Then $y \in U^C$, a $(1, 2)^*$-ψ-closed set.

This implies $(1, 2)^*$ -ψcl$(\{y\}) \subseteq U^C$ and therefore $(1, 2)^*$-ψcl$(\{x\}) \subseteq U^C$ or $x \in U^C$ which is a contradiction.

Hence case(ii) is not possible.

SECTION -2.4

(1,2)*- SEMI - g-REGULAR SPACES

Definition:2.4.1

A bitopological space (X, τ_1, τ_2) is said to be **$(1, 2)^*$-semi-regular** if for each $(1, 2)^*$-semi-closed set $A \subseteq X$ and each point $x \notin A$ there exist disjoint $(1, 2)^*$ -semi-open sets U and V such that $x \in U$ and $A \subseteq V$.

Definition:2.4.2

A bitopological space (X, τ_1, τ_2) is said to be **$(1, 2)^*$-semi-g-regular** if for each $(1, 2)^*$-sg-closed set A and each point $x \notin A$, there exist disjoint $(1, 2)^*$-semi-open sets $U, V \subseteq X$ such that $A \subseteq U$ and $x \in V$.

Lemma :2.4.3

A bitopological space (X, τ_1, τ_2) is $(1, 2)^*$ -semi-g-regular if and only if (X, τ_1, τ_2) is $(1, 2)^*$ -semi-regular and $(1, 2)^*$-semi-$T_{1/2}$.

Proof:

Let (X, τ_1, τ_2) be $(1, 2)^*$-semi-g-regular. Then clearly (X, τ_1, τ_2) is $(1, 2)^*$-semi-regular.

Let $A \subseteq X$ be $(1, 2)^*$-sg-closed.

For each $x \notin A$ there exists a $(1, 2)^*$ -semi-open set V_x containing x such that $V_x \cap A = \varphi$.

If $V = Y \{V_x / x \notin A\}$ then V is $(1, 2)^*$ -semi-open and $V = X - A$

Hence A is $(1, 2)^*$ -semi-closed.

Conversely,

Let (X, τ_1, τ_2) be $(1, 2)^*$-semi regular and $(1, 2)^*$-semi-$T_{1/2}$.

Then by definition of $(1, 2)^*$- semi regular,for each $(1, 2)^*$-semi-closed set $A \subseteq X$ and each point $x \notin A$ .There exist disjoint $(1,2)^*$ semi-open sets $U,V \subseteq X$ such that $A \subseteq U$ and $x \in V$.Hence A is $(1,2)^*$ semi-g-regular.

Remark:2.4.4

The following example shows that there exist $(1, 2)^*$ -semi-regular spaces which are not $(1, 2)^*$- semi-g-regular.

Example :2.4.5

Let X = {a , b, c, d} , τ_1= { φ, {a,b}, X }, τ_2= { φ, {c,d}, X },

$\tau_{1,2}$-open sets = { φ, {a,b}, {c,d}, X }$(1, 2)^*$ -semi-open sets = $\tau_{1,2}$-open sets and (X, τ_1, τ_2) is $(1, 2)^*$-semi-regular. Let A = {a} and x = b. A is $(1, 2)^*$-sg-closed, but A and x cannot be separated by disjoint $(1, 2)^*$ -semi-open sets.

Thus (X, τ_1, τ_2) is not $(1, 2)^*$ -semi-g-regular.

Remark:2.4.6

$(1, 2)^*$ -semi-regular and $(1, 2)^*$-semi-$T_{1/2}$-spaces are independent from each other.The bitopological space given in example 2.4.5 is $(1, 2)^*$ -semi-regular but not $(1, 2)^*$ -semi-$T_{1/2}$.

Example :2.4.7

LetX = {a, b, c, d} τ_1= { φ, {a},{a,b}, X } τ_2={φ,{d},{b,d},{c,d},{b,c,d},X}, $\tau_{1,2}$-open sets = { φ, {a,b}, {c,d}, X } ;$(1, 2)^*$ -semi-open sets =$\tau_{1,2}$-open sets. (X, τ_1, τ_2) is $(1, 2)^*$-semi-$T_{1/2}$ since all singleton subsets of X are $(1, 2)^*$-semi-open or $(1, 2)^*$ -semi-closed. But it is not $(1, 2)^*$ -semi-regular since b and the $(1, 2)^*$-semi-closed set {a ,c} cannot be separated by disjoint $(1, 2)^*$-semi-open subsets of X.

Definition:2.4.8

A subset A of (X, τ_1, τ_2) is called **$(1, 2)^*$ -semi-clopen** if A is both $(1, 2)^*$-semi-open and $(1, 2)^*$ -semi-closed. For any $(1, 2)^*$ - semi-open set V , $(1, 2)^*$ -scl(V) is always $(1, 2)^*$ -semi-clopen.

Theorem :2.4.9

For a bitopological space (X, τ_1, τ_2) the following are equivalent:

(i) (X, τ_1, τ_2) is $(1, 2)^*$ -semi-g-regular.

(ii) Every $(1, 2)^*$ - sg-open set is a union of $(1, 2)^*$-semi-clopen sets.

(iii)Every $(1, 2)^*$ -sg-closed set is an intersection of $(1, 2)^*$-semi-clopen sets.

Proof.

(i)$\Rightarrow$ (ii)

Let U be a $(1, 2)^*$ -sg-open set and let $x \in U$. If $A = X - U$, then A is $(1, 2)^*$ -sg-closed.

By assumption, there exist disjoint $(1, 2)^*$ -semi-open subsets $W_1$and W_2 of X such that $x \in W_1$ and $A \subseteq W_2$.

If $V = (1, 2)^*\text{-scl}(W_1)$ then V is $(1, 2)^*$-semi-clopen and $V \cap A \subseteq V \cap W_2 = \varphi$. It follows that $x \in V \subseteq X$.

Thus U is a union of $(1, 2)^*$-semi-clopen sets.

(ii) $\Rightarrow$ (iii)

Since every $(1, 2)^*$ - sg-open set is a union of $(1, 2)^*$-semi-clopen sets,then every $(1, 2)^*$ -sg-closed set is an intersection of $(1, 2)^*$-semi-clopen sets

(iii) $\Rightarrow$ (i)

Let A be $(1, 2)^*$ -sg-closed and let $x \notin A$. By assumption, there exists a $(1, 2)^*$-semi-clopen set V such that $A \subseteq V$ and $x \notin V$. If $U = X - V$, then U is a $(1, 2)^*$ -semi-open set containing x and $U \cap V = \varphi$. Thus (X, τ_1, τ_2) is $(1, 2)^*$-semi-g-regular.

Definition :2.4.10

A bitopological space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -(s,sg)-R_0 space** if $(1, 2)^*$ -scl($\{x\}$) $\subseteq$ U whenever U is $(1, 2)^*$ -sg-open and $x \in U$.

Theorem :2.4.11

Every $(1, 2)^*$-semi-g-regular bitopological space (X, τ_1, τ_2) is both $(1, 2)^*$ -semi-T_2 and $(1, 2)^*$-(s,sg)-R_0.

Theorem:2.4.12

For a bitopological space (X, τ_1, τ_2) the following are equivalent:

(i) (X, τ_1, τ_2) is $(1, 2)^*$-(s,sg)-R_0.

(ii) For each $(1, 2)^*$-sg-closed set F and $x \notin F$, $F \cap (1, 2)^*$-scl$(\{x\}) = \varphi$.

Proof:

(i) $\Rightarrow$ (ii)

Let F be a $(1, 2)^*$-sg-closed set and let $x \notin F$.

Then $x \in F^C$ and F^C is a $(1, 2)^*$ - sg-open set.

By (i), $(1, 2)^*$-scl$(\{x\}) \subseteq F^C$ or $F \in \cap (1, 2)^*$-scl$(\{x\}) = \varphi$.

(ii) $\Rightarrow$ (i)

Let U be a $(1, 2)^*$-sg-open set and let $x \in U$. Since $F = U^C$ is a $(1, 2)^*$-sg-closed set not containing x, by (ii), $U^C \cap 1, 2)^*$-scl$(\{x\}) = \varphi$.

This implies $U \supseteq (1, 2)^*$-scl$(\{x\})$ and hence (X, τ_1, τ_2) is $(1, 2)^*$-(s,sg)-R_0.

Theorem :2.4.13

Let the bitopological space (X, τ_1, τ_2) be $(1, 2)^*$-(s,sg)-R_0. Then

(i) For each $(1, 2)^*$-sg-closed set F and $x \notin F$, there exists a $(1, 2)^*$-sg-open set U such that $F \subseteq U$ and $x \notin U$.

(ii) For any two distinct points x and y of X, $(1, 2)^*$-scl$(\{x\}) = (1, 2)^*$-scl$(\{y\})$ or $(1, 2)^*$-scl$(\{x\}) \cap (1, 2)^*$-scl$(\{y\}) = \varphi$.

Proof.

(i). Let F be a $(1, 2)^*$-sg-closed set and let $x \notin F$. Then $x \in F^C$ and F^C is a $(1, 2)^*$-sg-open set.

Since X is $(1, 2)^*$-(s,sg)-R_0, $(1, 2)^*$-scl$(\{x\}) \subseteq F^C$.

If $U = ((1, 2)^*$-scl$(\{x\}))^c$, then U is a $(1, 2)^*$-sg-open set such that $F \subseteq U$ and $x \notin U$.

(ii). Let x and y be any two distinct points of X.

Case (a):

x is in $(1, 2)^*$-scl$(\{y\})$ and y is in $(1, 2)^*$-scl$(\{x\})$.

Then $(1, 2)^*$-scl$(\{x\}) \subseteq (1, 2)^*$-scl$(\{y\})$ and $(1, 2)^*$-scl$(\{y\}) \subseteq (1, 2)^*$-scl$(\{x\})$ or $(1, 2)^*$-scl$(\{x\}) = (1, 2)^*$-scl$(\{y\})$.

...

Case (b):

$x \notin (1, 2)^*$ -scl({y}) or $y \notin (1, 2)^*$ -scl({x}).

If $x \notin (1, 2)^*$ -scl({y}), then $x \in ((1, 2)^*$ -scl({y}))^c, a $(1, 2)^*$ -sg-open set.
Since X is $(1, 2)^*$ -(s,sg)-R_0, $(1, 2)^*$ -scl({x})$\subseteq$ ((1, 2)* -scl({y})^c or
$(1, 2)^*$ -scl({x})$\cap$(1, 2)* -scl({y}) = φ .
Statements (i) and (ii) in Theorem 2.4.13 need not imply that (X, τ_1, τ_2) is
$(1, 2)^*$ -(s,sg)-R_0.

Example :2.4.14

Let X = {a, b, c, d, e}; τ_1= { φ, {c,d,e}, X } τ_2= { φ, {a,b},{a,b,e},{a,b,c,d}, X}, $\tau_{1,2}$-open sets = { φ, {a,b}, {a,b,e},{c,d,e}, X }; $(1, 2)^*$ -semi-open sets ={φ,{a,b},{a,b,e},{c,d,e} {a,b,c,d},{a,b,c,e},{a,b,d,e} ,X } For this bitopological space (X, τ_1, τ_2) statements (i) and (ii) of Theorem 2.4.13 hold good but X is not $(1, 2)^*$ -(s,sg)-R_0.

SECTION – 2.5

(1,2)*- SEMI - g-NORMAL SPACES

Definition :2.5.1

A bitopological space (X, τ_1, τ_2) is said to be **(1, 2)* -semi-normal** if any two disjoint $(1, 2)^*$ -semi-closed sets can be separated by two disjoint $(1, 2)^*$ -semi-open sets.

Definition :2.5.2

A bitopological space (X, τ_1, τ_2) is said to be **(1, 2)* -semi-g-normal** if for every pair of disjoint $(1, 2)^*$-sg-closed subsets A and B of X, there exist disjoint $(1, 2)^*$-semi-open sets U, V $\subseteq$ X such that A $\subseteq$ U and B $\subseteq$ V .

Remark :2.5.3

Every $(1, 2)^*$ -semi-g-normal space is $(1, 2)^*$ -semi-normal. The converse is not true.

Definition :2.5.4

A bitopological space (X, τ_1, τ_2) is called **$(1, 2)^*$ -semi-symmetric** if for any two points $x, y \in X$, if $x \in (1, 2)^*$-scl($\{y\}$) then $y \in (1, 2)^*$ -scl($\{x\}$).

Theorem :2.5.5

A bitopological space (X, τ_1, τ_2) is $(1, 2)^*$ -semi-symmetric if and only if every singleton subset of X is $(1, 2)^*$-sg-closed.

Proof:

Let (X, τ_1, τ_2) be $(1, 2)^*$ -semi-symmetric and let $\{x\} \subseteq U$, a $(1, 2)^*$ -semi-open set.

Suppose $y \in (1, 2)^*$ -scl($\{x\}$) is not in U.

Then $(1, 2)^*$ -scl(y) $\subseteq X - U$. Since (X, τ_1, τ_2) is $(1, 2)^*$ -semi-symmetric,

$x \in (1, 2)^*$ -scl($\{y\}$) which implies $x \notin U$, a contradiction.

Therefore $x \in (1, 2)^*$ -scl($\{y\}$).

Conversely ,

Let $x \in (1, 2)^*$ -scl($\{y\}$).

Suppose $y \notin (1, 2)^*$ -scl($\{x\}$).

Then $\{y\}$ is $(1, 2)^*$-sg-closed implies $(1, 2)^*$ -scl($\{y\}$) $\subseteq X - (1, 2)^*$- scl($\{x\}$), a contradiction.

Remark :2.5.6

If a bitopological space (X, τ_1, τ_2) is $(1, 2)^*$-semi-normal and if $F \cap A = \varphi$ where F is $(1, 2)^*$ -semi-closed and A is $(1, 2)^*$ -sg-closed, then $F \cap (1, 2)^*$ -scl(A) $= \varphi$. Hence there exist disjoint $(1, 2)^*$ -semi-open sets $U, V \subseteq X$ such that $F \subseteq U$ and $A \subseteq V$

Theorem :2.5.7

(i) Every $(1, 2)^*$ -semi-normal, $(1, 2)^*$-semi-symmetric bitopological space (X, τ_1, τ_2) is $(1, 2)^*$ -semi-regular.

(ii) Every $(1, 2)^*$-semi-g-normal, $(1, 2)^*$ -semi-symmetric bitopological space (X, τ_1, τ_2) is $(1, 2)^*$-semi-g-regular.

Proof:

(i) Suppose that $A \subseteq X$ is $(1, 2)^*$-semi-closed and $x \notin A$. By Theorem 4.5, $\{x\}$ is $(1, 2)^*$ - sg-closed. By remark 4.6, A and $\{x\}$ can be separated by disjoint $(1, 2)^*$-semi-open sets.

(ii) Obvious.

Example :2.5.8

Let $X = \{a , b , c , d \}$, $\tau_1 = \{ \varphi, \{a,b\},\{c,d\}, X \}$ $\tau_2 = \{ \varphi, \{b,d\}, X\}$, $\tau_{1,2}$ -open sets $= \{ \varphi, \{a,b\}, \{b,d\},\{c,d\} \{a,b,d\},\{b,c,d\}, X \}$. $(1, 2)^*$ -semi-open sets = $\tau_{1,2}$ -open sets . (X, τ_1, τ_2) is $(1, 2)^*$ -g-normal but not $(1, 2)^*$ -semi-g-regular.

Theorem :2.5.9

For a bitopological space (X, τ_1, τ_2) the following are equivalent:

(i) (X, τ_1, τ_2) is $(1, 2)^*$ -semi-g-normal.

(ii) For every $(1, 2)^*$-sg-closed set A and every $(1, 2)^*$ -sg-open set U containing A, there is a $(1, 2)^*$-semi-clopen set V such that $A \subseteq V \subseteq U$.

Proof:

(i) $\Rightarrow$ (ii)

Let A be $(1, 2)^*$-sg-closed and U be $(1, 2)^*$ -sg-open with $A \subseteq U$

Now ,we have $A \cap (X - U) = \varphi$.

Hence there exist disjoint $(1, 2)^*$ -semi-open sets W_1 and W_2 such that $A \subseteq W_1$ and $X - U \subseteq W_2$.

Since W_1 is $(1, 2)^*$ -semi-open, $V = (1, 2)^*$- $scl(W_1)$ is a $(1, 2)^*$-semi-clopen set such that $A \subseteq V \subseteq U$.

(ii) $\Rightarrow$ (i)

This is obvious.

Definition :2.5.10

A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **$(1, 2)^*$ -semi-irresolute** if $f^{-1}(B)$ is $(1, 2)^*$ -semi-open in X for every $(1, 2)^*$ -semi-open set $B \subseteq Y$.

Definition :2.5.11

A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **(1, 2)* -pre-semi-closed** if f(F) is $(1, 2)^*$ -semi-closed in Y for every $(1, 2)^*$ -semi-closed set $F \subseteq X$.

Lemma :2.5.12

A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(1, 2)^*$ -pre-semi-closed if and only if $(1, 2)^*$ -scl(f(A)) $\subseteq$ f($(1, 2)^*$ -scl(A)) for every subset A of X.

Proof:

Obvious.

Lemma :2.5.13

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be $(1, 2)^*$ -semi-irresolute and $(1, 2)^*$ -pre-semi-closed. If $B \subseteq Y$ is $(1, 2)^*$ -sg-closed, then $f^{-1}(B) \subseteq X$ is $(1, 2)^*$ -sg-closed.

Proof:

Let $B \subseteq Y$ be $(1, 2)^*$ -sg-closed, and let $f^{-1}(B) \subseteq U$ where U is $(1, 2)^*$ -semi-open.

If V = Y - f(X - U), then $V \subseteq Y$ is $(1, 2)^*$ -semi-open since f is $(1, 2)^*$ -pre-semi-closed.

Moreover, we have $f^{-1}(V) \subseteq U$ and $B \subseteq V$.

Since B is $(1, 2)^*$ -sg-closed, we conclude that $(1, 2)^*$ -scl(B) $\subseteq V$.

Since f is $(1, 2)^*$-semi-irresolute, we now have $(1, 2)^*$ -scl($f^{-1}(B)$) $\subseteq f^{-1}((1, 2)^*$ -scl(B))

$\subseteq f^{-1}(V) \subseteq U$.

Hence $f^{-1}(B)$ is $(1, 2)^*$ -sg-closed.

Theorem:2.5.14

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be $(1, 2)^*$ -semi-irresolute, $(1, 2)^*$ -pre semi-closed and onto. If (X, τ_1, τ_2) is $(1, 2)^*$ -semi-g-normal then (Y, σ_1, σ_2) is $(1, 2)^*$-semi-g-normal.

Proof:

Let $B_1, B_2 \subseteq Y$ be two disjoint $(1, 2)^*$ -sg-closed sets.

...

By Lemma 2.5.9, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are $(1, 2)^*$ -sg-closed.

Hence there exist disjoint $(1, 2)^*$ -semi-open sets $U_1, U_2 \subseteq X$ such that $f^{-1}(B_1) \subseteq U_1$ and $f^{-1}(B_2) \subseteq U_2$.

As in the proof of Lemma 2.5.9, there exist $(1, 2)^*$ -semi-open sets V_1, $V_2 \subseteq Y$ such that $B_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq U_i$ for $i = 1,2$.

Hence $f^{-1}(V_1) \cap f^{-1}(V_2) = \varphi$ and since f is onto, we have $V_1 \cap V_2 = \varphi$,

i.e. (Y, σ_1, σ_2) is $(1, 2)^*$ -semi-g-normal.

Definition :2.5.15

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be **$(1, 2)^*$ - contra-semi-continuous** if the inverse image of every $\sigma_{1,2}$ closed set in (Y, σ_1, σ_2) is $(1, 2)^*$ -semi-open in (X, τ_1, τ_2) .

Definition:2.5.16

A bitopological space (X, τ_1, τ_2) is called **weakly $(1, 2)^*$ -sg-normal** if disjoint $(1, 2)^*$ -sg-closed sets can be separated by disjoint $\tau_{1,2}$ closed sets.

Definition :2.5.17

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be always **$(1, 2)^*$ -sg-closed** if the image of each $(1, 2)^*$ -sg-closed set in (X, τ_1, τ_2) is $(1, 2)^*$ -sg-closed in (Y, σ_1, σ_2).

Theorem:2.5.18

If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an injective $(1, 2)^*$ -contra-semi-continuous and always $(1, 2)^*$ -sg-closed function and (Y, σ_1, σ_2) is weakly $(1, 2)^*$ -sg-normal, then (X, τ_1, τ_2) is $(1, 2)^*$ -semi-g-normal.

Proof:

Suppose that $A_1, A_2 \subseteq X$ are $(1, 2)^*$ -sg-closed and disjoint.

Since f is always$(1, 2)^*$ -sg-closed and injective, $f(A_1)$, $f(A_2) \subseteq Y$ are $(1, 2)^*$ -sg-closed and disjoint.

Since (Y, σ_1, σ_2) is weakly $(1, 2)^*$ -sg-normal, $f(A_1)$ and $f(A_2)$ can be separated by disjoint $\sigma_{1,2}$-closed sets $B_1, B_2 \subseteq Y$.

Since f is $(1, 2)^*$ -contra-semi-continuous, A_1 and A_2 can be separated by disjoint $(1, 2)^*$ -semi-open sets $f^{-1}(B_1)$ and $f^{-1}(B_2)$. Thus (X, τ_1, τ_2) is$(1, 2)^*$ -semi-g-normal.