Fuzzy Subalgebras of Z-Algebras

2

In 1966, Imai and Iseki [25,26] introduced two new classes of abstract algebras: BCK-algebras and BCI-algebras. In 2017, Chandramouleeswaran et al.[18] introduced the concept of Z-algebras as a new structure of algebra based on propositional calculus. In 1965, Zadeh[73] introduced the fundamental concept of a fuzzy set which is a generalization of an ordinary set. In 1991, following the idea of fuzzy groups, Xi[71] introduced the notion of fuzzy BCK-algebras. In 2015, Christopher Jefferson and Chandramouleeswaran[19] applied fuzzy algebraic structures in BP-algebras.

This chapter is divided into two sections. In the first section, we discuss the notion of Fuzzy Z-Subalgebras in Z-algebras. In the second section, we discuss the notion of Fuzzy Z-Ideals in Z-algebras and obtain some of their properties.

2.1 Fuzzy Z-Subalgebras in Z-algebras

In this section, we define the notion of Fuzzy Z-Subalgebra of a Z-algebra and obtain some interesting results.

Definition 2.1.1: Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy Z-Subalgebra** of a Z-algebra X if, for all $x, y \in X$ the following condition is satisfied : $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

Example 2.1.2: Let X= {0, 1, 2, 3} be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Then $(X,*,0)$ is a Z-algebra.

Define a fuzzy set A_1 in X whose membership function μ_{A_1} is given by

$$\mu_{A_1}(x) = \begin{cases} 0.6 & \text{if} \quad x = 0 \\ 0.4 & \text{if} \quad x = 1 \\ 0.3 & \text{if} \quad x = 2, 3 \end{cases}$$

Then, A_1 is a fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 2.1.3: Intersection of any two fuzzy Z-Subalgebras of a Z-algebra X is again a fuzzy Z-Subalgebra.

Proof: Let A_1 and A_2 be fuzzy Z-Subalgebras of a Z-algebra X. Then, for all $x, y \in X$,

$$\begin{aligned} \mu_{A_1 \cap A_2}(x * y) &= \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \\ &\geq \min\{\min\{\mu_{A_1}(x), \mu_{A_1}(y)\}, \min\{\mu_{A_2}(x), \mu_{A_2}(y)\}\} \\ &= \min\{\min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \\ &= \min\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\} \end{aligned}$$

That is $\mu_{A_1 \cap A_2}(x * y) \geq \min\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\}$

Hence $A_1 \cap A_2$ is a fuzzy Z-Subalgebra of a Z-algebra X.

The following example illustrates Theorem 2.1.3.

Example 2.1.4: Consider Z-algebra X and fuzzy Z-Subalgebra A_1 as in Example 2.1.2 . The fuzzy set A_2 in X defined by $\mu_{A_2}(x) = \begin{cases} 0.7 & \text{if} \quad x = 0 \\ 0.5 & \text{if} \quad x = 1 \\ 0.2 & \text{if} \quad x = 2, 3 \end{cases}$ is also a fuzzy Z-Subalgebra of X.

Now , $$\mu_{A_1 \cap A_2}(x) = \begin{cases} 0.6 & \text{if} \quad x = 0 \\ 0.4 & \text{if} \quad x = 1 \\ 0.2 & \text{if} \quad x = 2, 3 \end{cases}$$

Then, $A_1 \cap A_2$ is a fuzzy Z-Subalgebra of a Z-algebra X.

We can generalize the theorem 2.1.3 as follows.

Corollary 2.1.5: Let $\{A_i | i \in \Omega\}$ where $i \in \Omega$ an index set , be a family of fuzzy Z-Subalgebras of a Z-algebra X. Then $\bigcap_{i \in \Omega} A_i$ is also a fuzzy Z-Subalgebra of X.

Theorem 2.1.6: Let A be a fuzzy Z-Subalgebra of a Z-algebra X such that $\mu_A(0) \neq 0$. Let $\mu_B : X \to [0,1]$ be a fuzzy set defined by $\mu_B(x) = \dfrac{\mu_A(x)}{\mu_A(0)}$ for all $x \in X$. Then B is a fuzzy Z-Subalgebra of X.

Proof: Let $x, y \in X$. Then,

$$\mu_B(x * y) = \frac{\mu_A(x * y)}{\mu_A(0)} \geq \frac{1}{\mu_A(0)} \min\{\mu_A(x), \mu_A(y)\} = \min\left\{\frac{\mu_A(x)}{\mu_A(0)}, \frac{\mu_A(y)}{\mu_A(0)}\right\} = \min\{\mu_B(x), \mu_B(y)\}.$$

Therefore B is a fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 2.1.7: If A is a Z-Subalgebra of a Z-algebra X, then the characteristic function χ_A defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is a fuzzy Z-Subalgebra of X.

Proof: Let $x, y \in X$.

Case (i) If $x, y \in A$ then $x * y \in A$. Hence $\chi_A(x) = 1, \chi_A(y) = 1$ and $\chi_A(x * y) = 1$. Thus, $\chi_A(x * y) \geq 1 = \min\{1,1\} = \min\{\chi_A(x), \chi_A(y)\}$, for all $x, y \in X$.

Case (ii) If both $x, y \notin A$, then $\chi_A(x) = 0$ and $\chi_A(y) = 0$. In this case, $\chi_A(x * y) \geq 0 = \min\{0,0\} = \min\{\chi_A(x), \chi_A(y)\}$, for all $x, y \in X$.

Case (iii) If $x \in A$, $y \notin A$, then $\chi_A(x) = 1$ and $\chi_A(y) = 0$. Then $\chi_A(x * y) \geq 0 = \min\{1,0\} = \min\{\chi_A(x), \chi_A(y)\}$, for all $x, y \in X$.

Case (iv) Interchanging the roles of x and y in case (iii) we can prove that A is a fuzzy Z-Subalgebra of X when $x \notin A$, $y \in A$.

Thus the theorem is completely proved.

The converse of the above theorem is also true.

Corollary 2.1.8: Let A be any subset of a Z-algebra X. If the characteristic function χ_A is a fuzzy Z-Subalgebra of X, then A is a Z-Subalgebra of X.

Proof: For any $x, y \in A$, $\chi_A(x) = 1 = \chi_A(y)$.

Therefore, $\chi_A(x * y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1,1\} = 1$. Hence $\chi_A(x * y) = 1 \Rightarrow x * y \in A$.

The theorem 2.1.7 can be generalized as follows.

Theorem 2.1.9: Let A be any Z-Subalgebra of a Z-algebra X and $\mu_A : X \to [0,1]$ be a fuzzy set defined by $\mu_A(x) = \begin{cases} t_0 & \text{if } x \in A \\ t_1 & \text{if } x \notin A \end{cases}$ with $t_0 > t_1$. Then A is a fuzzy Z-Subalgebra of X.

Theorem 2.1.10: A fuzzy set A of a Z-algebra X is a fuzzy Z-Subalgebra if and only if for every $t \in [0,1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is either empty or a Z-Subalgebra of X.

Proof: If A is a fuzzy Z-Subalgebra of X, for any $t \in [0,1]$, assume that $U(\mu_A; t) \neq \phi$

For any x , y$\in U(\mu_A; t)$, we have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$.

Then $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{t, t\} = t$

This implies $x * y \in U(\mu_A; t)$

That is, $U(\mu_A; t)$ is a Z-subalgebra of a Z-algebra X.

Conversely, assume that $U(\mu_A; t)$ is a Z-Subalgebra of a Z-algebra X.

Let x,y$\in$X and let $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$. Then x$\in U(\mu_A; t_1)$ and y$\in U(\mu_A; t_2)$.

If $t_1 \leq t_2$, then $U(\mu_A; t_2) \subseteq U(\mu_A; t_1)$ and so $y \in U(\mu_A; t_1)$.

Since $U(\mu_A; t_1)$ is a Z-Subalgebra of X, $x * y \in U(\mu_A; t_1)$.

Thus $\mu_A(x * y) \geq t_1 = \min\{\mu_A(x), \mu_A(y)\}$, proving that A is a fuzzy Z-Subalgebra of a Z-algebra X.

Definition 2.1.11: Let A be a fuzzy Z-Subalgebra of a Z-algebra X. For any $t \in [0,1]$, Z-Subalgebras $U(\mu_A; t)$ are called upper level Z-Subalgebras of A.

Remark 2.1.12: Henceforth, in this section, the upper level Z-Subalgebras of A will be referred as level Z-Subalgebras of A.

Theorem 2.1.13: Any Z-Subalgebra of a Z-algebra X can be realized as a level Z-Subalgebra of some fuzzy Z-Subalgebra of X.

Proof: Let S be a Z-Subalgebra of a Z-algebra X and A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

where $t \in [0,1]$ is fixed. Clearly $U(\mu_A; t) = S$.We consider the following cases:

Case (i): If $x, y \in S$ then $x * y \in S$.

Hence $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t$ and

$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = t$.

Case (ii): If $x, y \notin S$ then $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = 0$.

Then $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0$.

Case (iii): If at most one of $x, y \in S$ then atleast one of $\mu_A(x)$ and $\mu_A(y)$ is equal to 0.

Hence $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0$.

This shows that S is a level Z-Subalgebra of a Z-algebra X corresponding to the fuzzy Z-Subalgebra A of X.

Theorem 2.1.14: Let X be a Z-algebra. Then given any chain of Z-Subalgebras $S_0 \subset S_1 \subset \cdots \subset S_r = X$, there exists a fuzzy Z-Subalgebra A of X whose upper level Z-Subalgebras are exactly the Z-Subalgebras of this chain.

Proof : Consider a set of numbers $t_0 > t_1 > t_2 > \cdots > t_r$ **,** where each $t_i \in [0,1]$.

Let A be a fuzzy set defined by $\mu_A(S_0) = t_0$ and $\mu_A(S_i - S_{i-1}) = t_i$, $i = 1,2,...,r$.

Claim: A is a fuzzy Z-Subalgebra of a Z-algebra X.

Let $x, y \in X$. Then we classify it into two cases as follows:

Case (1): Let $x, y \in S_i - S_{i-1}$. Then by the definition of A, $\mu_A(x) = t_i = \mu_A(y)$.

Since S_i is a Z-Subalgebra of X, it follows that $x * y \in S_i$ and so either $x * y \in S_i - S_{i-1}$ or $x * y \in S_{i-1}$. In any case, we conclude that $\mu_A(x * y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$.

Case (2): For $i > j$, Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$.

Then $\mu_A(x) = t_i$; $\mu_A(y) = t_j$ and $x * y \in S_i$, since S_i is a Z-Subalgebra of X and $S_j \subset S_i$.

Hence $\mu_A(x * y) \geq t_j = \min\{\mu_A(x), \mu_A(y)\}$.

Thus A is a fuzzy Z-Subalgebra of a Z-algebra X.

From the definition of A, it follows that $\text{Im}(A) = \{t_0, t_1, \cdots, t_r\}$.

Hence the upper level Z-Subalgebras of A are given by the chain of Z-Subalgebras.

$U(\mu_A;t_0) \subset U(\mu_A;t_1) \subset U(\mu_A;t_2) \subset \cdots \subset U(\mu_A;t_r) = X$.

Now $U(\mu_A;t_0) = \{x \in X \mid \mu_A(x) \geq t_0\} = S_0$.

Finally , we prove that $U(\mu_A;t_i) = S_i$ for $i = 1,2,...,r$.

Clearly $S_i \subseteq U(\mu_A;t_i)$.

If $x \in U(\mu_A;t_i)$, then $\mu_A(x) \geq t_i$ which implies that $x \notin S_j$ for $j > i$.

Hence $\mu_A(x) \in \{t_1, t_2, \cdots, t_i\}$ and so $x \in S_k$ for some $k \leq i$.

As $S_k \subseteq S_i$, it follows that $x \in S_i \Rightarrow U(\mu_A;t_i) = S_i$ for $i = 0,1,2,\ldots r$.

This completes the proof.

Note: If X is a finite Z –algebra, then the number of Z-Subalgebras of X is finite whereas the number of level Z- Subalgebras of a fuzzy Z-Subalgebra A appears to be infinite. But since every level Z-Subalgebra is indeed Z-Subalgebra of X, not all these Z-Subalgebras are distinct. The next theorem characterizes this aspect.

Theorem 2.1.15: Let A be a fuzzy Z-Subalgebra of a Z-algebra X. Two level Z-Subalgebras $U(\mu_A;t)$ and $U(\mu_A;s)$ (with $t < s$) of A are equal if and only if there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Proof: Assume that $U(\mu_A;t) = U(\mu_A;s)$ for some $t < s$ and there exists $x \in X$ such that $t \leq \mu_A(x) < s$.

Then $U(\mu_A;s)$ is a proper subset of $U(\mu_A;t)$ which is a contradiction.

Hence there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Conversely, Suppose that there is no $x \in X$ such that $t \leq \mu_A(x) < s$. Since $t < s$, we get

$U(\mu_A;s) \subseteq U(\mu_A;t)$ (1)

If $x \in U(\mu_A;t)$ then $\mu_A(x) \geq t$ and so $\mu_A(x) > s$, because $\mu_A(x)$ does not lie between t and s.

Hence $x \in U(\mu_A;s)$.

Hence $U(\mu_A;t) \subseteq U(\mu_A;s)$ (2)

From (1) and (2) we get $U(\mu_A;t) = U(\mu_A;s)$.

Remark 2.1.16: As a consequence of **Theorem 2.1.15**, the level Z-Subalgebras of a fuzzy Z-Subalgebra A of a finite Z-algebra X form a chain $U(\mu_A;t_0) \subset U(\mu_A;t_1) \subset \cdots \subset U(\mu_A;t_r) = X$, where $t_0 > t_1 > t_2 > ... > t_r$.

Corollary 2.1.17: Let X be a finite Z-algebra and A be a fuzzy Z-Subalgebra of X. If $Im(A) = \{t_1, \cdots, t_n\}$, then the family of Z-Subalgebras $U(\mu_A;t_i), i = 1,2,...,n$, constitutes all the level Z-Subalgebras of A.

Proof: Let $t \in [0,1]$ and $t \notin Im(A)$. Suppose $t_1 < t_2 < \cdots < t_n$ without loss of generality.

If $t \leq t_1$, then $U(\mu_A;t_1) = X = U(\mu_A;t)$.

If $t > t_n$, then $U(\mu_A;t) = \phi$ obviously.

If $t_{i-1} < t < t_i$, then $U(\mu_A;t) = U(\mu_A;t_i)$ by **Theorem 2.1.15**. Thus for any $t \in [0,1]$, the level Z-Subalgebra is one of $\{U(\mu_A;t_i) \mid i = 1,2,\cdots,n\}$.

Lemma 2.1.18: Let X be a Z-algebra and A be a fuzzy Z-Subalgebra of X. If Im(A) is finite, say $\{t_1,t_2,...,t_n\}$, then for any $t_i, t_j \in Im(A)$, $U(\mu_A;t_i) = U(\mu_A;t_j)$ implies $t_i=t_j$.

Proof: Assume that $t_i \neq t_j$ and $t_i < t_j$.

If $x \in U(\mu_A;t_j)$ then $\mu_A(x) \geq t_j > t_i$.

Hence $x \in U(\mu_A;t_i)$

Let $x \in X$ such that $t_i < \mu_A(x) < t_j$.

Then $x \in U(\mu_A;t_i)$ but $x \notin U(\mu_A;t_j)$

Hence $U(\mu_A;t_j) \subset U(\mu_A;t_i)$ and

$U(\mu_A;t_j) \neq U(\mu_A;t_i)$ a contradiction.

Therefore $U(\mu_A;t_i) = U(\mu_A;t_j) \Rightarrow t_i = t_j$.

Theorem 2.1.19: Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras. If Im(A)=$\{t_1,t_2,...,t_r\}$ and Im(B)=$\{q_1,q_2,...,q_k\}$ where $t_1 \geq t_2 \geq ... \geq t_r$ and $q_1 \geq q_2 \geq ... \geq q_k$. Then

(i) $k = r$

(ii) $U(\mu_A;t_i) = U(\mu_B;q_i)$, $i = 1,2,\ldots,r$

(iii) If $x \in X$ such that $\mu_A(x) = t_i$ then $\mu_B(x) = q_i$ $i = 1,2,\ldots,r$.

Proof: Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras with F(A)=F(B) where F(A)= $\{U(\mu_A;t_i) | i = 1,2,...,r\}$ and F(B) = $\{U(\mu_B;q_i) | i = 1,2,...,k\}$.

Let Im(A) = $\{t_1,t_2,...,t_r\}$ where $t_1 \geq t_2 \geq ... \geq t_r$ (1)

and let Im(B) = $\{q_1, q_2,..., q_k\}$ where $q_1 \geq q_2 \geq ... \geq q_k$ (2)

From (1) we get $U(\mu_A;t_1) \subseteq U(\mu_A;t_2) \subseteq ... \subseteq U(\mu_A;t_r) = X$ (3)

From (2) we get $U(\mu_B;q_1) \subseteq U(\mu_B;q_2) \subseteq ... \subseteq U(\mu_B;q_k) = X$ (4)

To prove (i) : k = r

Suppose $k \neq r$, then we consider the following cases:

Case (i): $k > r$

Let $k > r$ then $U(\mu_A; t_i)= U(\mu_B; q_i)$ i=1,2,...,r

This shows that both t_i and $q_i \in \text{Im}(A)$

For i > r we observe that $t_i \notin \text{Im}(A)$ and hence, $U(\mu_A; t_i) \neq U(\mu_B; q_i)$, i= r+1, r+2,...,k.

Case (ii): $r > k$

Let $r > k$ then $U(\mu_A; t_i)= U(\mu_B; q_i)$ i=1,2,...,k

This shows that both t_i and $q_i \in \text{Im}(B)$.

For i > k we observe that $q_i \notin \text{Im}(B)$ and hence $U(\mu_A; t_i) \neq U(\mu_B; q_i)$, i=k+1,k+2,...,r.

From (3) and (4) we get $t_i \neq q_i$ for all i=1,2,...,r.

Hence we can find some i such that $U(\mu_A; t_i) \neq U(\mu_B; q_i)$.

This contradicts that F(A)=F(B).

Hence we conclude that k = r.

To prove (ii): By part (i), we have proved that k = r. Since A and B have identical family of level Z-Subalgebras, we have

$U(\mu_A; t_i)= U(\mu_B; q_i)$, i=1,2,...,r.

To prove (iii): Let $x \in X$ such that $\mu_A(x) = t_i$ and $\mu_B(x) = q_j$ where $0 \leq i \leq k$ and $0 \leq j \leq k$.

From (ii) follows that $x \in U(\mu_B;q_i)$, thus $\mu_B(x) \geq q_i$ and $q_j \geq q_i$

Therefore $U(\mu_B;q_j) \subseteq U(\mu_B;q_i)$

Since $x \in U(\mu_B;q_j) = U(\mu_A;t_j)$, we get $t_i = \mu_A(x) \geq t_j$, this

gives $U(\mu_B;q_i) = U(\mu_A;t_i) \subseteq U(\mu_A;t_j) = U(\mu_B;q_j)$

Thus $U(\mu_B;q_j) = U(\mu_B;q_i)$ and by above **lemma:2.1.18** we get $q_j = q_i$.

Hence $\mu_B(x) = q_i$.

Hence the proof.

Corollary 2.1.20: Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras. Then Im(A)=Im(B) implies A = B.

Proof: Let Im(A) = Im(B)= $\{q_1, q_2, ..., q_r\}$ where $q_1 \geq q_2 \geq ... \geq q_r$.

By **Theorem 2.1.19**, for any $x \in X$ there exists q_i such that $\mu_A(x) = q_i = \mu_B(x)$.

Thus $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

This implies A=B.

Theorem 2.1.21: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and let A be a fuzzy Z-Subalgebra of X with the sup property. Then the image of A denoted by h(A) is a fuzzy Z-Subalgebra of Y.

Proof: Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that $\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t)$; $\mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$.

Now,

$$\mu_{h(A)}(a *' b) = \sup_{t \in h^{-1}(a*'b)} \mu_A(t) \geq \mu_A(x_0 * y_0) \geq \min\{\mu_A(x_0), \mu_A(y_0)\} = \min\left\{\sup_{t \in h^{-1}(a)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t)\right\}$$

$$= \min\{\mu_{h(A)}(a), \mu_{h(A)}(b)\}$$

Hence h(A) is a fuzzy Z-Subalgebra of a Z-algebra Y.

Theorem 2.1.22: Let $h:(X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras. If A is a fuzzy Z-Subalgebra of Y then the pre-image of A denoted by $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X. Converse is true if h is an Z-epimorphism.

Proof: Let x, $y \in X$. Then,

$$\mu_{h^{-1}(A)}(x * y) = \mu_A(h(x * y)) = \mu_A(h(x) *' h(y)) \geq \min\{\mu_A(h(x)), \mu_A(h(y))\}$$

$$= \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}$$

Hence $\mu_{h^{-1}(A)}(x * y) \geq \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}$

Therefore, $h^{-1}(A)$ is a fuzzy Z-Subalgebra of a Z-algebra X.

On the other hand, assume that h is an Z-epimorphism and $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X. Let y_1, $y_2 \in Y$. Since h is an Z-epimorphism, there exists x_1, $x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

This implies $x_1 = h^{-1}(y_1)$ and $x_2 = h^{-1}(y_2)$.

Now, $\mu_A(y_1 *' y_2) = \mu_A(h(x_1) *' h(x_2)) = \mu_A(h(x_1 * x_2)) = \mu_{h^{-1}(A)}(x_1 * x_2)$

$$\geq \min\{\mu_{h^{-1}(A)}(x_1), \mu_{h^{-1}(A)}(x_2)\}$$

$$= \min\{\mu_A(h(x_1)), \mu_A(h(x_2))\}$$

$$= \min\{\mu_A(y_1), \mu_A(y_2)\}$$

Hence A is a fuzzy Z-Subalgebra of a Z-algebra Y.

Definition 2.1.23: Let h be an **Z-endomorphism** of Z-algebras and A be a fuzzy set in a Z-algebra X. We define a new fuzzy set A^h in X as $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$.

Theorem 2.1.24: Let h be an Z-endomorphism of a Z-algebra $(X, *, 0)$. If A be a fuzzy Z-Subalgebra of X. Then A^h is also a fuzzy Z-Subalgebra of X.

Proof: Let x, $y \in X$. Then,

$$\mu_{A^h}(x * y) = \mu_A(h(x * y)) = \mu_A(h(x) * h(y)) \geq \min\{\mu_A(h(x)), \mu_A(h(y))\} = \min\{\mu_{A^h}(x), \mu_{A^h}(y)\}$$

$$\Rightarrow \quad \mu_{A^h}(x * y) \geq \min\{\mu_{A^h}(x), \mu_{A^h}(y)\}$$

Hence A^h is a fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 2.1.25: If A and B are fuzzy Z-subalgebras of a Z-algebra X then $A \times B$ is also a fuzzy Z-Subalgebra of $X \times X$.

Proof: For any $(x_1,x_2), (y_1,y_2) \in X \times X$, we have

$$\begin{aligned}
\mu_{A\times B}((x_1,x_2)*(y_1,y_2)) &= \mu_{A\times B}(x_1*y_1, x_2*y_2) \\
&= \min\{\mu_A(x_1*y_1), \mu_B(x_2*y_2)\} \\
&\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1), \mu_B(x_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\} \\
&= \min\{\mu_{A\times B}(x_1,x_2), \mu_{A\times B}(y_1,y_2)\}
\end{aligned}$$

Hence $A \times B$ is also a fuzzy Z-Subalgebra of $X \times X$.

We can generalize the above theorem as follows.

Theorem 2.1.26: Let $\{X_i \mid i = 1,2,\ldots,n\}$ be a finite collection of Z-algebras and $X = \prod_{i=1}^{n} X_i$. Let A_i, $i = 1,2,\ldots,n$ be fuzzy Z-Subalgebras of X_i respectively. Then $A = \prod_{i=1}^{n} A_i$ is also a fuzzy Z-Subalgebra of X.

Theorem 2.1.27: If B is a fuzzy Z-subalgebra of a Z-algebra X then the strongest fuzzy relation A_B is a fuzzy Z-Subalgebra of $X \times X$.

Proof: For all $(x_1, y_1),(x_2, y_2) \in X \times X$,

Then
$$\begin{aligned}
\mu_{A_B}((x_1, y_1)*(x_2, y_2)) &= \mu_{A_B}(x_1 * x_2, y_1 * y_2) \\
&= \min\{\mu_B(x_1 * x_2), \mu_B(y_1 * y_2)\} \\
&\geq \min\{\min\{\mu_B(x_1), \mu_B(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_B(x_1), \mu_B(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\} \\
&= \min\{\mu_{A_B}(x_1, y_1), \mu_{A_B}(x_2, y_2)\}
\end{aligned}$$

Therefore A_B is a fuzzy Z-subalgebra of $X \times X$.

2.2 Fuzzy Z-Ideals in Z-algebras

In this section, we introduce the notion of Fuzzy Z-ideals in Z-algebras and prove some simple but elegant results.

Definition 2.2.1: Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy Z-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y in X,

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$

Example 2.2.2: Let X= {0, 1, 2, 3} be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

Then $(X,*,0)$ is a Z-algebra.

Define fuzzy sets A_1 and A_2 in X with membership functions μ_{A_1} and μ_{A_2} by $\mu_{A_1}(x)=0.9$ for all $x = 0,1,2,3$ and $\mu_{A_2}(x) = \begin{cases} 0.8 & \text{if } x=0 \\ 0.6 & \text{if } x=1 \\ 0.5 & \text{if } x=2,3 \end{cases}$. Then, A_1 is a fuzzy Z-ideal of X, while A_2 is not. For, $\mu_{A_2}(2) = 0.5 \ngeq 0.8 = \min\{0.8, 0.8\} = \min\{\mu_{A_2}(0), \mu_{A_2}(0)\} = \min\{\mu_{A_2}(2*0), \mu_{A_2}(0)\}$

Theorem 2.2.3: Arbitrary intersection of fuzzy Z-ideals of Z-algebra X is also a fuzzy Z-ideal.

Proof: Let $\{A_i \mid i \in \Omega\}$ be a family of fuzzy Z-ideals of a Z-algebra X.

For any x, y $\in$ X,

$\mu_{\underset{i\in\Omega}{\cap} A_i}(0) = \inf_{i\in\Omega}(\mu_{A_i}(0)) \geq \inf_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\underset{i\in\Omega}{\cap} A_i}(x)$

$\mu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x)) \geq \inf_{i\in\Omega}(\min\{\mu_{A_i}(x*y), \mu_{A_i}(y)\})$

$$= \min\{ \inf_{i\in\Omega}(\mu_{A_i}(x*y)),\ \inf_{i\in\Omega}(\mu_{A_i}(y)) \}$$

$$= \min\{ \mu_{\underset{i\in\Omega}{\cap} A_i}(x*y),\ \mu_{\underset{i\in\Omega}{\cap} A_i}(y)\}$$

Hence $\underset{i\in\Omega}{\cap} A_i$ is a fuzzy Z-ideal of a Z-algebra X.

Proposition 2.2.4: Let $\{A_i \mid i\in\Omega\}$ be a chain of fuzzy Z-ideals of a Z-algebra X. Then $\bigcup_{i\in\Omega} A_i$ is a fuzzy Z-ideal of X.

Proof: Let $x\in X$. Then $\mu_{\underset{i\in\Omega}{\cup} A_i}(0) = \sup_{i\in\Omega}(\mu_{A_i}(0)) \geq \sup_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\underset{i\in\Omega}{\cup} A_i}(x)$

Let $x, y\in X$. Then, we have

$$\mu_{\underset{i\in\Omega}{\cup} A_i}(x) = \sup_{i\in\Omega}(\mu_{A_i}(x)) \geq \sup_{i\in\Omega}\{\min\{\mu_{A_i}(x*y), \mu_{A_i}(y)\}\}$$

$$= \min\{\sup_{i\in\Omega}\mu_{A_i}(x*y), \sup_{i\in\Omega}\mu_{A_i}(y)\}$$

$$= \min\{\mu_{\underset{i\in\Omega}{\cup} A_i}(x*y), \mu_{\underset{i\in\Omega}{\cup} A_i}(y)\}$$

Hence $\bigcup_{i\in\Omega} A_i$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 2.2.5: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy Z-ideal if and only if for any $t\in[0,1]$, $U(\mu_A;t) = \{x\in X \mid \mu_A(x)\geq t\}$ is an Z-ideal of X where $U(\mu_A;t)\neq\phi$.

Proof: Suppose A is a fuzzy Z-ideal of a Z-algebra X and $U(\mu_A;t)\neq\phi$ for any $t\in[0,1]$.

Let $x\in U(\mu_A;t)$, then $\mu_A(x)\geq t$.

By definition of fuzzy Z-ideal, we have $\mu_A(0)\geq\mu_A(x)\geq t$. Thus $0\in U(\mu_A;t)$.

If $x*y\in U(\mu_A;t)$ and $y\in U(\mu_A;t)$, then $\mu_A(x*y)\geq t$ and $\mu_A(y)\geq t$.

By definition, we have $\mu_A(x)\geq\min\{\mu_A(x*y),\mu_A(y)\}\geq\min\{t,t\} = t$

Therefore $x\in U(\mu_A;t)$. Hence $U(\mu_A;t)$ is an Z-ideal of X.

Conversely, suppose that for each $t\in[0,1]$, $U(\mu_A;t)$ is either empty or an Z-ideal of a Z-algebra X.

For any $x\in X$, let $\mu_A(x) = t$. Then $x\in U(\mu_A;t)$.

Since $U(\mu_A;t)\neq\phi$ is an Z-ideal of X, we have $0\in U(\mu_A;t)$ and hence $\mu_A(0)\geq t = \mu_A(x)$.

Thus $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.

Assume $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$ is not true.

Then there exists $x_0, y_0 \in X$ such that $\mu_A(x_0) < \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$

Let $t_0 = \frac{1}{2}[\mu_A(x_0) + \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}]$

Then $\mu_A(x_0) < t_0 < \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$

This implies $x_0 * y_0, y_0 \in U(\mu_A; t_0)$ and $x_0 \notin U(\mu_A; t_0)$

But $U(\mu_A; t_0)$ is an Z-ideal of X. So $x_0 \in U(\mu_A; t_0)$ by the definition of Z-ideal.

This implies $\mu_A(x_0) \geq t_0$.

This is a contradiction.

Therefore $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$, for all $x, y \in X$.

Hence A is a fuzzy Z-ideal of a Z-algebra X.

Theorem 2.2.6: A fuzzy set A of a Z-algebra $(X, *, 0)$ is a fuzzy Z-ideal if and only if every nonempty upper level subset $U(\mu_A; q)$ of A, $q \in Im(A)$ is an Z-ideal.

Proof : Let A be a fuzzy Z-ideal of a Z-algebra X.

Claim: $U(\mu_A; q)$, $q \in Im(A)$ is an Z-ideal.

Since $U(\mu_A; q) \neq \phi$ there exists $x \in U(\mu_A; q)$ such that $\mu_A(x) \geq q$.

Since A is a fuzzy Z-ideal of X, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Hence for this $x \in U(\mu_A; q)$, $\mu_A(0) \geq q$, which shows that $0 \in U(\mu_A; q)$.

Now, for any $x, y \in X$, assume that $x * y \in U(\mu_A; q)$ and $y \in U(\mu_A; q)$

Then $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$.

This shows that, $\min\{\mu_A(x * y), \mu_A(y)\} \geq q$.

Since A is a fuzzy Z-ideal of X, $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \geq q$,

Thus $x \in U(\mu_A; q)$, this proves that $U(\mu_A; q)$ is an Z-ideal of a Z-algebra X.

Conversely, let $U(\mu_A; q)$, $q \in Im(A)$ be an Z-ideal of a Z-algebra X.

Claim: A is a fuzzy Z-ideal of a Z-algebra X.

Let $x, y \in X$. For any $q \in Im(A)$, let $q = \min\{\mu_A(x * y), \mu_A(y)\}$.

Therefore, $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$.

This shows that $x * y, y \in U(\mu_A; q)$.

Since $U(\mu_A; q)$ is an Z-ideal, we have $x \in U(\mu_A; q)$.

This proves that $\mu_A(x) \geq q = \min\{\mu_A(x * y), \mu_A(y)\}$

This shows that A is a fuzzy Z-ideal of a Z-algebra X.

Theorem 2.2.7: Let A be a fuzzy Z-ideal of Z-algebra X and let $x \in X$. Then $\mu_A(x) = t$ if and only if $x \in U(\mu_A; t)$ but $x \notin U(\mu_A; q)$ for all q > t.

Proof: Assume $\mu_A(x) = t$, so that $x \in U(\mu_A; t)$.

If possible, let $x \in U(\mu_A; q)$ for q > t.

Then $\mu_A(x) \geq q > t$.

This contradicts the fact that $\mu_A(x) = t$. Hence $x \notin U(\mu_A; q)$ for all q > t.

Conversely, let $x \in U(\mu_A; t)$, but $x \notin U(\mu_A; q)$ for all q > t.

$x \in U(\mu_A; t) \Rightarrow \mu_A(x) \geq t$.

Since $x \notin U(\mu_A; q)$ for all q > t, $\mu_A(x) = t$.

Theorem 2.2.8: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be a fuzzy Z-ideal of X with the sup property. Then image of A denoted by h(A) is a fuzzy Z-ideal of Y.

Proof: Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that

$$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) \; ; \; \mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t).$$

$$\mu_{h(A)}(0') = \sup_{t \in h^{-1}(0')} \mu_A(t) \geq \mu_A(0) \geq \mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) = \mu_{h(A)}(a)$$

$$\begin{aligned} \min\{\mu_{h(A)}(a *' b), \mu_{h(A)}(b)\} &= \min\left\{ \sup_{t \in h^{-1}(a*'b)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t) \right\} \\ &\leq \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\} \\ &\leq \mu_A(x_0) \end{aligned}$$

$$= \sup_{t \in h^{-1}(a)} \mu_A(t)$$

$$= \mu_{h(A)}(a)$$

This implies, $\mu_{h(A)}(a) \geq \min\{\mu_{h(A)}(a *' b), \mu_{h(A)}(b)\}$

Hence h(A) is a fuzzy Z-ideal of a Z-algebra Y.

Example 2.2.9: Consider the Z-algebras $(X, *, 0)$ and $(Y, *', 0')$ with the following Cayley tables 1 and 2 respectively.

Table 1

$*$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Table 2

$*'$	$0'$	**1**	**2**	**3**
$0'$	$0'$	1	2	3
1	$0'$	1	1	3
2	$0'$	1	2	1
3	$0'$	3	1	3

Now, the function $h : (X, *, 0) \to (Y, *', 0')$ such that

$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 3 & \text{if } x = 1,2,3 \end{cases}$ is a Z-homomorphism.

and a fuzzy set A in X defined by $\mu_A(x) = 0.4$ for all $x \in X$ is a fuzzy Z-ideal of X.

Then, the Z-homomorphic image of A, h(A) is a fuzzy Z-ideal of Y.

Theorem 2.2.10: Let $h : (X, *, 0) \to (Y, *', 0')$ be a Z-homomorphism of Z-algebras. If B is a fuzzy Z-ideal of Y, then $h^{-1}(B)$ is a fuzzy Z-ideal of X.

Proof: For any x $\in$ X, we have

$$\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$$

Let x, y $\in$ X. Then

$$\min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\} = \min\{\mu_B(h(x * y)), \mu_B(h(y))\}$$

$$= \min\{\mu_B(h(x) *' h(y)), \mu_B(h(y))\}$$

$$\leq \mu_B(h(x))$$

$$= \mu_{h^{-1}(B)}(x)$$

$$\Rightarrow \quad \mu_{h^{-1}(B)}(x) \geq \min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\}$$

Hence $h^{-1}(B)$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 2.2.11: Let $h:(X,*,0) \to (Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be a fuzzy set of Y. If $h^{-1}(B)$ is a fuzzy Z-ideal of X then B is a fuzzy Z-ideal of Y.

Proof: Let y $\in$ Y, there exists x $\in$ X such that h (x) = y. Then

$$\mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0')$$

This implies, $\quad \mu_B(0') \geq \mu_B(y)$

Let x, y $\in$ Y. Then there exists a, b $\in$ X such that h(a) = x and h (b) = y. It follows that

$$\mu_B(x) = \mu_B(h(a)) = \mu_{h^{-1}(B)}(a) \geq \min\{\mu_{h^{-1}(B)}(a * b), \mu_{h^{-1}(B)}(b)\} = \min\{\mu_B(h(a * b)), \mu_B(h(b))\}$$

$$= \min\{\mu_B(h(a) *' h(b)), \mu_B(h(b))\}$$

$$= \min\{\mu_B(x *' y), \mu_B(y)\}$$

Hence B is a fuzzy Z-ideal of a Z-algebra Y.

Theorem 2.2.12: Let h be an Z-endomorphism of a Z-algebra X and A be a fuzzy set in X. Then $A^h : X \to [0,1]$ defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$, is a fuzzy Z-ideal of X if A is a fuzzy Z-ideal .

Proof : Obvious.

Theorem 2.2.13: If A and B be fuzzy Z-ideals in a Z-algebra X then $A \times B$ is a fuzzy Z-ideal in $X \times X$.

Proof: Let $(x_1, x_2) \in X \times X$,

$$\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A\times B}(x_1, x_2)$$

Hence $\mu_{A\times B}(0,0) \geq \mu_{A\times B}(x_1, x_2)$ (1)

Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then,

$$\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\}$$

$$\geq \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\}$$

$$= \min\{\min\{\mu_A(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\}$$

$$= \min\{\mu_{A\times B}((x_1 * y_1), (x_2 * y_2)), \mu_{A\times B}(y_1, y_2)\}$$

$$= \min\{\mu_{A\times B}((x_1, x_2) * (y_1, y_2)), \mu_{A\times B}(y_1, y_2)\} \qquad (2)$$

Hence $\mu_{A\times B}(x_1, x_2) \geq \min\{\mu_{A\times B}((x_1, x_2) * (y_1, y_2)), \mu_{A\times B}(y_1, y_2)\}$

By (1) and (2) we get, $A \times B$ is a fuzzy Z-ideal in $X \times X$.

Theorem 2.2.14: Let A and B be fuzzy sets in a Z-algebra X such that $A \times B$ is a fuzzy Z-ideal of $X \times X$. Then,

(i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all x $\in$ X.

(ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$

(iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$ then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$

Proof: (i) If $\mu_A(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$ for some $x_1, x_2 \in X$.

Then, $\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \min\{\mu_A(0), \mu_B(0)\}$

$= \mu_{A\times B}(0,0)$, which is a contradiction.

Hence, either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all x $\in$ X.

(ii) Let $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Assume that there exists $x_1, x_2 \in$ X such that $\mu_B(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$.

Then, $\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} = \mu_B(0)$

$\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \mu_B(0) = \mu_{A\times B}(0,0)$

$\Rightarrow \mu_{A\times B}(x_1, x_2) > \mu_{A\times B}(0,0)$, which is a contradiction.

Hence either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$

(iii) will obtain by interchanging the roles of A and B in part (ii).

Theorem 2.2.15: Let A and B be fuzzy sets in a Z-algebra X and $A \times B$ is a fuzzy Z-ideal of $X \times X$ then either A or B is a fuzzy Z-ideal of X.

Proof : By Theorem 2.2.14 (i), we can assume that $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$. Then, by Theorem 2.2.14 (iii), either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.

Let $\mu_A(0) \geq \mu_B(x)$ for any $x \in X$, then

$$\mu_B(x) = \min\{\mu_A(0), \mu_B(x)\} = \mu_{A\times B}(0, x) \geq \min\{\mu_{A\times B}((0, x) * (0, y)), \mu_{A\times B}(0, y)\}$$

$$= \min\{\mu_{A\times B}((0 * 0), (x * y)), \mu_{A\times B}(0, y)\}$$

$= \min\{\mu_{A\times B}(0, x * y), \mu_{A\times B}(0, y)\}$

$= \min\{\min\{\mu_A(0), \mu_B(x * y)\}, \min\{\mu_A(0), \mu_B(y)\}\}$

$= \min\{\mu_B(x * y), \mu_B(y)\}$

Therefore, $\mu_B(x) \geq \min\{\mu_B(x * y), \mu_B(y)\}$, for all $x, y \in X$.

Hence B is a fuzzy Z-ideal of a Z-algebra X.

By Theorem 2.2.14 (i) and (ii) , assume that $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ and $\mu_B(0) \geq \mu_A(x)$ for any $x \in X$.

Then A is a fuzzy Z-ideal of a Z-algebra X.

This completes the proof.

Theorem 2.2.16: Let A be a fuzzy relation on a Z-algebra X and A_B be the strongest fuzzy relation on X, where B is a fuzzy set of X. If B is a fuzzy Z-ideal of a Z-algebra X, then A_B is a fuzzy Z-ideal of X x X.

Proof: Let $(x_1, x_2), (y_1, y_2) \in X \text{ x } X$. Then $\mu_{A_B}(0,0) = \min\{\mu_B(0), \mu_B(0)\} \geq \min\{\mu_B(x_1), \mu_B(x_2)\}$

$= \mu_{A_B}(x_1, x_2)$

and also $\mu_{A_B}(x_1, x_2) = \min\{\mu_B(x_1), \mu_B(x_2)\}$

$\geq \min\{\min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\}$

$= \min\{\min\{\mu_B(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\}$

$= \min\{\mu_{A_B}((x_1 * y_1), (x_2 * y_2)), \mu_{A_B}(y_1, y_2)\}$

$= \min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\}$

Therefore A_B is a fuzzy Z-ideal of X x X.

Theorem 2.2.17: Let A be a fuzzy relation on a Z-algebra X and B be a fuzzy set of X. If the strongest fuzzy relation A_B is a fuzzy Z-ideal of X x X, then B is a fuzzy Z-ideal of a Z-algebra X.

Proof : For all $(x_1, x_2), (y_1, y_2) \in X \text{ x } X$,

$\min\{\mu_B(0), \mu_B(0)\} = \mu_{A_B}(0,0) \geq \mu_{A_B}(x_1, x_2) = \min\{\mu_B(x_1), \mu_B(x_2)\}$

Then, $\mu_B(0) \geq \min\{\mu_B(x_1), \mu_B(x_2)\}$

$\Rightarrow$ $\mu_B(0) \geq \mu_B(x_1)$ or $\mu_B(0) \geq \mu_B(x_2)$ for all $x_1, x_2 \in X$.

Also, $\min\{\mu_B(x_1), \mu_B(x_2)\} = \mu_{A_B}(x_1, x_2)$

$\geq \min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\}$

$$= \min\{\mu_{A_B}((x_1 * y_1),(x_2 * y_2)), \mu_{A_B}(y_1, y_2)\}$$
$$= \min\{\min\{\mu_B(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\}$$
$$= \min\{\min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\}$$

Put $x_2 = y_2 = 0$, we get $\mu_B(x_1) \geq \min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}$

Hence B is a fuzzy Z-ideal of a Z-algebra X.