6 (i,j)-$\psi^{*}\alpha$-Closed Sets in Bitopological Spaces

6.1 Introduction

Kelly (1963) initiated the study of bitopological spaces and thereafter various topological concepts have been generalized to bitopological spaces. Fukutake (1985) introduced g-closed sets in bitopological spaces. Jelic (1990) introduced α-closed sets in bitopological spaces. Ramya and Parvathi (2013) introduced ψg-closed sets in bitopological spaces

In this chapter, a new class of sets called (i, j)-$\psi^{*}\alpha$-closed sets is introduced and their properties and characterizations are analyzed in bitopological spaces. A comparative study is carried out with already existing generalized notions of (i, j)-closed sets and (i, j)-$\psi^{*}\alpha$-closed sets. As an application of (i, j)-$\psi^{*}\alpha$-closed sets, four new spaces namely, (i, j)-${}_{\psi^*\alpha}T_c$-space, (i, j)-${}_{\psi^*\alpha}T_\alpha$-space, (i, j)-${}_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-${}_{\alpha g}T_{\psi^*\alpha}$-space are introduced and their interrelations are discussed.

6.2 (i, j)-$\psi^{*}\alpha$-closed sets

In this section, the concept of (i, j)-$\psi^{*}\alpha$-closed sets is defined and some of their characterizations and properties are studied.

Definition 6.2.1 A subset A of a bitopological space (X, τ_1, τ_2) is called **(i, j)-$\psi^{*}\alpha$-closed** if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-ψg-open in (X, τ_i), where i, j = 1, 2 and $i \neq j$.

The family of all (i, j)-$\psi^{*}\alpha$-closed sets in (X, τ_1, τ_2) is denoted by **$\psi^{*}\alpha C(i, j)$.**

Remark 6.2.2 By setting $\tau_1 = \tau_2$ in definition 6.2.1, an (i, j)-$\psi^{*}\alpha$-closed set is a $\psi^{*}\alpha$-closed set.

Example 6.2.3 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b},{a, b}, X}.Then ϕ, {c},{a, c}, {b, c}, X are (1, 2)-$\psi^{*}\alpha$-closed.

Proposition 6.2.4 Every τ_j-closed (resp. τ_j-α-closed, τ_j-regular closed) set is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2)

Proof: Let A be a τ_j-closed set (resp. τ_j-α-closed, τ_j-regular closed) and U be any ψg-open set in X containing A. Since A is τ_j-closed (resp. τ_j-α-closed, τ_j-regular closed),τ_j -cl(A) (resp. τ_j-αcl(A), τ_j-rcl(A)) = A $\subseteq$ U. But τ_j-αcl(A) $\subseteq$ τ_j-cl(A) (resp. τ_j-αcl(A), τ_j-rcl(A)). Therefore τ_j-αcl(A) $\subseteq$ U. Hence A is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2).

The converse of the above proposition is not true in general as seen from the following examples.

Example 6.2.5 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a},{a, b}, X}. Then the subset {b} is (1, 2)-$\psi^*\alpha$-closed but not τ_2-closed.

Example 6.2.6 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a, b}, X}. Then the subset {b, c} is (1, 2)-$\psi^*\alpha$-closed but not τ_2-α-closed.

Example 6.2.7 Let X = {a, b, c}, τ_1 = {ϕ, {a},{a, b}, X} and τ_2 = {ϕ, {a}, {a, b},{a, c}, X}. Then the subset {c} is (1, 2)-$\psi^*\alpha$-closed but not τ_2-regular closed

Proposition 6.2.8 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)-gp-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-open set in X containing A. Since every τ_i-open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. We know that τ_j-pcl(A) $\subseteq$ τ_j-αcl(A) $\subseteq$ U. Therefore A is (i, j)-gp-closed.

Example 6.2.9 Let X = {a, b, c, d}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {d}, {a, b},{a, b, d}, X}. Then the subset {a, c, d} is (1, 2)-gp-closed but not (1, 2)-$\psi^*\alpha$-closed.

Proposition 6.2.10 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)-gpr-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-regular open set in X containing A. Since every τ_i-regular open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. We know that τ_j-pcl(A) $\subseteq$ τ_j-αcl(A) $\subseteq$ U. Therefore A is (i, j)-gpr-closed.

Example 6.2.11 Let X = {a, b, c, d}, τ_1 = {ϕ, {a, b}, X} and τ_2 = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the subset {a, d} is (1, 2)-gpr-closed but not (1, 2)-$\psi^*\alpha$-closed.

Proposition 6.2.12 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)- $\tilde{g}_\alpha$-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-$^\#$gs-open set in X containing A. Since every τ_i-$^\#$gs-open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. Therefore A is (i, j)- $\tilde{g}_\alpha$-closed.

Example 6.2.13 Let X = {a, b, c, d}, τ_1 = {ϕ, {a}, {a, b}, X} and τ_2 = {ϕ, {d}, {a, b}, {a, b, d}, X}. Then the subset {b, c} is (1, 2)- $\tilde{g}_\alpha$-closed but not (1, 2)-$\psi^*\alpha$-closed.

Proposition 6.2.14 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)-gα-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-α-open set in X containing A. Since every τ_i-α-open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. Therefore A is (i, j)-gα-closed.

Example 6.2.15 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b, c}, X} and τ_2 = {ϕ, {a}, {a, b}, {a, c}, X}. Then the subsets {a, b} and {a, c} are (1, 2)-gα-closed but not (1, 2)-$\psi^*\alpha$-closed.

Proposition 6.2.16 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)-αg-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-open set in X containing A. Since every τ_i-open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. Therefore A is (i, j)-αg-closed.

Example 6.2.17 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b, c}, X} and τ_2 = {ϕ, {a}, X}. Then the subsets {a, b} and {a, c} are (1, 2)-αg-closed but not (1, 2)-$\psi^*\alpha$-closed.

Proposition 6.2.18 Every (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) is (i, j)-ψg-closed but not conversely.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and U be any τ_i-open set in X containing A. Since every τ_i-open set is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), τ_j-αcl(A) $\subseteq$ U. We know that τ_j -ψcl(A) $\subseteq$ τ_j-αcl(A) $\subseteq$ U and so τ_j -ψcl(A) $\subseteq$ U. Therefore A is (i, j)-ψg-closed.

Example 6.2.19 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a, b}, X}. Then the subsets {b}, {c}, {a, b} and {a, c} are (1, 2)-ψg-closed but not (1, 2)-$\psi^*\alpha$-closed.

Remark 6.2.20 The following examples show that (i, j)-α-closed set and (i, j)-$\psi^*\alpha$-closed set are independent.

Example 6.2.21 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, X}. Then the subset {a, c} is (1, 2)-α-closed but not (1, 2)-$\psi^*\alpha$-closed.

Example 6.2.22 Let X = {a, b, c}, τ_1 = {ϕ, {a, b}, X} and τ_2 = {ϕ, {a}, X}. Then the subsets {b}, {b, c} and {a, c} are (1, 2)-$\psi^*\alpha$-closed but not (1, 2)-α-closed.

Remark 6.2.23 The following examples show that (i, j)-regular closed set and (i, j)-$\psi^*\alpha$-closed set are independent.

Example 6.2.24 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, {a, b}, X}. Then the subset {a, c} is (1, 2)-regular closed but not (1, 2)-$\psi^*\alpha$-closed.

Example 6.2.25 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then the subset {b, c} is (1, 2)-$\psi^*\alpha$-closed but not (1, 2)-regular closed.

Remark 6.2.26 The following examples show that (i, j)-semi closed set and (i, j)-$\psi^*\alpha$-closed set are independent.

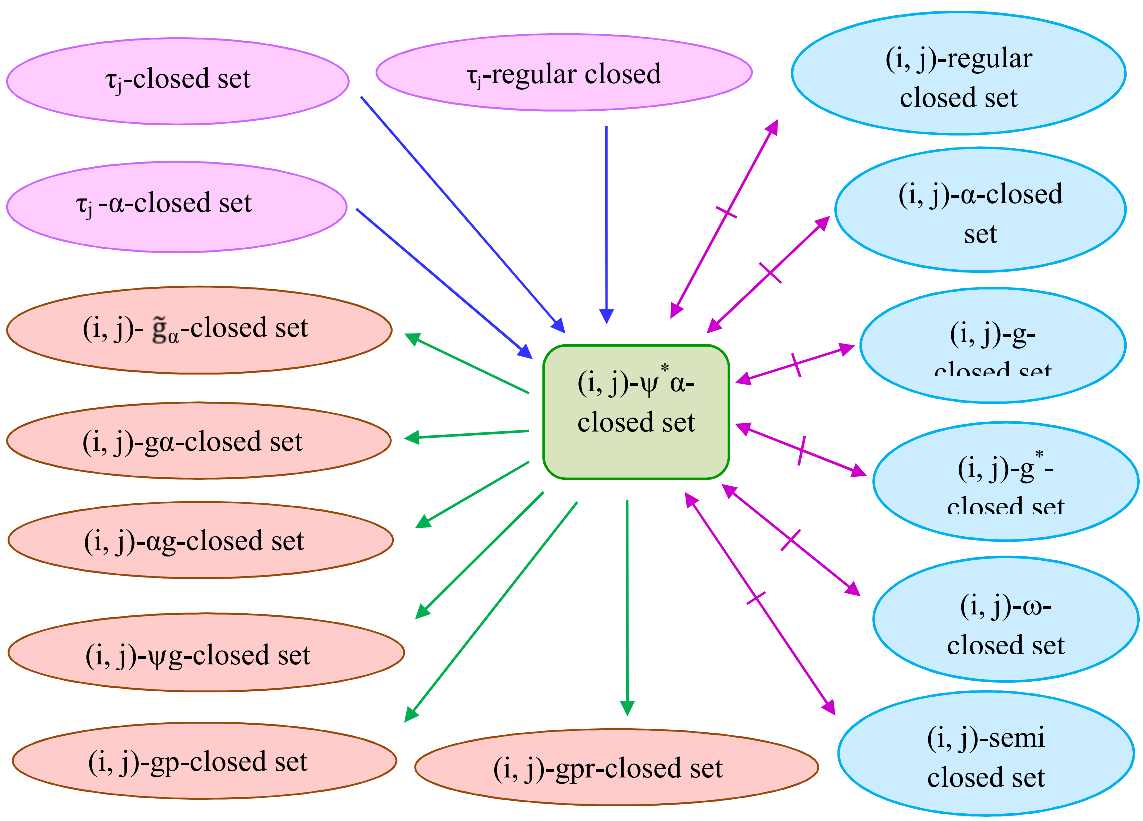
Example 6.2.27 Let X = {a, b, c}, τ_1= {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, X}. Then the subsets {a} and {a, c} are (1, 2)-semi closed but not (1, 2)-$\psi^*\alpha$-closed.

Example 6.2.28 Let X = {a, b, c}, τ_1 = {ϕ, {a, b}, X} and τ_2 = {ϕ, {a}, X}. Then the subsets {b}, {b, c} and {a, c} are (1, 2)-$\psi^*\alpha$ -closed but not (1, 2)-semi closed.

Remark 6.2.29 The following example show that(i, j)-$\psi^*\alpha$-closed set is independent of (i, j)- g-closed set,(i, j)-g^*-closed set and (i, j)- ω -closed set.

Example 6.2.30 Let X = {a, b, c, d}, τ_1 = {ϕ, {d}, {a, b}, {a, b, d}, X} and τ_2 = {ϕ, {a}, X}. Then the subset {a, c, d} is (1, 2)-g-closed, (1, 2)-g^*-closed and (1, 2)-ω-closed but not (1, 2)-$\psi^*\alpha$-closed. The subset {b} is (1, 2)-$\psi^*\alpha$-closed but not (1, 2)-g-closed, not (1, 2)-g^*-closed and not (1, 2)-ω-closed.

Remark 6.2.31 The above observations are depicted in the following diagram.



Theorem 6.2.32 If A is both τ_i-ψg-open and (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) then A is τ_j-α-closed.

Proof: Let A be both τ_j-ψg-open and (i, j)-$\psi^*\alpha$-closed. Since $A \subseteq A$, then τ_j-αcl(A) $\subseteq$ A. Therefore τ_j-αcl(A) = A. Consequently A is τ_j-α-closed.

Theorem 6.2.33 If A is (i, j)-$\psi^*\alpha$-closed and τ_i -ψg-open and F is τ_j-α-closed in (X, τ_1, τ_2) then $A \cap F$ is τ_j-α-closed.

Proof: Since A is (i, j)-$\psi^*\alpha$-closed and τ_i-ψg-open in (X, τ_1, τ_2), A is τ_j-α-closed (by **Theorem 6.2.32**). Since F is τ_j-α-closed, $A \cap F$ is τ_j-α-closed in (X, τ_1, τ_2).

Theorem 6.2.34 If A and B are (i, j)-$\psi^*\alpha$-closed sets, then $A \cup B$ is also an (i, j)-$\psi^*\alpha$-closed set.

Proof: Let A and B be (i, j)-$\psi^*\alpha$-closed sets and U be any τ_i -ψg-open set containing $A \cup B$. Then τ_j-αcl(A) $\subseteq$ U, τ_j-αcl(B) $\subseteq$ U, τ_j-αcl($A \cup B$) = τ_j-αcl(A) $\cup$ τ_j-αcl(B) $\subseteq$ U. Hence $A \cup B$ is (i, j)- $\psi^*\alpha$-closed.

Remark 6.2.35 The intersection of two (i, j)-$\psi^*\alpha$-closed sets need not be (i, j)-$\psi^*\alpha$-closed as seen from the following example.

Example 6.2.36 Let X = {a, b, c, d}, τ_1 = {ϕ, {a}, {a, b}, X} and τ_2 = {ϕ, {d}, {a, b}, {a, b, d}, X}. Then the subsets A= {a, b, c} and B = {b, c, d} are (1, 2)-$\psi^*\alpha$-closed but their intersection $A \cap B$ = {b, c} is not (1, 2)-$\psi^*\alpha$-closed.

Theorem 6.2.37 If a subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-closed then τ_j-αcl(A) – A contains no nonempty τ_i-ψg-closed set.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set and F be a τ_i-ψg-closed set contained in τ_j-αcl(A)–A. Since F^c is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed, τ_j-αcl(A) $\subseteq F^c$. Thus $F \subseteq [\tau_j\text{-}\alpha cl(A)]^c$. Hence $F \subseteq [\tau_j\text{-}\alpha cl(A)] \cap [\tau_j\text{-}\alpha cl(A)]^c = \phi$. Therefore F = ϕ. Hence τ_j-αcl(A) – A contains no nonempty τ_i-ψg-closed set.

Remark 6.2.38 The converse of the above theorem is not true as seen from the following example.

Example 6.2.39 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a, b, c\}, X\}$ and $\tau_2 = \{\phi, \{d\}, \{a, b\}, \{a, b, d\}, X\}$. Then $\psi GO(X, \tau_1) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$, $\psi^*\alpha C(1, 2) = \{\phi, \{c\}, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. If $A = \{a, b\}$, τ_j-$\alpha cl(A) - A = \{a, b, c\} - \{a, b\} = \{c\}$. But $\{a, b\}$ is not $(1, 2)$-$\psi^*\alpha$-closed.

Theorem 6.2.40 Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). Then A is τ_j-α-closed if and only if τ_j-$\alpha cl(A) - A$ is τ_i-ψg-closed.

Proof: Suppose that A is (i, j)-$\psi^*\alpha$-closed. Let A be τ_j-α-closed. Then τ_j-$\alpha cl(A) = A$. Therefore τ_j-$\alpha cl(A) - A = \phi$ is τ_i-ψg-closed.

Conversely, suppose that A is (i, j)-$\psi^*\alpha$-closed and τ_j-$\alpha cl(A) - A$ is τ_i-ψg-closed. Since A is (i, j)-$\psi^*\alpha$-closed τ_j-$\alpha cl(A) - A$ contains no nonempty τ_i-ψg-closed set (by **Theorem 6.2.37**). Since τ_j-$\alpha cl(A) - A$ is τ_i-ψg-closed, τ_j-$\alpha cl(A) - A = \phi$. Then τ_j-$\alpha cl(A) = A$. Hence A is τ_j-α-closed.

Theorem 6.2.41 Let A and B be subsets of (X, τ_1, τ_2) such that $A \subseteq B \subseteq \tau_j$-$\alpha cl(A)$. If A is (i, j)-$\psi^*\alpha$-closed, then B is (i, j)-$\psi^*\alpha$-closed.

Proof: Let A and B be subsets such that $A \subseteq B \subseteq \tau_j$-$\alpha cl(A)$. Suppose that A is (i, j)-$\psi^*\alpha$-closed. Let $B \subseteq U$ and U be τ_i-ψg-open in (X, τ_i). Then $A \subseteq U$. Since A is (i, j)-$\psi^*\alpha$-closed, τ_j-$\alpha cl(A) \subseteq U$. Since $B \subseteq \tau_j$-$\alpha cl(A)$, τ_j-$\alpha cl(B) \subseteq \tau_j$-$\alpha cl[\tau_j$-$\alpha cl(A)] = \tau_j$-$\alpha cl(A) \subseteq U$. Therefore B is (i, j)-$\psi^*\alpha$-closed.

Proposition 6.2.42 If A is an (i, j)-$\psi^*\alpha$-closed set, then τ_i-$\alpha cl(x) \cap A \neq \phi$ holds for each $x \in \tau_j$-$\alpha cl(A)$.

Proof: Let A be an (i, j)-$\psi^*\alpha$-closed set. If τ_i-$\alpha cl(x) \cap A = \phi$ for some $x \in \tau_j$-$\alpha cl(A)$. Then $A \subseteq [\tau_i$-$\alpha cl(x)]^c$. Since A is (i, j)-$\psi^*\alpha$-closed, τ_j-$\alpha cl(A) \subseteq [\tau_i$-$\alpha cl(x)]^c$. Also since $[\tau_i$-$\alpha cl(x)]^c$ is τ_i-α-open and hence $[\tau_i$-$\alpha cl(x)]^c$ is τ_i-ψg-open. This gives $x \notin \tau_j$-$\alpha cl(A)$, which is a contradiction.

The converse of the above proposition need not be true as seen from the following example.

Example 6.2.43 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then the subset A = {b} in (X, τ_1, τ_2) is not (1, 2)-$\psi^*\alpha$-closed. However τ_1-αcl(x) ∩ A ≠ ϕ holds for each x ∈ τ_2-αcl(A).

Remark 6.2.44 In general $\psi^*\alpha C(1, 2) \neq \psi^*\alpha C(2, 1)$ which can be seen from the following example.

Example 6.2.45 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b}, {a, b}, X}. Then $\psi^*\alpha C(1, 2)$ = {ϕ, {c}, {b, c}, {a, c}, X} and $\psi^*\alpha C(2, 1)$ = {ϕ, {b}, {c}, {b, c}, X}.This shows that $\psi^*\alpha C(1, 2) \neq \psi^*\alpha C(2, 1)$

Theorem 6.2.46 If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2) then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$.

Proof: Let A be a (2, 1)-$\psi^*\alpha$-closed set and U be any τ_1-ψg-open set containing A. Since $\tau_1 \subseteq \tau_2$, U is τ_2-ψg-open, τ_2-αcl(A) ⊆ τ_1-αcl(A). Thus τ_2-αcl(A) ⊆ U. Hence A is (1, 2)-$\psi^*\alpha$-closed. That is A ∈ $\psi^*\alpha C(1, 2)$.

The converse of the above theorem need not be true as seen from the following example.

Example 6.2.47 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$ but $\tau_1 \not\subseteq \tau_2$.

6.3 (i, j)-$\psi^*\alpha$-closure operator

In this section, (i, j)-$\psi^*\alpha$-closure of a set in bitopological spaces is introduced and their basic properties are analyzed.

Definition 6.3.1 An **(i, j)-$\psi^*\alpha$-closure of a subset A** of a bitopological space (X, τ_1, τ_2) denoted by **(i, j)-$\psi^*\alpha$cl(A)** is defined as follows:

(i, j)-$\psi^*\alpha$cl(A) = ∩{F ⊆ X : A ⊆ F and F is (i, j)-$\psi^*\alpha$-closed in (X, τ_i, τ_j)},where i, j = 1, 2 and i ≠ j.

Proposition 6.3.2 Let E and F be any two subsets of (X, τ_1, τ_2). Then the following results hold.

(a) (i, j)-$\psi^*\alpha cl(\phi)=\phi$ and (i, j)-$\psi^*\alpha cl(X)=X$

(b) If $E \subseteq F$, then (i, j)-$\psi^*\alpha cl(E) \subseteq$ (i, j)-$\psi^*\alpha cl(F)$

(c) $E \subseteq$ (i, j)-$\psi^*\alpha cl(E) \subseteq \tau_j$-cl(E)

(d) If E is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) then (i, j)-$\psi^*\alpha cl(E) = E$

(e) (i, j)-$\psi^*\alpha cl(E \cap F) \subseteq$ (i, j)-$\psi^*\alpha cl(E) \cap$ (i, j)-$\psi^*\alpha cl(F)$

(f) (i, j)-$\psi^*\alpha cl(E \cup F) =$ (i, j)-$\psi^*\alpha cl(E) \cup$ (i, j)-$\psi^*\alpha cl(F)$

(g) (i, j)-$\psi^*\alpha cl[$(i, j)-$\psi^*\alpha cl(E)] =$ (i, j)-$\psi^*\alpha cl(E)$

Proof:

(a) Since ϕ and X are (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), the results follows.

(b) Let $E \subseteq F$. Then by the definition of (i, j)-$\psi^*\alpha$-closure, (i, j)-$\psi^*\alpha cl(E) \subseteq$ (i, j)-$\psi^*\alpha cl(F)$

(c) From the definition of (i, j)-$\psi^*\alpha$ -closure, it follows that $E \subseteq$ (i, j)-$\psi^*\alpha cl(E)$. By **Proposition 6.2.4,** every τ_j-closed set is (i, j)-$\psi^*\alpha$-closed.Therefore $E \subseteq$ (i, j)-$\psi^*\alpha cl(E) \subseteq \tau_j$-cl(E)

(d) Follows from **(c)** and by the definition of (i, j)-$\psi^*\alpha$-closure.

(e) Since $E \cap F \subseteq E$ and $E \cap F \subseteq F$, by **(b)** (i, j)-$\psi^*\alpha cl(E \cap F) \subseteq$ (i, j)-$\psi^*\alpha cl(E)$, (i, j)-$\psi^*\alpha cl(E \cap F) \subseteq$ (i, j)-$\psi^*\alpha cl(F)$. Hence (i, j)-$\psi^*\alpha cl(E \cap F) \subseteq$ (i, j)-$\psi^*\alpha cl(E) \cap$ (i, j)-$\psi^*\alpha cl(F)$.

(f) Since $E \subseteq E \cup F$ and $F \subseteq E \cup F$, by **(b)** (i, j)-$\psi^*\alpha cl(E) \subseteq$ (i, j)-$\psi^*\alpha cl(E \cup F)$ and (i, j)-$\psi^*\alpha cl(F) \subseteq$ (i, j)-$\psi^*\alpha cl(E \cup F)$.Therefore (i, j)-$\psi^*\alpha cl(E) \cup$ (i, j)-$\psi^*\alpha cl(F) \subseteq$ (i, j)-$\psi^*\alpha cl(E \cup F)$. To prove the reverse inclusion, let $x \in [$(i, j)-$\psi^*\alpha cl(E \cup F)]$ and suppose that $x \notin [$(i, j)-$\psi^*\alpha cl(E) \cup$ (i, j)-$\psi^*\alpha cl(F)]$. Then $x \notin [$(i, j)-$\psi^*\alpha cl(E)]$ and

$x \notin [(i, j)\text{-}\psi^*\alpha cl(F)]$. Therefore there exist (i, j)-$\psi^*\alpha$-closed sets U and V such that $E \subseteq U$, $F \subseteq V$, $x \notin U$ and $x \notin V$. Hence $E \cup F \subseteq U \cup V$ and $x \notin [U \cup V]$. By **Theorem 6.2.34,** $U \cup V$ is a (i, j)-$\psi^*\alpha$-closed set and hence $x \notin [(i, j)\text{-}\psi^*\alpha cl(E \cup F)]$, which is a contradiction. Hence $(i, j)\text{-}\psi^*\alpha cl(E \cup F) \subseteq (i, j)\text{-}\psi^*\alpha cl(E) \cup (i, j)\text{-}\psi^*\alpha cl(F)$. Therefore $(i, j)\text{-}\psi^*\alpha cl(E \cup F) = (i, j)\text{-}\psi^*\alpha cl(E) \cup (i, j)\text{-}\psi^*\alpha cl(F)$.

(g) Follows from the **Definition 6.3.1.**

Theorem 6.3.3 The closure operator (i, j)-$\psi^*\alpha$-closure is a Kuratowski closure operator on (X, τ_1, τ_2).

Proof: As $(i, j)\text{-}\psi^*\alpha cl(\phi) = \phi$, $E \subseteq (i, j)\text{-}\psi^*\alpha cl(E)$, $(i, j)\text{-}\psi^*\alpha cl(E \cup F) = (i, j)\text{-}\psi^*\alpha cl(E) \cup (i, j)\text{-}\psi^*\alpha cl(F)$ and $(i, j)\text{-}\psi^*\alpha cl[(i, j)\text{-}\psi^*\alpha cl(E)] = (i, j)\text{-}\psi^*\alpha cl(E)$, (i, j)-$\psi^*\alpha$-closure is a Kuratowski closure operator on (X, τ_1, τ_2).

6.4 (i, j)-$\psi^*\alpha$-open sets

In this section, (i, j)-$\psi^*\alpha$-open sets in bitopological spaces is introduced and their properties are studied.

Definition 6.4.1 A subset A of a bitopological space (X, τ_1, τ_2) is called **(i, j)-$\psi^*\alpha$-open** if its complement is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

The family of all (i, j)-$\psi^*\alpha$-open sets in (X, τ_1, τ_2) is denoted by **$\psi^*\alpha O(i, j)$.**

Example 6.4.2 Let X = {a, b, c}, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then ϕ, {a},{b}, {a, b} are (1, 2)-$\psi^*\alpha$-open.

Definition 6.4.3 An **(i, j)-$\psi^*\alpha$-interior of a subset A** of a bitopological space (X, τ_1, τ_2) denoted by **(i, j)-$\psi^*\alpha int(A)$** is defined as follows:

$(i, j)\text{-}\psi^*\alpha int(A) = \cup \{F \subseteq X : F \subseteq A$ and F is (i, j)-$\psi^*\alpha$-open in $(X, \tau_i, \tau_j)\}$, where i, j = 1, 2 and $i \neq j$.

Proposition 6.4.4 In a bitopological space (X, τ_1, τ_2) the following statements are true.

(i) Every τ_j-open set is (i, j)-$\psi^*\alpha$-open.

(ii) Every τ_j-α-open set is (i, j)-$\psi^*\alpha$-open.

(iii) Every τ_j-regular open set is (i, j)-$\psi^*\alpha$-open.

(iv) Every (i, j)-$\psi^*\alpha$-open set is (i, j)-gp-open.

(v) Every (i, j)-$\psi^*\alpha$-open set is (i, j)-gpr-open

(vi) Every (i, j)-$\psi^*\alpha$-open set is (i, j)- $\tilde{g}_\alpha$-open.

(vii) Every (i, j)-$\psi^*\alpha$-open set is (i, j)-gα-open.

(viii) Every (i, j)-$\psi^*\alpha$-open set is (i, j)-αg-open.

(ix) Every (i, j)-$\psi^*\alpha$-open set is (i, j)-ψg-open.

The converse of the statements in the above proposition are not true in general as seen from the examples **6.2.5, 6.2.6, 6.2.7, 6.2.9, 6.2.11,6.2.13,6.2.15,6.2.17 and 6.2.19** by considering their respective open sets**.**

Theorem 6.4.5 A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-open if and only if F $\subseteq$ τ_j-αint(A) whenever F $\subseteq$ A and F is τ_i -ψg-closed in (X, τ_1, τ_2)

Proof: Suppose that A is (i, j)-$\psi^*\alpha$-open. Let F $\subseteq$ A and F be τ_i-ψg-closed. Then $A^c \subseteq F^c$ and F^c is τ_i-ψg-open. Since A^c is (i, j)-$\psi^*\alpha$-closed, τ_j-αcl(A^c) $\subseteq$ F^c. Since τ_j-αcl(A^c) = τ_j-(αint(A))^c, τ_j-(αint(A))$^c \subseteq F^c$. Hence F $\subseteq$ τ_j-αint(A).

Conversely, suppose that F $\subseteq$ τ_j-αint(A) whenever F $\subseteq$ A and F is τ_i-ψg-closed in (X, τ_1, τ_2). Let U be τ_i-ψg-open and $A^c \subseteq U$. Then U^c is τ_i-ψg-closed and $U^c \subseteq A$. Hence by assumption $U^c \subseteq \tau_j$-αint(A). Therefore τ_j-(αint(A))$^c \subseteq U$. That is τ_j-αcl(A^c) $\subseteq$ U. Therefore A^c is (i, j)-$\psi^*\alpha$-closed. Hence A is (i, j)-$\psi^*\alpha$-open.

Theorem 6.4.6 If a subset A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), then τ_j-αcl(A) – A is (i, j)-$\psi^*\alpha$-open.

Proof: Suppose that A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Let F $\subseteq$ τ_j-αcl(A) – A and F be τ_i-ψg-closed. Since A is (i, j)-$\psi^*\alpha$-closed, τ_j-αcl(A) – A does not contain nonempty τ_i-ψg-closed set (by **Theorem 6.2.37).** Hence F = ϕ. Thus F $\subseteq$ τ_j-αint[τ_j-αcl(A) – A]. Hence τ_j -αcl(A) – A is (i, j)-$\psi^*\alpha$-open.

Theorem 6.4.7 If a set A is (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2) then G = X whenever G is τ_i-ψg-open and τ_j-αint(A) $\cup$ A^c $\subseteq$ G.

Proof: Suppose that A is (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2), G is τ_i-ψg-open and τ_j-αint(A) $\cup$ A^c $\subseteq$ G. Then G^c $\subseteq$ $\{\tau_j\text{-}\alpha int(A) \cup A^c\}^c$ = τ_j-αcl(A^c) – A^c. Since A^c is (i, j)-$\psi^*\alpha$-closed, τ_j-αcl(A^c) – A^c contains no nonempty τ_i-ψg-closed set in (X, τ_1, τ_2) (by **Theorem 6.2.37**). Therefore G^c = ϕ. Hence G = X.

Remark 6.4.8 The converse of the above theorem is not true in general as seen from the following example.

Example 6.4.9 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a, b}, X}. Let A = {c} and G = X. Then G is τ_1-ψg-open, τ_2-αint(A) $\cup$ A^c = ϕ $\cup$ {a, b} = {a, b} $\subseteq$ G, but A = {c} is not (1, 2)-$\psi^*\alpha$-open.

Theorem 6.4.10 Let (X, τ_1, τ_2) be a bitopological space. If x $\in$ X, then singleton {x} is either τ_i-ψg-closed or (i, j)-$\psi^*\alpha$-open.

Proof: Let x $\in$ X and suppose that {x} is not τ_i-ψg-closed. Then X – {x} is not τ_i-ψg-open. Consequently, X is the only τ_i-ψg-open set containing the set X – {x}. Therefore τ_j-αcl(X – {x}) $\subseteq$ X and so X – {x} is (i, j)-$\psi^*\alpha$ -closed. Hence {x} is (i, j)-$\psi^*\alpha$-open.

6.5 Applications of (i, j)-$\psi^*\alpha$-closed sets

As an application of (i, j)-$\psi^*\alpha$-closed set, four new spaces namely, (i, j)-${}_{\psi^*\alpha}T_c$-space, (i, j)-${}_{\psi^*\alpha}T_\alpha$-space, (i, j)-${}_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-${}_{\alpha g}T_{\psi^*\alpha}$-space are introduced and their interrelations are studied.

Definition 6.5.1 A bitopological space (X, τ_1, τ_2) is called an

(i) **(i, j)-${}_{\psi^*\alpha}T_c$-space** if every (i, j)-$\psi^*\alpha$-closed subset of (X, τ_1, τ_2) is τ_j-closed in X.

(ii) **(i, j)-${}_{\psi^*\alpha}T_\alpha$-space** if every (i, j)-$\psi^*\alpha$-closed subset of (X, τ_1, τ_2) is τ_j-α-closed in X.

(iii) **(i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space** if every (i, j)-gα-closed subset of (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2)

(iv) **(i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space** if every (i, j)-αg-closed subset of (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2)

Remark 6.5.2 Every (i, j)-$_{\psi^*\alpha}T_c$-space is an (i, j)-$_{\psi^*\alpha}T_\alpha$-space but not conversely.

Example 6.5.3 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b, c}, X} and τ_2= {ϕ, {a}, {a, b}, X}. Then (X, τ_1, τ_2) is an (i, j)-$_{\psi^*\alpha}T_\alpha$-space but not an (i, j)-$_{\psi^*\alpha}T_c$-space, since the subset {b} is (1, 2)-$\psi^*\alpha$-closed but not τ_2-closed.

Remark 6.5.4 Every (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space is an (i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space but not conversely.

Example 6.5.5 Let X = {a, b, c}, τ_1 = {ϕ, {a}, X} and τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}. Then (X, τ_1, τ_2) is an (i, j)-$_{g\alpha}T_{\psi^*\alpha}$ -space but not an (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space, since the subset {a, b} is (1, 2)-αg-closed but not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 6.5.6 The following examples show that (i, j)-$_{\psi^*\alpha}T_c$-space is independent of (i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space.

Example 6.5.7 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then (X, τ_1, τ_2) is an (i, j)-$_{\psi^*\alpha}T_c$-space but not an (i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space, since the subset {a, c} is (1, 2)-gα-closed and (1, 2)-αg-closed but not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

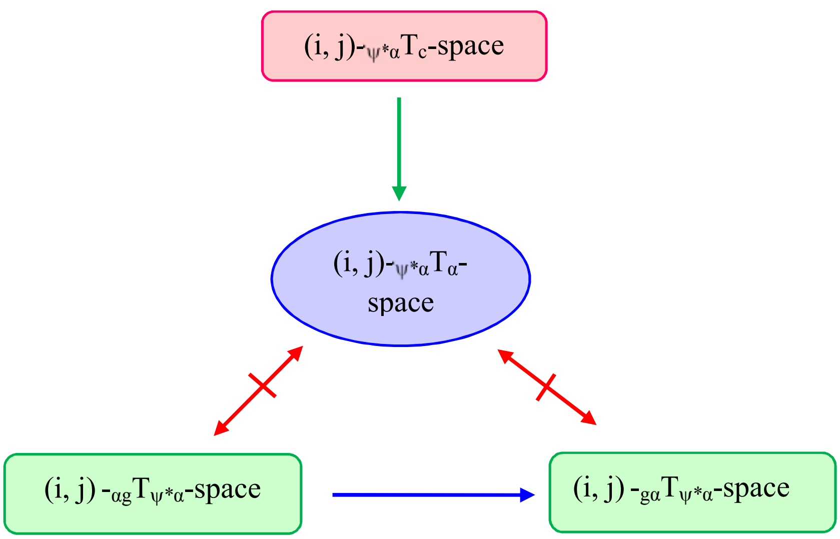
Example 6.5.8 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b},{a, b}, {a, c}, X} and τ_2 = {ϕ, {a}, X}. Then (X, τ_1, τ_2) is an (i, j) - $_{g\alpha}T_{\psi^*\alpha}$ - space and an (i, j) - $_{\alpha g}T_{\psi^*\alpha}$ - space but not an (i, j)-$_{\psi^*\alpha}T_c$-space, since the subsets {b}and {c} are (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) but not τ_2-closed.

Remark 6.5.9 The following examples show that (i, j)-$_{\psi^*\alpha}T_\alpha$-space is independent of (i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space.

Example 6.5.10 Let X = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, {b, c}, X}. Then (X, τ_1, τ_2) is an (i, j)-$_{\psi^*\alpha}T_\alpha$-space but not an (i, j)-$_{g\alpha}T_{\psi^*\alpha}$-space and (i, j)-$_{\alpha g}T_{\psi^*\alpha}$-space, since the subset {a, c} is (1, 2)-gα-closed and (1, 2)-αg-closed but not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Example 6.5.11 Let X = {a, b, c}, τ_1 = {ϕ, {a, b}, X} and τ_2 = {ϕ, {a}, {a b}, X}. Then (X, τ_1, τ_2) is an (i, j)-$_{g\alpha}T_{\psi^*\alpha}$ -space and (i, j)- $_{\alpha g}T_{\psi^*\alpha}$ -space but not an (i, j)- $_{\psi^*\alpha}T_\alpha$-space, since the subsets {b}and {a, c} are (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) but not τ_2 -α -closed.

Remark 6.5.12 The above results are given in the following diagram.



Theorem 6.5.13 For a space (X, τ_1, τ_2) the following statements are equivalent

(i) (X, τ_1, τ_2) is an (i, j)-$_{\psi^*\alpha}T_\alpha$-space

(ii) For each x ∈ X, {x} is either τ_i-ψg-closed or τ_j-α-open, for i ≠ j.

Proof: (i) $\Rightarrow$ (ii) Suppose that {x} is not a τ_i-ψg-closed subset for some x $\in$ X. Then X – {x} is not τ_i -ψg-open and hence X is the only τ_i-ψg-open set containing X – {x}. This implies that X – {x} is (i, j)-$\psi^*\alpha$-closed. Since (X, τ_1, τ_2) is an (i, j)-${}_{\psi^*\alpha}T_\alpha$-space, X – {x} is τ_j-α-closed or equivalently {x} is τ_j-α-open.

(ii) $\Rightarrow$ (i) Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and x $\in$ τ_j-αcl(A). We show that x $\in$ A. By (ii), {x} is either τ_i -ψg-closed or τ_j-α-open.

Case 1: Assume that {x} is τ_j-α-open. Then X – {x} is τ_j-α-closed. If x $\notin$ A, then A $\subseteq$ X – {x}. Since x $\in$ τ_j-αcl(A) and x $\in$ X – {x},it is a contradiction. Hence x $\in$ A.

Case 2: Assume that {x} is τ_i-ψg-closed and x $\notin$ A. Then τ_j-αcl(A) – A contains a τ_i -ψg-closed set {x}. This contradicts **Theorem 6.2.37.** Therefore x $\in$ A.