Chapter 5

Continuity Concepts using λ_g^δ-closed sets

5.1 Introduction

Topology is concerned with questions either directly or indirectly concerned with continuity. General Topologists have introduced and investigated many different generalizations of continuous functions. In this chapter, various types of continuities using λ_g^δ-closed sets namely λ_g^δ-continuity, quasi λ_g^δ-continuity, perfectly λ_g^δ-continuity, totally λ_g^δ-continuity, strongly λ_g^δ-continuity and contra λ_g^δ-continuity are defined with their properties being discussed.

5.2 λ_g^δ-continuity

In this section λ_g^δ-continuous functions in topological spaces are introduced and some interesting properties are obtained.

Definition 5.2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ***λ_g^δ-continuous*** if the inverse image of every open set in (Y, σ) is λ_g^δ-open in (X, τ).

Theorem 5.2.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be λ_g^δ-continuous iff inverse image of a closed set in (Y, σ) is λ_g^δ-closed in (X, τ).*

Theorem 5.2.3. *If for any function $f : X \to Y$ and $A \subseteq X$ such that $f(\lambda_g^\delta cl(A)) \subseteq \lambda_g^\delta cl(f(A))$ then the inverse image of every δ-closed set in Y is λ_g^δ-closed in X.*

Proof. Let B be a δ-closed set in Y and let $A = f^{-1}(B)$. This implies, $f(A) = f(f^{-1}(B)) \subseteq B$. Let $x \in \lambda_g^\delta cl(A)$ then $f(x) \subseteq f(\lambda_g^\delta cl(A)) \subseteq \lambda_g^\delta cl(f(A)) \subseteq \lambda_g^\delta cl(B) \subseteq cl_\delta(B) = B$. This implies $x \in f^{-1}(B) = A$. Thus $\lambda_g^\delta cl(A) \subseteq A$ and hence A is λ_g^δ-closed in X. □

Theorem 5.2.4. *If a function $f : X \to Y$ is λ_g^δ-continuous then for each $x \in X$ and each open set V of $f(x)$, there exists a λ_g^δ-open set U of x such that $f(U) \subseteq V$.*

Proof. Let $x \in X$ and V be a open set of $f(x)$. Then $U = f^{-1}(V)$ is a λ_g^δ-open set of x such that $f(U) \subseteq V$. □

Proposition 5.2.5. If $f : (X, \tau) \to (Y, \sigma)$ is λ_g^δ-continuous then for each $x \in X$ and each open set V containing $f(x)$, there is a λ_g^δ-open(resp. $g\delta$-open, $g\delta s$-open and δgs-open) set U containing x such that $f(U) \subseteq V$.

Proof. Let $x \in X$ and let V be an open set containing $f(x)$ then by hypothesis $f^{-1}(V)$ is a λ_g^δ-open set containing x. Let $U = f^{-1}(V)$ then $f(U) = f(f^{-1}(V)) \subseteq V$. □

Theorem 5.2.6. *For a function $f : X \to Y$, the following conditions are true.*

(i) f *is* λ_g^δ-*continuous* $\Rightarrow$

(ii) $\lambda_g^\delta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, *for each* $B \subseteq Y \Rightarrow$

(iii) $f(\lambda_g^\delta cl(A)) \subseteq cl(f(A))$, *for each* $A \subseteq X \Rightarrow$

(iv) $f^{-1}(int(B)) \subseteq \lambda_g^\delta int(f^{-1}(B))$, *for each* $B \subseteq Y$.

Proof. (i) $\Rightarrow$ (ii) Let $B \subseteq Y$. $cl(B)$ is closed in $Y \Rightarrow f^{-1}(cl(B))$ is λ_g^δ-closed in X. Therefore $f^{-1}(cl(B)) = \lambda_g^\delta cl(f^{-1}(cl(B))) \supseteq \lambda_g^\delta cl(f^{-1}(B))$.
(ii) $\Rightarrow$ (iii) Let $A \subseteq X$ then $f(A) \subseteq Y$. By (ii), $f^{-1}(cl(f(A))) \supseteq \lambda_g^\delta cl(f^{-1}(f(A))) \supseteq \lambda_g^\delta cl(A)$ and hence $f(\lambda_g^\delta cl(A)) \subseteq cl(f(A))$.
(iii) $\Rightarrow$ (iv) Let $B \subseteq Y$. By (iii), $f(\lambda_g^\delta cl(X \setminus f^{-1}(B))) \subseteq cl(f(X \setminus f^{-1}(B))) \Rightarrow f(X \setminus$

$\lambda_g^\delta int(f^{-1}(B))) \subseteq cl(Y \setminus B) = Y \setminus int(B) \Rightarrow X \setminus \lambda_g^\delta int(f^{-1}(B)) \subseteq f^{-1}(Y \setminus int(B)) \Rightarrow f^{-1}(int(B)) \subseteq \lambda_g^\delta int(f^{-1}(B))$.

□

Remark 5.2.7. In the above theorem, (iv) ⇒ (i) if $f^{-1}(B) = \lambda_g^\delta int f^{-1}(B) \Rightarrow f^{-1}(B)$ is λ_g^δ-open.

Proposition 5.2.8. Every super continuous function is a λ_g^δ-continuous function but not conversely.

Proof. Let $f : X \to Y$ be a super continuous function. Let V be any open set in Y. Since f is super continuous, $f^{-1}(V)$ is δ-open in X. Since every δ-open set is λ_g^δ-open, we have $f^{-1}(V)$ is λ_g^δ-open in X. Hence f is λ_g^δ-continuous. □

Example 5.2.9. Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{X, \phi, \{a\}, \{a, b\}\}$. Let a function $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Then f is λ_g^δ-continuous but not super continuous, since the inverse image of the open sets $\{a, b\}$, $\{a\}$ of (Y, σ) are $\{b, c\}$ and $\{c\}$ respectively which are both λ_g^δ-open but not δ-open in (X, τ).

Proposition 5.2.10. Every λ_g^δ-continuous function is $g\delta$-continuous but not conversely.

Proof. Let $f : X \to Y$ be a λ_g^δ-continuous function. Let V be any open set in Y. Since f is λ_g^δ-continuous, $f^{-1}(V)$ is λ_g^δ-open in X. Since every λ_g^δ-open set is $g\delta$-open, $f^{-1}(V)$ is $g\delta$-open in X. Therefore f is $g\delta$-continuous. □

Example 5.2.11. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is $g\delta$-continuous, since the inverse image of the all open sets are $g\delta$-open in (X, τ) but not λ_g^δ-continuous since the inverse image of the open set $\{a, c\}$, $f^{-1}\{a, c\} = \{a, c\}$ is not λ_g^δ-open in (X, τ).

Proposition 5.2.12. Every λ_g^δ-continuous function is δgs-continuous but not conversely

Proof. Let $f : X \to Y$ be a λ_g^δ-continuous function. Let V be any open set in Y. Since f is λ_g^δ-continuous, $f^{-1}(V)$ is λ_g^δ-open in X. Since every λ_g^δ-open set is δgs-open and thus $f^{-1}(V)$ is δgs-open in X. Hence f is δgs-continuous. □

Example 5.2.13. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is δgs-continuous, since the inverse image of the all open sets are δgs-open in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.

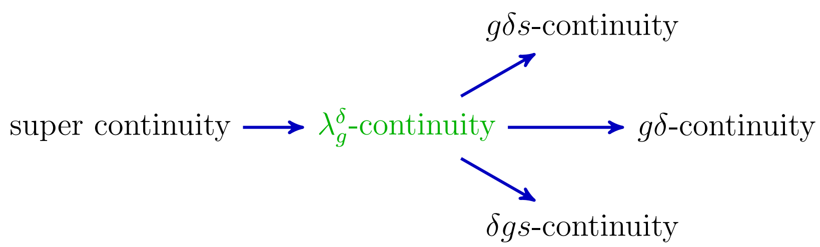
Proposition 5.2.14. Every λ_g^δ-continuous function is $g\delta s$-continuous but not conversely.

Proof. Let $f : X \to Y$ be a λ_g^δ-continuous function. Let V be any open set in Y. Since f is λ_g^δ-continuous, $f^{-1}(V)$ is λ_g^δ-open in X. Since every λ_g^δ-open set is $g\delta s$-open and thus $f^{-1}(V)$ is $g\delta s$-open in X. Hence f is $g\delta s$-continuous. □

Example 5.2.15. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is $g\delta s$-continuous , since the inverse image of the all open sets are $g\delta s$-open in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.

Remark 5.2.16. The following diagram portrays the dependence relation between the newly introduced continuity and other already existing continuities.

Figure 5.1:



Remark 5.2.17. The following examples show that the notion of λ_g^δ-continuity and continuity (resp. α-continuity, pre-continuity, semi-continuity) are independent of each

other.

Remark 5.2.18. λ_g^δ-continuity and continuity are independent of each other which is shown in the following examples.

Example 5.2.19. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is continuous, since the inverse image of the all open sets are open in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.

Example 5.2.20. Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{X, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is λ_g^δ-continuous but not continuous, since the inverse image of the only closed set $\{b, c\}$ of (Y, σ) is $\{b, c\}$ which is λ_g^δ-closed in (X, τ) but not closed in (X, τ).

Remark 5.2.21. λ_g^δ-continuity and α-continuity are independent of each other which is shown in the following examples.

Example 5.2.22. Let X, Y, τ, σ and f be defined as in Example 5.2.20. Then f is λ_g^δ-continuous but not α-continuous, as the inverse image of the only closed set $\{b, c\}$ of (Y, σ), $f^{-1}\{b, c\} = \{b, c\}$ is λ_g^δ-closed in (X, τ) but not α-closed in (X, τ).

Example 5.2.23. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is α-continuous, since the inverse image of the all closed sets are α-closed in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.

Remark 5.2.24. λ_g^δ-continuity and pre-continuity are independent of each other as shown in the following examples.

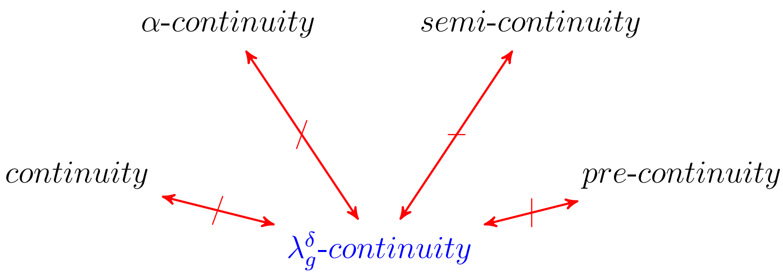
Example 5.2.25. Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is λ_g^δ-continuous but not pre-continuous, as the inverse image of $\{a, c\}$ of (Y, σ), $f^{-1}\{a, c\} = \{a, c\}$ is λ_g^δ-closed in (X, τ) but not pre-closed in (X, τ).

Example 5.2.26. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is pre-continuous, since the inverse image of the all closed sets are pre-closed in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.

Remark 5.2.27. λ_g^δ-continuity and semi-continuity are independent of each other as shown in the following examples.

Example 5.2.28. Let X, Y, τ, σ and f be defined as in Example 5.2.20. Then f is λ_g^δ-continuous but not semi-continuous, as the inverse image of the only closed set $\{b, c\}$ of (Y, σ), $f^{-1}\{b, c\} = \{b, c\}$ is λ_g^δ-closed in (X, τ) but not semi-closed in (X, τ).

Example 5.2.29. Let X, Y, τ, σ and f be defined as in Example 5.2.11. Then f is semi-continuous, since the inverse image of the all closed sets are semi-closed in (X, τ) but not λ_g^δ-continuous as seen in Example 5.2.11.



Remark 5.2.30. The composition of two λ_g^δ-continuous functions need not be a λ_g^δ-continuous function as seen from the following example.

Example 5.2.31. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{X, \phi, \{a, b\}\}$ and $\eta = \{X, \phi, \{c\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ as identity functions. Then f and g are λ_g^δ-continuous functions but $g \circ f$ is not λ_g^δ-continuous as $f^{-1}(g^{-1}\{a, b\}) = \{a, b\}$ is not λ_g^δ-closed in (X, τ).

Theorem 5.2.32. *For functions $f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \eta)$ and $g \circ f : (X, \tau) \to (Z, \eta)$, we have the following:*

(i) If f is λ_g^δ-continuous and g is super continuous then $g \circ f$ is λ_g^δ-continuous,

(ii) If f is super continuous and g is super continuous then $g \circ f$ is λ_g^δ-continuous,

(iii) *If f is λ_g^δ-continuous and g is continuous then $g \circ f$ is λ_g^δ-continuous(resp. $g\delta$-continuous, $g\delta s$-continuous and δgs-continuous).*

Proof. (i) Let V be closed in (Z, η). Since g is super continuous, $g^{-1}(V)$ is δ-closed in (Y, σ). As every δ-closed set is closed, $g^{-1}(V)$ is closed in (Y, σ). Since f is λ_g^δ-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is λ_g^δ-closed in (X, τ). Hence $g \circ f$ is λ_g^δ-continuous.

(ii) Let V be closed in (Z, η). Since g is super continuous, $g^{-1}(V)$ is δ-closed in (Y, σ). As every δ-closed set is closed, $g^{-1}(V)$ is closed in (Y, σ). Since f is super continuous, $f^{-1}(g-1(V)) = (g \circ f)^{-1}(V)$ is δ-closed in (X, τ). As every δ-closed set is λ_g^δ-closed, $(g \circ f)^{-1}(V)$ is λ_g^δ-closed in (X, τ). Hence $g \circ f$ is λ_g^δ-continuous.

(iii) Similar to (i) and (ii).

□

5.3 λ_g^δ-continuity via Separation Spaces

This section deals with the influence of the five newly introduced separation spaces on λ_g^δ-closed sets. These special spaces are helpful in reversing the relational properties of λ_g^δ-continuity.

Proposition 5.3.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a λ_g^δ-continuous function and (X, τ) be a ${}_{\lambda_g^\delta}T_\delta$-space. Then f is super continuous.

Proof. Let V be closed in (Y, σ). Since f is λ_g^δ-continuous, $f^{-1}(V)$ is λ_g^δ-closed in (X, τ). Since (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$-space, in which every λ_g^δ-closed set is δ-closed, $f^{-1}(V)$ is δ-closed in (X, τ). Hence f is super continuous. □

Proposition 5.3.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a $g\delta$-continuous function and (X, τ) be a ${}_{g\delta}T_{\lambda_g^\delta}$-space. Then f is λ_g^δ-continuous.

Proof. Let V be a closed set in (Y, σ). Since f is $g\delta$-continuous, $f^{-1}(V)$ is $g\delta$-closed in (X, τ). Since (X, τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$-space, $f^{-1}(V)$ is λ_g^δ-closed in (X, τ). Hence f is λ_g^δ-

continuous. □

Proposition 5.3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a δgs-continuous function and (X, τ) be a ${}_{\delta gs}T_{\lambda_g^\delta}$-space. Then f is λ_g^δ-continuous.

Proof. Let V be a closed set in (Y, σ). Since f is δgs-continuous, $f^{-1}(V)$ is δgs-closed in (X, τ). Since (X, τ) is a ${}_{\delta gs}T_{\lambda_g^\delta}$-space, $f^{-1}(V)$ is λ_g^δ-closed in (X, τ). Hence f is λ_g^δ-continuous. □

Proposition 5.3.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a $g\delta s$-continuous function and (X, τ) be a ${}_{g\delta s}T_{\lambda_g^\delta}$-space. Then f is λ_g^δ-continuous.

Proof. Let V be a closed set in (Y, σ). Since f is $g\delta s$-continuous, $f^{-1}(V)$ is $g\delta s$-closed in (X, τ). Since (X, τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$-space, $f^{-1}(V)$ is λ_g^δ-closed in (X, τ). Hence f is λ_g^δ-continuous. □

5.4 Quasi λ_g^δ-continuity and Perfectly λ_g^δ-continuity

This section deals with five different types of continuities. A significant result comprising of various compositions of these continuities is established in the end of this section.

Definition 5.4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called ***quasi λ_g^δ-continuous*** if the inverse image of every λ_g^δ-open set in (Y, σ) is open in (X, τ).

Proposition 5.4.2. A function $f : X \to Y$ is quasi λ_g^δ-continuous if and only if the inverse image of every λ_g^δ-closed set in (Y, σ) is closed in (X, τ).

Theorem 5.4.3. *For a function $f : X \to Y$, the following are equivalent:*

(i) quasi λ_g^δ-continuous;

(ii) $f^{-1}(B)$ is closed in X, for every λ_g^δ-closed B in Y.

(iii) For each $x \in X$ and each λ_g^δ-open set B containing $f(x)$, there exists an open set A containing x such that $f(A) \subseteq B$.

Proof. (i) $\Leftrightarrow$ (ii) is obvious.
(i) $\Rightarrow$ (iii) Let $x \in X$ and let B be an open set containing $f(x)$ then by hypothesis $f^{-1}(B)$ is a λ_g^δ-open set containing x. Let $A = f^{-1}(B)$ then $f(A) = f(f^{-1}(B)) \subseteq B$.
(iii) $\Rightarrow$ (i) Let B be λ_g^δ-open in Y with $x \in f^{-1}(B) \Rightarrow f(x) \Rightarrow B$ then by hypothesis there exists an open set A containing x such that $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$. The result follows as $f^{-1}(B)$ can be written as the union of open sets. □

Definition 5.4.4. A function $f : (X, \tau) \to (Y, \sigma)$ is called ***perfectly λ_g^δ-continuous*** if the inverse image of every λ_g^δ-open set in (Y, σ) is clopen in (X, τ).

Proposition 5.4.5. A function $f : X \to Y$ is perfectly λ_g^δ-continuous if and only if the inverse image of every λ_g^δ-closed set in Y is clopen in X.

Theorem 5.4.6. *For a function $f : X \to Y$, the following statements are true.*

(i) Every strongly continuous function is a quasi λ_g^δ-continuous function but not conversely.

(ii) Every perfectly λ_g^δ-continuous function is a quasi λ_g^δ-continuous function but not conversely.

(iii) Every strongly continuous function is a perfectly λ_g^δ-continuous function.

Proof. (i) Let $f : X \to Y$ be a strongly continuous function and B be a λ_g^δ-closed set in Y. By hypothesis, $f^{-1}(B)$ is clopen in X. This proves f is a quasi λ_g^δ-continuous function.

(ii) Let $f : X \to Y$ be a perfectly λ_g^δ-continuous function and B be a λ_g^δ-closed set in Y. By hypothesis, $f^{-1}(B)$ is clopen in X and thus closed in X. This proves f is a quasi λ_g^δ-continuous function.

(iii) Let $f : X \to Y$ be a strongly continuous function and B be a λ_g^δ-closed set in Y. By hypothesis, $f^{-1}(B)$ is clopen in X. This proves f is a perfectly λ_g^δ-continuous function.

□

Example 5.4.7. Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Take $A = \{a, b\}$ in (Y, σ) then f is quasi λ_g^δ-continuous but not strongly continuous since $A = \{a, b\}$ is λ_g^δ-open in (Y, σ) whereas $f^{-1}(A) = \{a, b\}$ is open but not closed in (X, τ).

Example 5.4.8. Let X, Y, τ, σ, f and A be defined as in Example 5.4.7. Then f is quasi λ_g^δ-continuous but not perfectly λ_g^δ-continuous since A is λ_g^δ-open in (Y, σ) whereas $f^{-1}(A)$ is open but not closed in (X, τ).

Theorem 5.4.9. *Let (X, τ) be a partition space[7], (Y, σ) be a topological space and $f : (X, \tau) \to (Y, \sigma)$ be any function. Then the following are equivalent.*

(i) f is perfectly λ_g^δ-continuous;

(ii) f is quasi λ_g^δ-continuous.

Proof. (i) $\Rightarrow$ (ii) Already proved.
(ii) $\Rightarrow$ (i) Follows from the definition of a partition space in which every open set is closed. □

Proposition 5.4.10. Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then their composition $g \circ f : X \to Z$ is

(i) perfectly λ_g^δ-continuous if g is perfectly λ_g^δ-continuous and f is continuous,

(ii) continuous if g is λ_g^δ-continuous and f is quasi λ_g^δ-continuous.

(iii) continuous if g is λ_g^δ-continuous and f is perfectly λ_g^δ-continuous.

Proof. (i) Let $g : Y \to Z$ be perfectly λ_g^δ-continuous and $f : X \to Y$ be continuous. Let V be a λ_g^δ-closed set in Z. Then $g^{-1}(V)$ is open as well as closed in Y. Since f is continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in X. Hence $g \circ f$ is perfectly λ_g^δ-continuous.

(ii) Let $g : Y \to Z$ be λ_g^δ-continuous and $f : X \to Y$ be quasi λ_g^δ-continuous. Let V be a closed set in Z. Then $g^{-1}(V)$ is λ_g^δ-closed in Y. Further, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is closed in X. Hence $g \circ f$ is continuous.

(iii) Let $g : Y \to Z$ be λ_g^δ-continuous and $f : X \to Y$ be perfectly λ_g^δ-continuous. Let V be a closed set in Z. Then $g^{-1}(V)$ is λ_g^δ-closed in Y. Further, since f is perfectly λ_g^δ-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in X and thereby closed in X. Hence $g \circ f$ is continuous.

□

5.5 Totally λ_g^δ-continuity, Strongly λ_g^δ-continuity and Contra λ_g^δ-continuity

Definition 5.5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) ***totally λ_g^δ-continuous*** if the inverse image of every open subset of (Y, σ) is λ_g^δ-clopen in (X, τ).

(ii) ***strongly λ_g^δ-continuous*** if the inverse image of every subset of (Y, σ) is λ_g^δ-clopen in (X, τ).

(iii) ***contra λ_g^δ-continuous*** if the inverse image of every closed set of (Y, σ) is λ_g^δ-open in (X, τ).

Proposition 5.5.2. A function $f : (X, \tau) \to (Y, \sigma)$ is

(i) totally λ_g^δ-continuous iff $f^{-1}(V)$ is λ_g^δ-clopen in (X, τ) for every closed set V in (Y, σ).

(ii) contra λ_g^δ-continuous iff $f^{-1}(V)$ is λ_g^δ-closed in (X, τ) for every open set V in (Y, σ).

Theorem 5.5.3. *If (X, τ) and (Y, σ) are any two topological spaces with a function $f : (X, \tau) \to (Y, \sigma)$ then the following statements are true.*

(i) Every strongly λ_g^δ-continuous function is a totally λ_g^δ-continuous function but not conversely.

(ii) Every totally λ_g^δ-continuous function is a contra λ_g^δ-continuous function but not conversely.

(iii) *Every strongly λ_g^δ-continuous function is a contra λ_g^δ-continuous function but not conversely.*

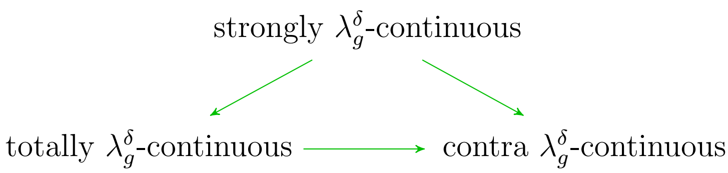
Proof. (i) Let $f : X \to Y$ be a strongly λ_g^δ-continuous function and B be a open subset of Y. By hypothesis, $f^{-1}(B)$ is λ_g^δ-clopen in X. This proves f is a totally λ_g^δ-continuous function.

(ii) Let $f : X \to Y$ be a totally λ_g^δ-continuous function and B be a closed subset of Y. By hypothesis, $f^{-1}(B)$ is λ_g^δ-clopen in X. This proves f is a contra λ_g^δ-continuous function.

(iii) Let $f : X \to Y$ be a strongly λ_g^δ-continuous function and B be a open subset of Y. By hypothesis, $f^{-1}(B)$ is λ_g^δ-clopen in X. This proves f is a contra λ_g^δ-continuous function.

□

Figure 5.2:



Example 5.5.4. Let $X = Y = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{X, \phi, \{a, b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is totally λ_g^δ-continuous but not strongly λ_g^δ-continuous since the inverse image of the closed set $\{d\}$ is λ_g^δ-clopen in (X, τ) but the inverse image of $\{a\}$ is not λ_g^δ-clopen in (X, τ).

Example 5.5.5. Let X, Y, τ and σ be defined as in Example 5.5.4. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = d, f(b) = b, f(c) = c$ and $f(d) = a$. Then f is contra λ_g^δ-continuous but not totally λ_g^δ-continuous as the inverse image of $\{d\}$ is $\{a\}$ and $\{a\}$ is λ_g^δ-open but not λ_g^δ-closed in (X, τ).

Example 5.5.6. Let X, Y, τ, σ and f be defined as in Example 5.5.5. Then f is contra λ_g^δ-continuous but not strongly λ_g^δ-continuous as the inverse image of $\{b\}$ is $\{b\}$ itself and $\{b\}$ is not λ_g^δ-clopen in (X, τ).

Theorem 5.5.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a totally λ_g^δ-continuous function, where (Y, σ) is a partition space. Then f is a strongly λ_g^δ-continuous function.*

Proof. Follows from the definition of a partition space. □

Remark 5.5.8. The composition of two contra λ_g^δ-continuous functions need not be a contra λ_g^δ-continuous function.

Example 5.5.9. Let $X = Y = Z = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{X, \phi, \{a, b\}\}$and $\eta = \{X, \phi, \{c\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c, f(b) = a, f(c) = b$ and $g : (Y, \sigma) \to (Z, \eta)$ as an identity function. Then f and g are contra λ_g^δ-continuous functions but $g \circ f$ is not contra λ_g^δ-continuous as $f^{-1}(g^{-1}\{c\}) = \{a\}$ is not λ_g^δ-closed in (X, τ).

Remark 5.5.10. A significant result on the composition of various continuities is established in the following theorem.

Theorem 5.5.11. *For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta), f : (X, \tau) \to (Y, \sigma), g : (Y, \sigma) \to (Z, \eta)$ and $g \circ f : (X, \tau) \to (Z, \eta)$ the following results are true.*

(i) If f is quasi λ_g^δ-continuous and g is super continuous then $g \circ f$ is continuous.

(ii) If f is a quasi λ_g^δ-continuous and g is a totally λ_g^δ-continuous then $g \circ f$ is a perfectly continuous.

(iii) If f is perfectly λ_g^δ-continuous and g is super continuous then $g \circ f$ is perfectly continuous.

(iv) If f is continuous and g is perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) then $g \circ f$ is perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous).

(v) If f is super continuous and g is perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) then $g \circ f$ is perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous)

(vi) *If f is a perfectly λ_g^δ-continuous and g is a strongly λ_g^δ-continuous then $g \circ f$ is a strongly continuous.*

(vii) *If f is a strongly λ_g^δ-continuous and g is any function then $g \circ f$ is a strongly λ_g^δ-continuous.*

(viii) *If f is a quasi λ_g^δ-continuous and g is a contra λ_g^δ-continuous then $g \circ f$ is a contra continuous function.*

(ix) *If f is a contra λ_g^δ-continuous and g is a continuous(resp. super continuous) then $g \circ f$ is a contra λ_g^δ-continuous.*

(x) *If f is a totally λ_g^δ-continuous and g is a continuous(resp. super continuous) then $g \circ f$ is a totally λ_g^δ-continuous.*

Proof. (i) Let $f : X \to Y$ be quasi λ_g^δ-continuous function and $g : Y \to Z$ be a super continuous function. Let W be a closed subset of Z. Then $g^{-1}(W)$ is δ-closed in Y. Thus $g^{-1}(W)$ is λ_g^δ-closed in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is closed in X. Hence $g \circ f$ is a continuous function.

(ii) Let $f : X \to Y$ be quasi λ_g^δ-continuous function and $g : Y \to Z$ be a totally λ_g^δ-continuous function. Let W be a open subset of Z. Then $g^{-1}(W)$ is λ_g^δ-clopen in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen in X. Hence $g \circ f$ is a perfectly continuous function.

(iii) Let $f : X \to Y$ be perfectly λ_g^δ-continuous function and $g : Y \to Z$ be a super continuous function. Let W be a open subset of Z. Then $g^{-1}(W)$ is δ-open in Y and hence λ_g^δ-open in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen in X. Hence $g \circ f$ is a perfectly continuous function.

(iv) Let $f : X \to Y$ be a continuous function and $g : Y \to Z$ be a perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) function. Let W be a λ_g^δ-open subset of Z. Then $g^{-1}(W)$ is clopen(resp. open) in Y and hence $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen(resp. open) in X. Hence $g \circ f$ is a perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) function.

(v) Let $f : X \to Y$ be a super continuous function and $g : Y \to Z$ be a perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) function. Let W be a λ_g^δ-open subset of Z. Then $g^{-1}(W)$ is clopen(resp. open) in Y. Then $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is δ-closed as well as δ-open in X. Thereby it is λ_g^δ-closed as well as λ_g^δ-open in X. Hence $g \circ f$ is a perfectly λ_g^δ-continuous(resp. quasi λ_g^δ-continuous) function.

(vi) Let $f : X \to Y$ be perfectly λ_g^δ-continuous function and $g : Y \to Z$ be a strongly λ_g^δ-continuous function. Let W be any subset of Z. Then $g^{-1}(W)$ is λ_g^δ-clopen in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen in X. Hence $g \circ f$ is a strongly continuous function.

(vii) Let $f : X \to Y$ be a strongly λ_g^δ-continuous function and $g : Y \to Z$ be a quasi λ_g^δ-continuous function. Let W be any subset of Z. Then $g^{-1}(W)$ is a subset in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is λ_g^δ-clopen in X. Hence $g \circ f$ is a strongly λ_g^δ-continuous function.

(viii) Let $f : X \to Y$ be a quasi λ_g^δ-continuous function and $g : Y \to Z$ be a contra λ_g^δ-continuous function. Let W be any closed subset of Z. Then $g^{-1}(W)$ is λ_g^δ-open in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open in X. Hence $g \circ f$ is a contra continuous function.

(ix) Let $f : X \to Y$ be a contra λ_g^δ-continuous function and $g : Y \to Z$ be a continuous(resp. super continuous) function. Let W be any closed subset of Z. Then $g^{-1}(W)$ is closed(resp. δ-closed) in Y. Thereby it is closed in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is λ_g^δ-open in X. Hence $g \circ f$ is a contra λ_g^δ-continuous function.

(x) Let $f : X \to Y$ be a totally λ_g^δ-continuous function and $g : Y \to Z$ be a continuous(resp. super continuous) function. Let W be any open subset of Z. Then $g^{-1}(W)$ is open(resp. δ-open) in Y. Thereby it is open in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is λ_g^δ-clopen in X. Hence $g \circ f$ is a totally λ_g^δ-continuous function.

□

5.6 λ_g^δ-irresoluteness and contra λ_g^δ-irresoluteness

Definition 5.6.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) λ_g^δ***-irresolute*** if $f^{-1}(B)$ is λ_g^δ-closed in (X, τ) for every λ_g^δ-closed set B in (Y, σ).

(ii) ***contra*** λ_g^δ***-irresolute*** if $f^{-1}(B)$ is λ_g^δ-closed in (X, τ) for every λ_g^δ-open set B in (Y, σ).

Theorem 5.6.2. *For a function $f : (X, \tau) \to (Y, \sigma)$ with (Y, σ) being an almost weakly Hausdorff space, every λ_g^δ-continuous function is λ_g^δ-irresolute.*

Proof. Let $f : X \to Y$ be a λ_g^δ-continuous function and B be a λ_g^δ-closed subset of Y. By Lemma 2.3.15, B is closed in Y. Then $f^{-1}(B)$ is λ_g^δ-closed in X and hence f is a λ_g^δ-irresolute function. □

Remark 5.6.3. λ_g^δ-irresolute functions and irresolute functions are independent of each other.

Example 5.6.4. Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Then f is λ_g^δ-irresolute but not irresolute since the inverse image of $\{a\}$ is $\{c\}$ and $\{c\}$ is not semi-open in (X, τ).

Example 5.6.5. Let $X = Y = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma = \{X, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is irresolute but not λ_g^δ-irresolute since the inverse image of $\{a\}$ is $\{a\}$ which is not λ_g^δ-closed in (X, τ).

Theorem 5.6.6. *Composition of two λ_g^δ-irresolute functions is a λ_g^δ-irresolute function.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be any two λ_g^δ-irresolute functions. Let W be a λ_g^δ-closed subset of Z. Then $g^{-1}(W)$ is λ_g^δ-closed in Y. Therefore $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is λ_g^δ-closed in X. Hence $g \circ f$ is λ_g^δ-irresolute. □

Theorem 5.6.7. *For topological spaces $(X,\tau),(Y,\sigma),(Z,\eta), f:(X,\tau)\to(Y,\sigma)$, $g:(Y,\sigma)\to(Z,\eta)$ and $g\circ f:(X,\tau)\to(Z,\eta)$ the following results are true.*

(i) If f is a λ_g^δ-irresolute function and g is a λ_g^δ-continuous(resp. totally λ_g^δ-continuous, strongly λ_g^δ-continuous, contra λ_g^δ-continuous, contra λ_g^δ-irresolute) function then $g\circ f$ is a λ_g^δ-continuous (resp. totally λ_g^δ-continuous, strongly λ_g^δ-continuous, contra λ_g^δ-continuous, contra λ_g^δ-irresolute) function.

(ii) If f is a contra λ_g^δ-irresolute(resp. strongly continuous, quasi λ_g^δ-continuous, perfectly λ_g^δ-continuous) function and g is a λ_g^δ-irresolute function then $g\circ f$ is a contra λ_g^δ-irresolute(resp. strongly continuous, quasi λ_g^δ-continuous, perfectly λ_g^δ-continuous) function.

Proof. Follow from the definitions. □

Remark 5.6.8. From the previous theorem, we observe that λ_g^δ-irresoluteness acts as a mirror in reflecting the type of continuity in their compositions.

Theorem 5.6.9. *If $f:(X,\tau)\to(Y,\sigma)$ is λ_g^δ-irresolute then*

(i) For $A\subseteq X, f(\lambda_g^\delta cl(A))\subseteq cl_\delta(f(A))$.

(ii) For $B\subseteq Y, \lambda_g^\delta cl(f^{-1}(B))\subseteq f^{-1}(cl_\delta(A))$.

Proof. (i) Let $A\subseteq X$ then $cl_\delta(f(A))$ is δ-closed in Y and thus λ_g^δ-closed in Y. Since f is λ_g^δ-irresolute, $f^{-1}(cl_\delta(f(A)))$ is λ_g^δ-closed in X. $A\subseteq f^{-1}(f(A))\subseteq f^{-1}(cl_\delta(f(A)))\Rightarrow \lambda_g^\delta cl(A)\subseteq \lambda_g^\delta cl(f^{-1}(cl_\delta(f(A))))=f^{-1}(cl_\delta(f(A)))\Rightarrow f(\lambda_g^\delta cl(A))\subseteq cl_\delta(f(A))$.

(ii) Let $B\subseteq Y$ then $cl_\delta(B)$ is δ-closed in Y and thus λ_g^δ-closed in Y. By hypothesis, $f^{-1}(cl_\delta(B))$ is λ_g^δ-closed in X. Since $B\subseteq cl_\delta(B), f^{-1}(B)\subseteq f^{-1}(cl_\delta(B))\Rightarrow \lambda_g^\delta cl(f^{-1}(B))\subseteq \lambda_g^\delta cl(f^{-1}(cl_\delta(B)))=f^{-1}(cl_\delta(B))$.

□

Theorem 5.6.10. *If $f:(X,\tau)\to(Y,\sigma)$ is surjective, λ_g^δ-irresolute and closed with (X,τ) being an almost weakly Hausdorff space then every λ_g^δ-closed set is closed in (Y,σ).*

Proof. Let B be λ_g^δ-closed in Y. By hypothesis, $f^{-1}(B)$ is λ_g^δ-closed in X. Since X is almost weakly Hausdorff, $f^{-1}(B)$ is closed in X. Once again by hypothesis, B is closed in Y. Since f is surjective, this is true for any arbitrary B in Y. Therefore every λ_g^δ-closed set is closed in Y. □