

CHAPTER V

SOFT LA – SEMIGROUPS

Definition : 5.1

A **groupoid** is an algebraic structure (S, \cdot) consisting of a non-empty set S and a binary operation ' \cdot ', which satisfies the following axioms

- i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in S$. (associative property)
- ii. There exist an element $e \in S$ called identity element such that $a \cdot e = e \cdot a = a$ for any $a \in S$.
- iii. There exist an element $a^{-1} \in S$ called inverse of a for every $a \in S$, so that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Definition : 5.2

A groupoid (S, \cdot) is called a **left almost semigroup (LA-semigroup)** if it satisfies left invertive law,

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a \text{ for all } a, b, c \in S.$$

Definition : 5.3

A non-empty subset B of an LA-semigroup S is called a **LA-subsemigroup** of S if $BB \subseteq B$. i-e $a, b \in B$ for all $a, b \in B$.

Definition : 5.4

A non-empty subset I of an LA-semigroup S is called a **left (right) ideal** over S , if $SI \subseteq I$ ($IS \subseteq I$) and is called an **ideal** if it is a left and a right ideal over S .

Definition : 5.5

Let (F, A) and (G, B) be any two soft sets over an LA-semigroup S . Then, the **restricted product** of (F, A) and (G, B) is defined as the soft set

$(H, C) = (F, A) \odot (G, B)$, where $C = A \cap B$ and H is a set valued function from C to $P(S)$ defined as $H(e) = F(e)G(e)$ for all $e \in C$.

Definition : 5.6

A soft set (F, A) over S is called a **soft LA-semigroup** over S if

$$(F, A) \odot (F, A) \sqsubseteq (F, A)$$

Theorem : 5.7

A soft set (F, A) over S is a soft LA-semigroup if and only if each $F(a) \neq \phi$ is an fuzzy LA-subsemigroup of S for all $a \in A$.

Proof:

Let (F, A) be a soft LA-semigroup over S .

By definition, $(F, A) \odot (F, A) \sqsubseteq (F, A)$, and so $F(a)F(a) \subseteq F(a)$ for all $a \in A$. It follows that for all $a \in A$, $F(a) \neq \phi$ is an LA-subsemigroup of S .

Conversely, assume that (F, A) is a soft set over S such that for all $a \in A$, $F(a)$ is an LA-subsemigroup of S whenever $F(a) \neq \phi$. As $F(a)$ is LA-subsemigroup of S , therefore for each $a \in A$, $F(a)F(a) \subseteq F(a)$. Thus $(F, A) \odot (F, A) \sqsubseteq (F, A)$. Hence (F, A) is a soft LA-semigroup over S .

Theorem : 5.8

Let (F, A) and (G, B) be two soft LA-semigroups over S . Then $(F, A) \wedge (G, B)$ is also a soft LA-semigroup over S , if it is non-empty.

Proof :

Since $(F, A) \wedge (G, B) = (H, A \times B)$ and $H(a, b) = F(a) \cap G(b)$. As $F(a)$ and $G(b)$ are LA-subsemigroup of S , therefore either $F(a) \cap G(b) = \emptyset$ or $F(a) \cap G(b)$ is an LA-subsemigroup of S . Consequently, $(H, A \times B)$ is a soft LA-semigroup of S .

Theorem : 5.9

Let (F, A) and (G, B) be two soft LA-semigroups over S . Then their extended intersection $(F, A) \cap_E (G, B)$ is a soft LA-semigroup over S .

Proof :

Since (F, A) and (G, B) are soft LA-semigroups over S , so their extended intersection over S is a soft set (H, C) where $C = A \cup B$ and for all $c \in C$ it is defined as

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B \\ G(c), & \text{if } c \in B - A \\ F(c) \cap G(c), & \text{if } c \in A \cap B \end{cases}$$

Since for all $c \in C$ either $c \in A - B$ or $c \in B - A$ or $c \in A \cap B$. If $c \in A - B$ then $H(c) = F(c)$ and if $c \in B - A$ then $H(c) = G(c)$ and if $c \in A \cap B$ then $H(c) = F(c) \cap G(c)$. In all the three cases $H(c)$ is an LA-subsemigroup of S . Hence $(H, C) = (F, A) \cap_E (G, B)$ is a soft LA-semigroup over S .

Theorem : 5.10

Let (F, A) and (G, B) be two soft LA-semigroups over S such that $A \cap B = \emptyset$. Then their restricted intersection $(F, A) \cap_R (G, B)$ is a soft LA-semigroup over S .

Proof :

Since (F, A) and (G, B) are soft LA-semigroups over S , so their extended intersection over S is a soft set (H, C) where $C = A \cap B = \emptyset$ and

$$H(c) = F(c) \cap G(c) \text{ for all } c \in C.$$

Since $F(c)$ and $G(c)$ are both LA-subsemigroups of S and intersection of two LA-subsemigroups of S is either empty or an LA-subsemigroup of S . Hence $F(c) \cap G(c)$ is either empty or an LA-subsemigroup of S .

Hence $(H, C) = (F, A) \cap_R (G, B)$ is soft LA-semigroup over S .

Theorem : 5.11

Let (F, A) and (G, B) be two soft LA-semigroups over S such that $A \cap B = \emptyset$. Then $(F, A) \cup_E (G, B)$ is also a soft LA-semigroup over S .

Proof :

Since $(F, A) \cup_E (G, B) = (H, C)$ and $A \cap B = \emptyset$, so for all $c \in C = A \cup B$ either $c \in A - B$ or $c \in B - A$. If $c \in A - B$ then $L(c) = F(c)$ and if $c \in B - A$ then $H(c) = G(c)$. In both cases $H(c)$ is an LA-subsemigroup of S . Therefore (H, C) is a soft LA-semigroup over S .

Definition : 5.12

If (F, A) is a soft LA-semigroup over S , then we say that (F, A) is a soft LA-semigroup with left identity e if each $F(a) \neq \emptyset$ is an LA-subsemigroup with left identity e , where e is the left identity of S .

Theorem : 5.13

If (F, A) is a soft LA-semigroup with left identity e over S , then $(F, A) \delta (F, A) = (F, A)$.

Proof :

By definition, $(F, A) \odot (F, A) = (H, C)$,

where $C = A \cap A = A$ and $H(a) = F(a) F(a)$ for all $a \in C = A$. By definition $F(a)$ is an LA-subsemigroup of S with left identity by e . Then,

$$F(a)F(a) = F(a)$$

$$\Rightarrow H(a) = F(a) \text{ and also } C = A$$

Hence $(H, C) = (F, A)$.

Definition : 5.14

Let (F, A) and (G, B) be two soft sets over an LA-semigroup S . Then the **operation $*$ for soft sets** is defined as $(F, A) * (G, B) = (H, A \times B)$ where $H(a, b) = F(a)G(b)$ for all $a \in A, b \in B$ and $A \times B$ is the cartesian product of the sets A and B .

Theorem : 5.15

Let (F, A) and (G, B) be two soft LA-semigroups over S . Then $(F, A) * (G, B)$ is also a soft LA semigroup over S .

Proof :

By definition, $(F, A) * (G, B) = (H, A \times B)$ and

$$H(a, b) = F(a)G(b) \text{ for } a \in A, b \in B.$$

To prove $(F, A) * (G, B)$ is also a soft LA semigroup over S , we show that $H(a, b)$ is an LA-subsemigroup of S . Let $x, y \in H(a, b)$. This implies $x = a_1b_1$ and $y = a_2b_2$, where $a_1a_2 \in F(a), b_1b_2 \in G(b)$. Now by using the medial law we have

$$\begin{aligned} xy &= (a_1b_1)(a_2b_2) \\ &= (a_1b_1)(a_2b_2) \in F(a)G(b) = H(a, b) \end{aligned}$$

so $H(a, b)$ is an LA-subsemigroup of S . Consequently $(F, A) * (G, B)$ become a soft LA-semigroup over S .

Definition : 5.16

A soft set (F, A) over an LA-semigroup S is called a **soft left (right) ideal** over S if $\tilde{A}_S \odot (F, A) \sqsubseteq (F, A)$ ($(F, A) \odot \tilde{A}_S \sqsubseteq (F, A)$), where \tilde{A}_S is the absolute soft set over S .

A soft set over S is a **soft ideal** if it is both a soft left and a soft right ideal over S .

Theorem : 5.17

A soft set (F, A) over S is a soft ideal over S if and only if $F(a) \neq \emptyset$ is an ideal of S for all $a \in A$.

Proof :

Let (F, A) over S be a soft ideal over S . Then

$$\tilde{A}_S \odot (F, A) \sqsubseteq (F, A) \text{ and } (F, A) \odot \tilde{A}_S \sqsubseteq (F, A).$$

This gives us that $SF(a) \subseteq F(a)$ and $F(a)S \subseteq F(a)$ for all $a \in A$. Thus it follows that $F(a)$ is an ideal of S whenever it is non-empty. Conversely, suppose that (F, A) is a soft set over S such that for all $a \in A$, $F(a)$ is an ideal of S whenever $F(a) \neq \emptyset$. As $F(a)$ is an ideal of S , so

$$SF(a) \subseteq F(a) \text{ and } F(a)S \subseteq F(a) \text{ for all } a \in A.$$

This implies that $\tilde{A}_S \odot (F, A) \sqsubseteq (F, A)$ and $(F, A) \odot \tilde{A}_S \sqsubseteq (F, A)$. Thus (F, A) is a soft ideal over S .

Theorem : 5.18

Let (F, A) and (G, B) be two soft ideals over S . Then $(F, A) \wedge (G, B)$ is also a soft ideal over S .

Proof :

Since $(F, A) \wedge (G, B) = (H, A \times B)$ and $H(a, b) = F(a) \cap G(b)$. Now $F(a)$ and $G(b)$ are ideals of S , clearly $F(a) \cap G(b)$ is an ideal of S . Therefore $(H, A \times B)$ is a soft ideal over S .

Theorem : 5.19

Let (F, A) and (G, B) be two soft ideals over S . Then $(F, A) \vee (G, B)$ is also a soft ideal over S .

Proof :

Since $(F, A) \vee (G, B) = (H, A \times B)$ and $H(a, b) = F(a) \cup G(b)$. Now $F(a)$ and $G(b)$ are ideals of S , clearly $F(a) \cup G(b)$ is an ideal of S . Therefore $(H, A \times B)$ is a soft ideal over S .

Theorem : 5.20

Let (F, A) and (G, B) be two soft ideals over S , such that $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is also a soft ideal over S , contained in both (F, A) and (G, B) .

Proof :

By definition, $(F, A) \cap_R (G, B) = (H, C)$,
where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ where $F(c)$ and $G(c)$ are ideals of S .
As intersection of two ideals of an LA-semigroups is again an ideal of S , so $H(c) = F(c) \cap G(c)$ is an ideal of S . Also it is clear that $A \cap B \subseteq A$, $A \cap B \subseteq B$, $H(c) \subseteq F(c)$ and $H(c) \subseteq G(c)$. Therefore $(H, C) \sqsubseteq (F, A)$ and $(H, C) \sqsubseteq (G, B)$.

Theorem : 5.21

Let (F, A) and (G, B) be two soft ideals over S . Then $(F, A) \cap_E (G, B)$ is also a soft ideal over S .

Proof :

As $(F, A) \cap_E (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B \\ G(c), & \text{if } c \in B - A \\ F(c) \cap G(c), & \text{if } c \in A \cap B \end{cases}$$

If $c \in A - B$ then $H(c) = F(c)$ and if $c \in B - A$ then $H(c) = G(c)$ and if $c \in A \cap B$ then $H(c) = F(c) \cap G(c)$. In each case $H(c)$ is an ideal of S . Hence (H, C) is a soft ideal over S .

Theorem : 5.22

Let (F, A) and (G, B) be two soft ideals over S . Then $(F, A) \cup_E (G, B)$ is also a soft ideal over S , containing (F, A) and (G, B) .

Proof :

As $(F, A) \cup_E (G, B) = (H, C)$, where $C = A \cup B$. For $c \in C$ either $c \in A - B$ or $c \in B - A$ or $c \in A \cap B$. If $c \in A - B$ then $H(c) = F(c)$, if $c \in B - A$ then $H(c) = G(c)$ and if $c \in A \cap B$ then $H(c) = F(c) \cup G(c)$. In all the cases, (H, C) is a soft ideal over S . Clearly $(F, A) \sqsubseteq (H, C)$ and $(G, B) \sqsubseteq (H, C)$.

Theorem : 5.23

Let (F, A) and (G, B) be two soft ideals over S . Then $(F, A) \cup_R (G, B)$ is also a soft ideal over S .

Proof :

As $(F, A) \cup_R (G, B) = (H, C)$, where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$. As union of two ideals in an LA-semigroup is again an ideal, so $H(c) = F(c) \cup G(c)$ is an ideal of S . Consequently, (H, C) becomes soft ideal over S .

Theorem : 5.24

Let (F, A) and (G, B) be two soft ideals over S , where S is an LA-semigroup with left identity e . Then $(F, A) * (G, B)$ is also a soft ideal over S .

Proof :

Since $(F, A) * (G, B) = (H, A \times B)$ and $H(a, b) = F(a)G(b)$. We show that $H(a, b)$ is an ideal of S . For this, let $x \in H(a, b) = F(a)G(b)$. This implies that $x = yz$ where $y \in F(a)$ and $z \in G(b)$. Now for $s \in S$,

$$\begin{aligned}
 sx &= s(yz) \\
 &= (es)(yz) \\
 &= (ey)(sz) \text{ (by using medial law)} \\
 &= y(sz) \in F(a)G(b) = H(a, b)
 \end{aligned}$$

Also

$$xs = (yz)s$$

$$= (yz)(es)$$

$$= (ye)(zs) \in F(a)G(b) = H(a, b)$$

Hence $H(a, b)$ is an ideal of S . consequently, $(F, A) * (G, B)$ becomes a soft ideal over S .

If S is an LA-semigroup without left identity then $(F, A) * (G, B)$ is not necessarily soft ideal over S .

Definition : 5.25

Let (F, A) and (G, B) be two soft LA-semigroups over S , then their **cartesian product** is defined as $(F, A) \tilde{\times} (G, B) = (H, A \times B)$, where $H(a, b) = F(a) \times G(b)$ for all $(a, b) \in A \times B$.

Theorem : 5.26

Let S and T be two LA-semigroups and (F, A) and (G, B) is also a soft ideal over $S \times T$.

Proof :

As $(F, A) \tilde{\times} (G, B) = (H, A \times B)$ and $H(a, b) = F(a) \times G(b)$, for all $(a, b) \in A \times B$.

As (F, A) and (G, B) are soft ideals over S and T respectively. Therefore $F(a)$ and $G(b)$ are ideals of S and T respectively, then $F(a) \times G(b)$ is an ideal of $S \times T$. That is, $H(a, b)$ is an ideal of $S \times T$. Consequently, $(H, A \times B)$ is a soft ideal over $S \times T$.

Definition : 5.27

Let (G, B) be a soft subset of a soft LA-semigroup (F, A) over S . Then (G, B) is called a **soft LA-subsemigroup** of (F, A) , if $G(b)$ is an LA-subsemigroup of $F(b)$ for all $b \in B$.

The soft ideal of a soft LA-semigroup defined above is different from usual soft ideal over an LA-semigroup S . The following example is to depict this fact.

Theorem : 5.28

If (F, A') and (G, B') are soft ideals of soft LA-semigroup (F, A) and (G, B) over LA-semigroups S and T respectively, then $(F, A') \tilde{\times} (G, B')$ is a soft ideal of soft LA-semigroup $(F, A) \times (G, B)$ over $S \times T$, whenever $(F, A') \times (G, B')$ is non-null.

Proof :

Since (F, A') and (G, B') are soft ideals of soft LA-semigroups (F, A) and (G, B) , therefore $A' \subseteq A$ and $B' \subseteq B$, clearly $A' \times B' \subseteq A \times B$. Also $F(a')$ is an ideal of $F(a')$ for all $a' \in A'$ and $G(b')$ is an ideal of $G(b')$ for all $b' \in B'$, therefore $F(a') \times G(b')$ is an ideal of $F(a') \times G(b')$. Hence $(F, A') \times (G, B')$ is a soft ideal of soft LA-semigroup $(F, A') \tilde{\times} (G, B')$ over $S \times T$.

Theorem : 5.29

Let (F, A) be a soft LA-semigroup over S and $\{(H_i, B_i); i \in I\}$ is a non-empty family of soft LA-subsemigroup of (F, A) . Then:

- i. $\bigcap_{i \in I} (H_i, B_i)$ is a soft LA-subsemigroup of (F, A) , whenever $\bigcap_{i \in I} (H_i, B_i)$ is non-null and non-empty.

- ii. $\bigwedge_{i \in I} (H_i, B_i)$ is a soft LA-subsemigroup of $\bigwedge_{i \in I} (F, A)$, whenever $\bigwedge_{i \in I} (H_i, B_i)$ is non-null.
- iii. If $B_i \cap B_j = \emptyset$ for all $i, j \in I$ then $\bigcup_{i \in I} (H_i, B_i)$ is a soft LA-subsemigroup of (F, A) .

Theorem : 5.30

Let (F, A) be a soft LA-semigroup over S and $\{(H_i, B_i) ; i \in I\}$ is a non-empty family of soft ideals of (F, A) then

- i. $\bigcap_{i \in I} (H_i, B_i)$ is a soft ideal of (F, A) .
- ii. $\bigwedge_{i \in I} (H_i, B_i)$ is a soft ideal of $\bigwedge_{i \in I} (F, A)$.
- iii. $\bigcup_{i \in I} (H_i, B_i)$ is a soft ideal of (F, A) .
- iv. $\bigvee_{i \in I} (H_i, B_i)$ is a soft ideal of $\bigvee_{i \in I} (F, A)$.

Theorem : 5.31

If (F, A) is a soft LA-semigroup with left identity e over S and (G, B) is a soft right ideal of (F, A) , then (G, B) is soft left ideal of (F, A) .

Proof :

As (G, B) is a soft right ideal of (F, A) , $B \subseteq A$ and $G(b)$ is a right ideal of $F(b)$ for all $b \in B$. We need to show that $G(b)$ is a left ideal of $F(b)$ for all $b \in B$. For

this let $x \in G(b)$, $y \in F(b)$. As $e \in F(b)$, so $yx = (ey)x = (xy)e$ belongs to $G(b)$. Implies that $G(b)$ is left ideal of $F(b)$ for all $b \in B$. Hence (G, B) is a left ideal of (F, A) .

Theorem : 5.32

If (G, B) is a soft right ideal of a soft LA-semigroup (F, A) with left identity over S , then $(G, B) \odot (G, B)$ is a soft ideal of (F, A) .

Proof :

By theorem 5.31 (G, B) becomes a soft left ideal hence (G, B) is a soft ideal of (F, A) . By definition $(G, B) \odot (G, B) = (H, C)$ where $C = B \cap B = B$ and for all $b \in B, H(b) = G(b)G(b)$. Clearly $C \subseteq A$. we show that $H(b)$ is a right ideal of $F(b)$.

Let $x \in H(b) = G(b)G(b)$ then $x = ij$ where $i, j \in G(b)$ and $y \in F(b)$ then $xy = (ij)y = (yj)i \in G(b)G(b) = H(b)$, implies that $H(b)$ is right ideal of $F(b)$. That is (H, C) is a soft right ideal of (F, A) . This implies $(G, B) \odot (G, B)$ is a soft right ideal of (F, A) . Also by theorem.5.31 $(G, B) \odot (G, B)$ is a soft left ideal of (F, A) .

Hence $(G, B) \odot (G, B)$ is a soft ideal of (F, A) .