Review of Literature

REVIEW OF LITERATURE

Over the last decades there has been a considerable interest in the Linear algebra and Operator theory in the so-called Linear preserver problems.

Linear preserver problems concern the characterizations of linear operators in matrix algebras that leave certain functions, subsets or relations invariant.

The earliest papers on the study of linear preserver problems such as Frobenius [17] date back to 1897. Frobenius showed that every linear map $f : \mathbf{M}_n(\mathbb{C}) \rightarrow \mathbf{M}_n(\mathbb{C})$ preserving the determinant has certainly the form $f(A) = PAQ$ or $f(A) = PA^tQ$, where $P,Q \in GL_n(\mathbb{C})$ with $\det(PQ) = 1$, where $\mathbf{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries in the complex number field $\mathbb{C}$ and $GL_n(\mathbb{C})$ is the general linear group. Since then linear preserver problems have been studied extensively, and many interesting results have been discovered.

During the last three decades many authors investigated linear transformations on more general algebraic structures, such as matrices over rings and semirings. There is much literature on the study of linear operators that preserve the rank of matrices over several semirings [3,6,9,10,12,14,21,24]. Nonnegative matrices also have been the subject of research by many authors [6,12,14,24].

Beasley, L.B., Gregory and Pullman, N.J., [6] obtained characterizations of linear operators which preserve the rank of nonnegative real matrices.

In [24], Song, S.Z., and Hwang, S.G., characterized spanning column ranks and their preservers of nonnegative matrices.

Beasley, L.B., Song, S.Z., Kang, K.T., and Sarma, B.K., [12] treated column ranks of nonnegative real matrices and characterized their preservers.

But there are few papers on the characterizations of linear operators preserving the perimeter of matrices.

Beasley, L.B., et al. characterized those linear operators preserving the rank and perimeter of Boolean rank-1 matrices [10].

The set of linear operators preserving the perimeter of matrices of rank k (≥ 2) over the nonnegative reals is considered by Song, S.Z., and Kang, K.T., [26].

The linear operators that strongly preserve regular matrices over semirings including the binary Boolean algebra, the nonnegative reals, the nonnegative integers and the fuzzy scalars are studied by Song, S.Z., Kang, K.T., Beasley, L.B., and Sze, N.S., [29].

Determinant, singularity, and non-singularity preservers for matrices over semirings is studied by Beasley, L.B., Guterman, A.E., Lee, S.G., and Song, S.Z., [11].

Linear operators that preserve column rank of fuzzy matrices is characterized by Song, S.Z., [23].

Inverse-preserving linear maps between spaces of matrices over fields is studied by Zhang, X., [32].

Characterizations of rank-preserving operators on field-valued matrices are determined for fuzzy matrices and for matrices over related semirings by Beasley, L.B., and Pullman, N.J., [8].

There is much literature on the study of linear preserver problem for matrices.

In this Review of Literature, a brief survey of some of the articles published on linear preserver problems for matrices are given.

1. Linear operators that preserve column rank of fuzzy matrices

Seok-Zun Song, (1994) [23]

For each $m \geq 2$ and $n \geq 3$, the linear operators, T, on the set of $m \times n$ fuzzy matrices that preserve column rank are characterized. That is, T preserves column rank if and only if T strongly preserves column rank 1 and it preserves column rank 3. Some other characterizations of column rank preserving operators are also given.

2. Regular matrices and their strong preservers over semirings

Seok-Zun Song, Kyung-Tae Kang, LeRoy B.Beasley, and Nung-Sing Sze, (2008) [29]

In this paper, the linear operators that strongly preserve regular matrices over semirings including the binary Boolean algebra, the nonnegative reals, the nonnegative integers and the fuzzy scalars are studied.

3. Linear preservers of perimeters of nonnegative real matrices

Seok-Zun Song and Kyung-Tae Kang, (2008) [26]

For a nonnegative real matrix A of rank 1, A can be factored as $\mathbf{ab}^t$ for some vectors **a** and **b**. The perimeter of A is the number of nonzero entries in both **a** and **b**. If B is a matrix of rank k, then B is the sum of k matrices of rank 1. The perimeter of B is the minimum of the sums of perimeters of k matrices of rank 1, where the minimum is taken over all possible rank-1 decompositions of B. In this paper, characterizations of the linear operators which preserve perimeters 2 and k for some $k \geq 4$ is obtained.

4. Determinant preservers for matrices over semirings

LeRoy B.Beasley, Alexander E.Guterman, Sang-Gu Lee, and Seok-Zun Song, (2003) [11]

In this paper, semiring versions of classical theorems by Frobenius and Dieudonne on determinant, singularity and non-singularity preservers are obtained.

5. Linear preservers of Boolean nilpotent matrices

Seok-Zun Song, Kyung-Tae Kang, and Young-Bae Jun, (2006) [27]

For an $n \times n$ Boolean matrix A, A is called nilpotent if $A^m \equiv O$ for some positive integer m. In this paper, linear operators that strongly preserve nilpotent matrices over Boolean algebras are characterized.

6. Linear operators strongly preserving idempotent matrices over semirings

LeRoy B.Beasley and Norman J.Pullman, (1992) [5]

In this paper, the problem of characterizing those linear operators L on the matrices over a semiring such that L(X) is idempotent if and only if X is considered. Complete characterizations are obtained for many semirings, including nonnegative reals, nonnegative integers, two element Boolean algebra, and fuzzy scalars.

7. Linear operators preserving idempotence on matrices spaces over skew-fields

Liu Shaowu and Yuan Guifang, (1997) [22]

Let R and R_1 be skew-fields with centers F and F_1 , where $F \subset F_1$ and $|F|>2$. By $\mathbf{M}_n(R)$ and $\mathbf{I}_n(R)$ we denote the F-space of all $n \times n$ matrices over R and the set of all idempotent matrices in $\mathbf{M}_n(R)$, respectively. If a linear map L from $\mathbf{M}_n(R)$ to $\mathbf{M}_m(R_1)$ satisfies $L(\mathbf{I}_n(R)) \subset \mathbf{I}_m(R_1)$ we call L an idempotence preserver. In this paper, the forms of idempotence preservers are determined.

8. General algebra and linear transformations preserving matrix invariants

Alexander E.Guterman and A.V.Mikhalev, (2006) [20]

In this peaper, the interrelations between the theory of linear transformations preserving matrix invariants and different branches of mathematics are surveyed.

9. Transformations preserving matrix invariants over semirings

Alexander E.Gutterman, (2007) [19]

In this paper, the problems corresponding to transformations preserving matrix invariants over semirings are discussed.

10. Linear maps preserving idempotence on nest algebras

Jian Lian Cui and Jin Chuan Hou, (2004) [16]

In this paper, the rank-1-preserving linear maps on nest algebras of Hilbert-space operators are discussed. Several characterizations of such linear maps are obtained.

11. Fuzzy rank-preserving operators

LeRoy B.Beasley and Norman J.Pullman, (1986) [8]

In this paper, characterizations of rank-preserving operators on field-valued matrices are determined for fuzzy matrices and for matrices over related semirings.

12. Invertible linear maps preserving {1}-inverses of matrices over principal ideal domain

Chang Jiang Bu, (2006) [15]

Let R be a principal ideal domain, chR = 2, n >1, $\mathbf{M}_n(R)$ be the $n \times n$ full matrix algebra over R. For $A \in \mathbf{M}_n(R)$, if there exists $X \in \mathbf{M}_n(R)$ such that $A \times A = A$, then X is called a {1}-inverse of A. f denotes any invertible linear map preserving {1}-inverses from $\mathbf{M}_n(R)$ to itself. In this paper, it is proved that f is an invertible linear map on $\mathbf{M}_n(R)$ preserving {1}-inverses if and only if f satisfies any one of the following two conditions :

(i) there exists a matrix $P \in GL_n(R)$ such that

$$f(A) = PAP^{-1} \quad \text{for all } A \in \mathbf{M}_n(R),$$

(ii) there exists a matrix $P \in GL_n(R)$ such that

$$f(A) = PA^tP^{-1} \quad \text{for all } A \in \mathbf{M}_n(R),$$

where, $GL_n(R)$ is the general linear group.

13. Inverse-preserving linear maps between spaces of matrices over fields

Xian Zhang, (2005) [32]

Suppose F is a field different from F_2, the field with two elements. Let $\mathbf{M}_n(F)$ and $\mathbf{S}_n(F)$ be the spaces of $n \times n$ symmetric matrices over F, respectively. For any G_1, $G_2 \in \{ \mathbf{S}_n(F), \mathbf{M}_n(F)\}$, we say that a linear map f from G_1 to G_2 is inverse-preserving if $f(X)^{-1} = f(X^{-1})$ for every invertible $X \in G_1$. Let $\mathbf{L}(G_1,G_2)$ denote the set of all inverse preserving linear maps from G_1 to G_2. In this paper, the sets $\mathbf{L}(\mathbf{S}_n(F), \mathbf{M}_n(F))$, $\mathbf{L}(\mathbf{S}_n(F), \mathbf{S}_n(F))$, $\mathbf{L}(\mathbf{M}_n(F), \mathbf{M}_n(F))$ and $\mathbf{L}(\mathbf{M}_n(F), \mathbf{S}_n(F))$ are characterized.

14. Linear maps that preserving M-P inverses of matrices between matrix spaces over fields

Hai-Yan Wu, Chong-Guang Cao, and Wei Zhang, (2008) [31]

Suppose F is a field of characteristic not 2. Let n and m be two arbitrary positive integers with $n \geq 2$. We denote by $\mathbf{M}_n(F)$ and $\mathbf{S}_n(F)$ the space of $n \times n$ full matrices and the space of $n \times n$ symmetric matrices over F, respectively. For any matrix $A \in \mathbf{M}_n(F)$, we denote by A^T the transpose of A. Assume that σ is an involutorial automorphism on F. We denote by A^+ the M-P inverse of $A \in \mathbf{M}_n(F)$ which is the unique solution of the equations :

$AXA = A$, $XAX = A$, $(AX)^* = AX$, $(XA)^* = XA$, where $A^* = (A^\sigma)^T$. All linear maps from $\mathbf{S}_n(F)$ to $\mathbf{M}_m(F)$ preserving M-P inverses of matrices are characterized first, and thereby all linear maps from $\mathbf{S}_n(F)(\mathbf{M}_n(F))$ to $\mathbf{S}_m(F)(\mathbf{M}_m(F))$ preserving M-P inverses of matrices are characterized, respectively.

15. Additive rank-1 preservers between spaces of Hermitian matrices

Xiang-Yu Cao and Xian Zhang, (2008) [18]

Suppose $\mathbb{R}$ is the real number field, $\mathbb{C}$ is the complex number field, and m, n are integers with $\min\{m,n\} \geq 2$. Denote by $\mathbf{S}_n(\mathbb{R})$ (respectively, $\mathbf{H}_n(\mathbb{C})$) the $\mathbb{R}$-linear space of all $n \times n$ real symmetric (respectively, complex Hermitian) matrices. In this paper, the structure of all additive rank-1 preservers from $\mathbf{S}_m(\mathbb{R})$ (respectively, $\mathbf{H}_m(\mathbb{C})$) to $\mathbf{H}_n(\mathbb{C})$ are described, and thereby, the general form of all additive rank preservers from $\mathbf{S}_m(\mathbb{R})$ (respectively, $\mathbf{H}_m(\mathbb{C})$) to $\mathbf{H}_n(\mathbb{C})$ is determined.