Chapter I

CHAPTER – I

TRIANGULAR FUZZY NUMBERS AND TRIANGULAR FUZZY NUMBER MATRICES

Definition : 1.1 : Triangular Fuzzy Number

A **triangular fuzzy number** denoted by $\tilde{M} = < m, \alpha, \beta >$, has the membership function

$$\mu_{\tilde{M}}(x) = \begin{cases} 0 & \text{for } x \leq m - \alpha \\ 1 - \dfrac{m-x}{\alpha} & \text{for } m - \alpha < x < m \\ 1 & \text{for } x = m \\ 1 - \dfrac{x-m}{\beta} & \text{for } m < x < m + \beta \\ 0 & \text{for } x \geq m + \beta \end{cases}$$

The point m, with membership grade of 1, is called the mean value and α, β are the left hand and right hand spreads of M respectively. Also it can be defined as follows :

A triangular fuzzy number represented as $\tilde{A} = (a^{\ell}, \tilde{a}, a^{u} : \mu_{\tilde{A}})$, where a^{ℓ}, $\tilde{a}$, a^{u} are all real values, $\mu_{\tilde{A}}$ denotes the membership grade or height and $\mu_{\tilde{A}} \in [0, 1]$. a^{ℓ}, a^{u} are the left hand and right hand spreads of the mean value $\tilde{a}$ respectively.

Definition : 1.2

A triangular fuzzy number is said to be **symmetric** if both its spreads are equal, i.e., if $\alpha = \beta$ and it is denoted by $\tilde{M} = < m, \alpha >$.

Definition : 1.3 : Standard Arithmetic Operations of Triangular Fuzzy Numbers using Function Principle

Let $\tilde{A} = < a_1, a_2, a_3 >$ and $\tilde{B} = < b_1, b_2, b_3 >$ then,

(i) **Addition :** $\tilde{A} + \tilde{B} = < a_1 + b_1, a_2 + b_2, a_3 + b_3 >$

(ii) **Subtraction :** $\tilde{A} - \tilde{B} = < a_1 - b_3, a_2 - b_2, a_3 - b_1 >$

(iii) **Multiplication :** $\tilde{A} \times \tilde{B} = < \min (a_1 b_1, a_1 b_3, a_3 b_1, a_3 b_3), a_2 b_2, \max (a_1 b_1, a_1 b_3, a_3 b_1, a_3 b_3)>$

(iv) **Division :** $\tilde{A} / \tilde{B} = < \min (a_1 / b_1, a_1 / b_3, a_3 / b_1, a_3 / b_3), a_2 / b_2, \max (a_1 / b_1, a_1 / b_3, a_3 / b_1, a_3 / b_3)>$

Definition : 1.4 : Arithmetic Operations of Triangular Fuzzy Numbers due to Dubois and Prade [14]

Let $\tilde{M} = < m, \alpha, \beta >$ and $\tilde{N} = < n, \gamma, \delta >$ be two triangular fuzzy numbers.

(1) **Addition :** $\tilde{M} + \tilde{N} = < m + n, \alpha + \gamma, \beta + \delta >$

(2) **Scalar Multiplication :** Let λ be a scalar, $\lambda\tilde{M} = < \lambda m, \lambda\alpha, \lambda\beta >$, when $\lambda \geq 0$. $\lambda\tilde{M} =< \lambda m, -\lambda\beta, -\lambda\alpha >$, when $\lambda \leq 0$. I n particular, $-\tilde{M} = < -m, \beta, \alpha >$

(3) **Subtraction :** $\tilde{M} - \tilde{N} = < m, \alpha, \beta > - < n, \gamma, \delta > = < m - n, \alpha + \delta, \beta + \gamma >$

For two TFNs $\tilde{M}$ and $\tilde{N}$, their addition, subtraction and scalar multiplication, i.e., $\tilde{M} + \tilde{N}$, $\tilde{M} - \tilde{N}$ and $\lambda\tilde{M}$ are all triangular fuzzy matrices.

(4) **Multiplication :** It can be shown that the shape of the membership function of $\tilde{M}.\tilde{N}$ is not necessarily a triangular, but if the spreads of M and N are small compared to their mean value m and n then the shape of membership function is closed to a triangle. A good approximation is as follows :

(a) When $\tilde{M} \geq 0$ and $\tilde{N} \geq 0$ ($\tilde{M} \geq 0$, if $m \geq 0$)

$\tilde{M}.\tilde{N} = \langle m, \alpha, \beta \rangle.\langle n, \gamma, \delta \rangle = \langle mn, m\gamma + n\alpha, m\delta + n\beta \rangle$

(b) When $\tilde{M} \leq 0$ and $\tilde{N} \geq 0$

$\tilde{M}.\tilde{N} = \langle m, \alpha, \beta \rangle.\langle n, \gamma, \delta \rangle = \langle mn, n\alpha - m\delta, n\beta - m\gamma \rangle$

(c) When $\tilde{M} \leq 0$ and $\tilde{N} \leq 0$

$\tilde{M}.\tilde{N} = \langle m, \alpha, \beta \rangle.\langle n, \gamma, \delta \rangle = \langle mn, -n\beta - m\delta, -n\alpha - m\gamma \rangle$

When spreads are not small compared with mean values, the following is a better approximation :

$\langle m, \alpha, \beta \rangle. \langle n, \gamma, \delta \rangle = \langle mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta \rangle$ for $\tilde{M} > 0$, $\tilde{N} > 0$.

Throughout this chapter we use the previous definition.

Also, $\langle m, \alpha, \beta \rangle.\langle 0, 0, 0 \rangle = \langle 0, 0, 0 \rangle$

Now we define inverse of a triangular fuzzy number based on the definition of multiplication.

(5) Inverse : The inverse of a triangular fuzzy number $\tilde{M} = \langle m, \alpha, \beta \rangle$, $m > 0$ is defined as, $\tilde{M}^{-1} = \langle m, \alpha, \beta \rangle^{-1} = \langle m^{-1}, \beta m^{-2}, \alpha m^{-2} \rangle$.

This is also an approximate value of $\tilde{M}^{-1}$ and it is valid only a neighbourhood of 1 / m. Division of $\tilde{M}$ by $\tilde{N}$ is given by

$$\frac{\tilde{M}}{\tilde{N}} = \tilde{M}.\tilde{N}^{-1}$$

Since inverse and product both are approximate, the division is also an approximate value. The formal definition of division is given below.

(6) Division

$$\frac{\tilde{M}}{\tilde{N}} = \tilde{M}.\tilde{N}^{-1} = \langle m, \alpha, \beta \rangle.\langle n^{-1}, \delta n^{-2}, \gamma n^{-2} \rangle = \left\langle \frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right\rangle$$

From the definition of multiplication of triangular fuzzy numbers, the power of any triangular fuzzy number $\tilde{M}$ is defined in the following way.

(7) Exponentiation

Using the definition of multiplication it can be shown that M^n is given by

$$\tilde{M}^n = < m, \alpha, \beta >^n$$
$$= < m^n, -nm^{n-1}\beta, -nm^{n-1}\alpha >, \text{ when n is negative,}$$
$$= < m^n, nm^{n-1}\alpha, nm^{n-1}\beta >, \text{ when n is positive.}$$

Remark : 1.5

Consider two triangular fuzzy numbers with a common mean value. Then subtraction produces a triangular fuzzy number whose mean value is zero and the spreads are the sum of both the spreads of computed triangular fuzzy number. The quotient of same triangular fuzzy numbers is a triangular fuzzy number having mean value one. The inverse of a triangular fuzzy number whose mean value is zero does not exist and we cannot divide by such a number. The addition and multiplication of triangular fuzzy numbers are both commutative and associative. But the distributive law does not always hold.

For example, if $\tilde{A} = < 2, 0.5, 0.5 >$, $\tilde{B} = < 3, 0.8, 0.7 >$ $\tilde{C} = < 5, 1, 2 >$ and $\tilde{D} = < -5, 2, 1 >$, then

$$\tilde{A}.(\tilde{B}+\tilde{C}) = \tilde{A}.\tilde{B} + \tilde{A}.\tilde{C} \text{ holds but, } \tilde{A}.(\tilde{C}+\tilde{D}) \neq \tilde{A}.\tilde{C} + \tilde{A}.\tilde{D}$$

Definition : 1.6 : Triangular Fuzzy Number Matrix (TFNM)

A **triangular fuzzy number matrix** of order m x n is defined as $A = (\tilde{M}_{ij})_{mxn}$, where $\tilde{M}_{ij} = < m_{ij}, \alpha_{ij}, \beta_{ij} >$ is the ij^{th} element of A, m_{ij} is the mean value of $\tilde{M}_{ij}$ and α_{ij}, β_{ij} are the left and right spreads of $\tilde{M}_{ij}$ respectively.

Definition : 1.7 : Operations on TFNM

Let $A = (\tilde{A}_{ij})$ and $B = (\tilde{B}_{ij})$ be two triangular fuzzy number matrices of same order. Then we have the following.

(i) $A + B = (\tilde{A}_{ij} + \tilde{B}_{ij})$

(ii) $A - B = (\tilde{A}_{ij} - \tilde{B}_{ij})$

(iii) For $A = (\tilde{A}_{ij})_{mxn}$ and $B = (\tilde{B}_{ij})_{nxp}$, $A.B = (\tilde{C}_{ij})_{mxp}$, where $\tilde{C}_{ij} = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots p$.

(iv) $A' = (\tilde{A}_{ji})$ (the transpose of A)

(v) $k.A = (k\tilde{A}_{ij})$, where k is a scalar.

Definition : 1.8

A triangular fuzzy number matrix is said to be a

(1) **pure null triangular fuzzy number matrix** if all its entries are zero, i.e., all elements are < 0, 0, 0 >. This matrix is denoted by 0.

(2) **fuzzy null triangular fuzzy number matrix** if all elements are of the form $\tilde{A}_{ij} = < 0, \xi_1, \xi_2 >$, where $\xi_1.\xi_2 \neq 0$.

Definition : 1.9

A square triangular fuzzy number matrix is said to be a

(1) **pure unit triangular fuzzy number matrix** if $\tilde{A}_{ii} = < 1, 0, 0 >$ and $\tilde{A}_{ij} = < 0, 0, 0 >$, $i \neq j$, for all i, j. It is denoted by I.

(2) **fuzzy unit triangular fuzzy number matrix** if $\tilde{A}_{ii} = < 1, \xi_1, \xi_2 >$ and $\tilde{A}_{ij} = < 0, \xi_3, \xi_4 >$ for $i \neq j$, for all i, j, where $\xi_1 . \xi_2 \neq 0$, $\xi_3.\xi_4 \neq 0$.

(3) **pure triangular TFNM** if either $\tilde{A}_{ij} = < 0, 0, 0 >$ for all $i > j$ or $\tilde{A}_{ij} = < 0, 0, 0 >$ for all $i < j$, $i, j = 1, 2, \ldots, n$.

(4) **fuzzy triangular TFNM** if either $\tilde{A}_{ij} = < 0, \xi_1, \xi_2 >$ for all $i > j$ or $\tilde{A}_{ij} = < 0, \xi_1, \xi_2 >$ for all $i < j$; $i, j = 1, 2, \ldots, n$ and $\xi_1.\xi_2 \neq 0$.

(5) **symmetric TFNM** if A = A′, i.e., if $\tilde{A}_{ij} = \tilde{A}_{ji}$ for all i, j.

(6) **pure Skew-Symmetric** if A= – A′ and $\tilde{A}_{ii}$ = < 0, 0, 0 >, i.e., if $\tilde{A}_{ij} = -\tilde{A}_{ji}$ for all i, j and $\tilde{A}_{ii}$ = < 0, 0, 0 >.

(7) **fuzzy Skew-Symmetric** if A = – A′ and $\tilde{A}_{ii}$ = < 0, ξ_1, ξ_2 >, i.e., if $\tilde{A}_{ij} = -\tilde{A}_{ji}$ for all i, j and $\tilde{A}_{ii}$ = < 0, ξ_1, ξ_2 >, $\xi_1.\xi_2 \neq 0$.

Definition : 1.10

A pure triangular TFNM A = $(\tilde{A}_{ij})$ is said to be **pure upper triangular TFNM** when $\tilde{A}_{ij}$ = < 0, 0, 0 > for all i > j and is said to be a **pure lower triangular TFNM** if $\tilde{A}_{ij}$ = < 0, 0, 0 > for all i < j.

Property : 1.11

For any three TFNMs A, B and C of order m x n and scalars k, ℓ we have :

(1) A + B = B + A,

(2) A + (B + C) = (A + B) + C,

(3) A + A = 2A,

(4) A – A is a fuzzy null TFNM,

(5) A + 0 = A – 0 = A,

(6) (A′)′ = A,

(7) (A + B)′ = A′ + B′,

(8) (A.B)′ = B′.A′,

(9) k(ℓA) = (kℓ) A,

(10) k(A + B) = kA + kB,

(11) (k + ℓ) A = kA + ℓA

(12) k(A – B) = kA – kB

(13) (k.A)′ = k.A′,

(14) (k.A + ℓ.B)′ = k.A′ + ℓ.B′.

Property : 1.12

Let A be a square TFNM then

(i) A.A′ and A′.A are both symmetric,

(ii) A + A′ is symmetric,

(iii) A – A′ is fuzzy skew-symmetric.

Definition : 1.13 : Trace of TFNM

The trace of a square TFNM A = $(\tilde{A}_{ij})$, denoted by tr(A), is the sum of the principal diagonal elements. In other words, tr(A) = $\sum_{i=1}^{n} \tilde{A}_{ii}$.

Property : 1.14

Let A = $(\tilde{A}_{ij})$and B = $(\tilde{B}_{ij})$be any two square TFNMs of order n x n then,

(i) tr(A) + tr(B) = tr(A + B)

(ii) tr(A) = tr(A′)

(iii) tr(A.B) = tr(B.A).

Proof

(i) Let A = $(\tilde{A}_{ij})_{nxn}$ and B = $(\tilde{B}_{ij})_{nxn}$ be two TFNMs where $\tilde{A}_{ij}$ = < m_{ij}, α_{ij}, β_{ij} > and $\tilde{B}_{ij}$ = < n_{ij}, γ_{ij}, δ_{ij} >. Now, tr(A) = $\sum_{i=1}^{n} \tilde{A}_{ii}$ = < M, T_L, T_U >, where $M = \sum_{i=1}^{n} m_{ii}$, $T_L = \sum_{i=1}^{n} \alpha_{ii}$ and $T_U = \sum_{i=1}^{n} \beta_{ii}$.

Similarly, tr(B) = < N, P_L, P_U > where $N = \sum_{i=1}^{n} n_{ii}$, $P_L = \sum_{i=1}^{n} \gamma_{ii}$ and $P_U = \sum_{i=1}^{n} \delta_{ii}$.

Therefore, tr(A) + tr(B) = < M, T_L, T_U > + < N, P_L, P_U >

= < M + N, $T_L + P_L$, $T_U + P_U$ >.

Again, let A + B = D = $(\tilde{D}_{ij})$, where

$\tilde{D}_{ij} = < m_{d_{ij}}, \alpha_{d_{ij}}, \beta_{d_{ij}} >$, $m_{d_{ij}} = m_{ij} + n_{ij}$, $\alpha_{d_{ij}} = \alpha_{ij} + \gamma_{ij}$ and

$\beta_{d_{ij}} = \beta_{ij} + \delta_{ij}$.

$$\text{Now, tr(D)} = \sum_{i=1}^{n} \tilde{D}_{ii} = \sum_{i=1}^{n} < m_{ij} + n_{ij}, \alpha_{ij} + \gamma_{ij}, \beta_{ij} + \delta_{ij} >$$

$$= < M + N, T_L + P_L, T_U + P_U >$$

Hence, tr(A) + tr(B) = tr(A + B).

(ii) The proof is trivial.

(iii) Let A = $(\tilde{A}_{ij})$ and B = $(\tilde{B}_{ij})$ be two TFNMs of order n x n, where $\tilde{A}_{ij} = < m_{ij}, \alpha_{ij}, \beta_{ij} >$ and $\tilde{B}_{ij} = < n_{ij}, \gamma_{ij}, \delta_{ij} >$. Also, let C = A.B = $(\tilde{C}_{ij})$, then $\tilde{C}_{ij} = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$

$$\text{Now, tr(A.B)} = \sum_{i=1}^{n} \tilde{C}_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{ki}$$

Again, let B.A = D = $(\tilde{D}_{ij})$where $\tilde{D}_{ij} = \sum_{k=1}^{n} \tilde{B}_{ik}.\tilde{A}_{kj}$.

$$\text{Now tr(B.A)} = \sum_{i=1}^{n}\sum_{k=1}^{n} \tilde{B}_{ik}.\tilde{A}_{ki}$$

By intercharging the indices i and k, we get

$$\text{tr(B.A)} = \sum_{k=1}^{n}\sum_{i=1}^{n} \tilde{A}_{ik}.\tilde{B}_{ki}$$

Hence, tr(A.B) = tr(B.A).

Property : 1.15

The product of two pure upper triangular TFNMs of order n x n is a pure upper triangular TFNM.

Proof

Let $A = (\tilde{A}_{ij})$ and $B = (\tilde{B}_{ij})$ be two upper triangular TFNMs where $\tilde{A}_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $\tilde{B}_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$. Since A and B are pure upper triangular TFNMs then $\tilde{A}_{ij} = \langle 0, 0, 0 \rangle$ and $\tilde{B}_{ij} = \langle 0, 0, 0 \rangle$ for all $i > j$; $i, j = 1, 2, \ldots, n$.

Let $A.B = C = (\tilde{C}_{ij})$, where $\tilde{C}_{ij} = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$

$$= \sum_{k=1}^{n} \langle m_{ik}, \alpha_{ik}, \beta_{ik} \rangle . \langle n_{kj}, \gamma_{kj}, \delta_{kj} \rangle$$

We shall now show that $\tilde{C}_{ij} = \langle 0, 0, 0 \rangle$ if $i > j$; $i, j = 1, 2, \ldots, n$.

For $i > j$ we have $\tilde{A}_{ik} = \langle 0, 0, 0 \rangle$ for $k = 1, 2, \ldots, i-1$ and similarly $\tilde{B}_{kj} = \langle 0, 0, 0 \rangle$ for $k = i, i+1, \ldots, n$.

Therefore,
$$\tilde{C}_{ij} = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$$
$$= \sum_{k=1}^{i-1} \tilde{A}_{ik}.\tilde{B}_{kj} + \sum_{k=i}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$$
$$= \langle 0, 0, 0 \rangle$$

Now
$$\tilde{C}_{ii} = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj}$$
$$= \sum_{k=1}^{i-1} \tilde{A}_{ik}.\tilde{B}_{kj} + \tilde{A}_{ii} . \tilde{B}_{ii} + \sum_{k=i+1}^{n} \tilde{A}_{ik}.\tilde{B}_{ki}$$
$$= \langle 0, 0, 0 \rangle$$
$$= \tilde{A}_{ii}.\tilde{B}_{ii}, \text{ since } \tilde{A}_{ik} = \langle 0, 0, 0 \rangle \text{ for } k = 1, 2, \ldots, i-1$$

and

$$\tilde{B}_{ki} = \langle 0, 0, 0 \rangle \text{ for } k = i+1, i+2, \ldots, n.$$

Hence the result follows.

Property : 1.16

The product of two pure lower triangular TFNMs of order n x n is also a pure lower triangular TFNMs.

Definition : 1.17 : Determinant of TFNM

The triangular fuzzy determinant of a TFNM A of order n x n is denoted by | A | or det(A) and is defined as,

$$|A| = \sum_{\sigma \in S_n} \text{Sgn}\,\sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > \cdots\cdots\cdots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$

$$= \sum_{\sigma \in S_n} \text{Sgn}\,\sigma \prod_{i=1}^{n} \tilde{A}_{i\sigma(i)},$$

where $\tilde{A}_{i\sigma(i)} = < m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} >$ are TFNs and S_n denotes the symmetric group of all permutations of the indices {1, 2, ..., n} and Sgn σ = 1 or − 1 according as the permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$ is even or odd respectively.

Definition : 1.18 : Adjoint

Let A = $(\tilde{A}_{ij})$ be a square TFNM and B = (A_{ij}) be a square TFNM whose elements are the cofactors of the corresponding elements in | A | then the transpose of B is called the adjoint or adjugate of A and it is equal to (A_{ji}). The adjoint of A is denoted by adj(A).

Property : 1.19

Let A = $(\tilde{A}_{ij})$ be a TFNM of order n x n.

(i) If all the elements of a row (column) of A are < 0, 0, 0 >, then | A | = < 0, 0, 0 >.

(ii) If a row (column) be multiplied by a scalar k, then | A | is multiplied by k.

(iii) If A is triangular TFNM, then

$$| A | = \prod_{i=1}^{n} < m_{ii}, \alpha_{ii}, \beta_{ii} >.$$

Proof

(i) Let A = $\tilde{A}_{ij}$ be a square TFNM of order n x n where $\tilde{A}_{ij}$ = < m_{ij}, α_{ij}, β_{ij} >.

We define the determinants $E_1, E_2, \ldots, E_n$ as follows :

$E_i(A) = \sum_{j=1}^{n} \tilde{A}_{ij} . A_{ij} = \sum_{j=1}^{n} < m_{ij}, \alpha_{ij}, \beta_{ij} >. A_{ij}$, where A_{ij} is the cofactor of $\tilde{A}_{ij}$ in the determinant A.

Obviously, $E_1(A) = E_2(A) = \ldots = E_n(A)$ = | A |. Let all the elements of r^{th} row, $1 \leq r \leq n$, be < 0, 0, 0 >. Then $E_r(A)$ = < 0, 0, 0 > since a_{rj} = < 0, 0, 0 > for all j = 1, 2, …, n.

Therefore, | A | = $E_r(A)$ = < 0, 0, 0 >.

(ii) If k = 0, then the result is obviously true since | A | = < 0, 0, 0 > when A has a zero row.

Let B = $(\tilde{B}_{ij})_{n \times n}$ where $\tilde{B}_{ij}$ = < n_{ij}, γ_{ij}, δ_{ij} > be obtained from an n x n TFNM A $= (\tilde{A}_{ij})$ by multiplying its r^{th} row by a scalar $k \neq 0$. Obviously < n_{ij}, γ_{ij}, δ_{ij} > = < m_{ij}, α_{ij}, β_{ij} >, for all $i \neq 0$ and < n_{rj}, γ_{rj}, δ_{rj} > = < km_{rj}, $k\alpha_{rj}$, $k\beta_{rj}$ >, when k is positive and < n_{rj}, γ_{rj}, δ_{rj} > = < km_{rj}, $k\beta_{rj}$, $k\alpha_{rj}$ >, when k is negative.

Then by definition,

$$| B | = \sum_{\sigma \in S_n} \text{Sgn}\ \sigma < n_{1\sigma(1)}, \gamma_{1\sigma(1)}, \delta_{1\sigma(1)} > < n_{2\sigma(2)}, \gamma_{2\sigma(2)}, \delta_{2\sigma(2)} > \ldots\ldots < n_{r\sigma(r)}, \gamma_{r\sigma(r)}, \delta_{r\sigma(r)} > \ldots\ldots < n_{n\sigma(n)}, \gamma_{n\sigma(n)}, \delta_{n\sigma(n)} >.$$

When k is positive scalar,

$$| B | = \sum_{\sigma \in S_n} \text{Sgn } \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} >$$

$$< m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} > \ldots\ldots\ldots < km_{r\sigma(r)}, k\alpha_{r\sigma(r)}, k\beta_{r\sigma(r)} > \ldots\ldots\ldots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$

$$= k \sum_{\sigma \in S_n} \text{Sgn } \sigma \prod_{i=1}^{n} < m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} >$$

$$= k | A |.$$

When k is negative scalar,

$$| B | = \sum_{\sigma \in S_n} \text{Sgn } \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} >$$

$$< m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} > \ldots\ldots\ldots < km_{r\sigma(r)}, k\beta_{r\sigma(r)}, k\alpha_{r\sigma(r)} > \ldots\ldots\ldots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$

$$= k \sum_{\sigma \in S_n} \text{Sgn } \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} >$$

$$< m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} > \ldots\ldots\ldots < m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \beta_{r\sigma(r)} > \ldots\ldots\ldots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$

$$= k \sum_{\sigma \in S_n} \text{Sgn } \sigma \prod_{i=1}^{n} < m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} >$$

$$= k | A |$$

Hence the result follows.

(iii) Let $A = (\tilde{A}_{ij})_{nxn}$ be a square triangular TFNM for $i < j$, i.e., $a_{ij} = < 0, 0, 0 >$ for $i < j$. Take a term t of | A |,

$$t = < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > < m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} > \ldots\ldots\ldots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >.$$

Let $\sigma(1) \neq 1$, i.e., $1 < \sigma(1)$ and so that, $m_{1\sigma(1)} = 0$, $\alpha_{1\sigma(1)} = 0$ and $\beta_{1\sigma(1)} = 0$. Consequently, t = < 0, 0, 0 >.

Now, let $\sigma(1) = 1$ but $\sigma(2) \neq 2$. Then it is obvious that $\sigma(2) > 2$ therefore, $m_{2\sigma(2)} = 0$, $\alpha_{2\sigma(2)} = 0$ and $\beta_{2\sigma(2)} = 0$ and hence t = < 0, 0, 0 >. This means that each term of | A | is < 0, 0, 0 > if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$.

i.e., $m_{i\sigma(i)} = 0$, $\alpha_{i\sigma(i)} = 0$ and $\beta_{i\sigma(i)} = 0$ for $\sigma(i) \neq i$.

Therefore, $| A | = < m_{11}, \alpha_{11}, \beta_{11} > < m_{22}, \alpha_{22}, \beta_{22} > \ldots\ldots < m_{nn}, \alpha_{nn}, \beta_{nn} >$

$$= \prod_{i=1}^{n} < m_{ii}, \alpha_{ii}, \beta_{ii} >.$$

Property : 1.20

Let A be a square TFNM if any two rows (or columns) of A are interchanged then determinant | A | of A changes the sign of | A |.

Proof

Let $A = (\tilde{A}_{ij})$ be a TFNM of order n x n. If $B = (\tilde{B}_{ij})_{nxn}$ is obtained from A by interchanging the r^{th} and s^{th} row (r < s) of A, then it is clear that $\tilde{B}_{ij} = \tilde{A}_{ij}$, $i \neq r$, $i \neq s$ and $\tilde{B}_{rj} = \tilde{A}_{sj}$, $\tilde{B}_{sj} = \tilde{A}_{rj}$

Now,

$$| B | = \sum_{\sigma \in S_n} \text{Sgn}\,\sigma\; \tilde{B}_{1\sigma(1)}\, \tilde{B}_{2\sigma(2)} \ldots \tilde{B}_{r\,\sigma(r)} \ldots \tilde{B}_{s\sigma(s)} \ldots \tilde{B}_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{Sgn}\,\sigma\; \tilde{A}_{1\sigma(1)}\, \tilde{A}_{2\sigma(2)} \ldots \tilde{A}_{s\sigma(r)} \ldots \tilde{A}_{r\sigma(s)} \ldots \tilde{A}_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{Sgn}\,\sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > < m_{s\sigma(r)}, \alpha_{s\sigma(r)}, \beta_{s\sigma(r)} > \ldots$$

$$< m_{r\sigma(s)}, \alpha_{r\sigma(s)}, \beta_{r\sigma(s)} > \ldots\ldots < m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >.$$

$$\text{Let } \lambda = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n \\ 1 & 2 & \ldots & s & \ldots & r & \ldots & n \end{pmatrix}$$

Then λ is a transposition interchanging r and s and Sgn $\lambda = -1$.

Let $\sigma\lambda = \phi$. As σ runs through all permutations on {1, 2, …, n}, ϕ also runs over the same permutations, because $\sigma_1 \lambda = \sigma_2 \lambda$ or $\sigma_1 = \sigma_2$.

Now,

$$\phi = \sigma\lambda$$

$$= \begin{pmatrix} 1 & 2 & \dots & r & \dots & s & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(r) & \dots & \sigma(s) & \dots & \sigma(n) \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & \dots & r & \dots & s & \dots & n \\ 1 & 2 & \dots & s & \dots & r & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & r & \dots & s & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(s) & \dots & \sigma(r) & \dots & \sigma(n) \end{pmatrix}$$

Therefore $\phi i = \sigma i$, $i \neq r$, $i \neq s$ and $\phi(r) = \sigma(s)$, $\phi(s) = \phi(r)$.

$\because$ λ is odd, ϕ is even or odd according as ϕ is odd or even. Therefore, $\text{Sgn}\ \phi = -\ \text{Sgn}\ \sigma$.

Then,

$$|B| = \sum_{\sigma \in S_n} \text{Sgn}\ \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > < m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} > \dots$$

$$< m_{s\sigma(r)}, \alpha_{s\sigma(r)}, \beta_{s\sigma(r)} > \dots < m_{r\sigma(s)}, \alpha_{r\sigma(s)}, \beta_{r\sigma(s)} > \dots$$

$$< m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$

$$= -\sum_{\sigma \in S_n} \text{Sgn}\ \phi < m_{1\phi(1)}, \alpha_{1\phi(1)}, \beta_{1\phi(1)} > < m_{2\phi(2)}, \alpha_{2\phi(2)}, \beta_{2\phi(2)} > \dots$$

$$< m_{s\phi(r)}, \alpha_{s\phi(r)}, \beta_{s\phi(r)} > \dots < m_{n\phi(n)}, \alpha_{n\phi(n)}, \beta_{n\phi(n)} >$$

$$= -|A|.$$

Property : 1.21

If A is a square TFNM then $|A| = |A'|$.

Proof

Let $A = (\tilde{A}_{ij})_{nxn}$ be a square TFNM and $A' = B = (\tilde{B}_{ij})_{nxn}$. Then,

$$|B| = \sum_{\sigma \in S_n} \text{Sgn}\ \sigma\ \tilde{B}_{1\sigma(1)} . \tilde{B}_{2\sigma(2)} \dots \tilde{B}_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{Sgn}\ \sigma\ \tilde{A}_{\sigma(1)1} . \tilde{A}_{\sigma(2)2} \dots \tilde{A}_{\sigma(n)n}$$

Let ϕ be the permutation of {1, 2, ..., n} such that $\phi\sigma = I$, the identity permutation. Then $\phi = \sigma^{-1}$. Since σ runs over the whole set of permutations, ϕ also runs over the same set of permutations. Let $\sigma(i) = j$ then $i = \sigma^{-1}(j)$ and $a_{\sigma(i)i} = a_{j\phi(j)}$ for all i, j. Therefore,

$$
\begin{aligned}
|B| &= \sum_{\sigma\in S_n} \text{Sgn}\,\sigma < m_{\sigma(1)1}, \alpha_{\sigma(1)1}, \beta_{\sigma(1)1} > < m_{\sigma(2)2}, \alpha_{\sigma(2)2}, \beta_{\sigma(2)2} > \ldots \\
&\quad < m_{\sigma(n)n}, \alpha_{\sigma(n)n}, \beta_{\sigma(n)n} > \\
&= \sum_{\phi\in S_n} \text{Sgn}\,\phi < m_{1\phi(1)}, \alpha_{1\phi(1)}, \beta_{1\phi(1)} > < m_{2\phi(2)}, \alpha_{2\phi(2)}, \beta_{2\phi(2)} > \ldots \\
&\quad < m_{n\phi(n)}, \alpha_{n\phi(n)}, \beta_{n\phi(n)} > \\
&= |A|
\end{aligned}
$$

Hence, $|A| = |A'|$.

Property : 1.22

Let A be a square TFNM of order n x n,

(i) If A is symmetric then adj(A) is symmetric.

(ii) If A is a fuzzy null TFNM then adj(A) is a fuzzy null TFNM.

(iii) If A is a pure unit TFNM then adj(A) is a pure unit TFNM.

(iv) If A is a fuzzy unit TFNM then adj(A) is a fuzzy unit TFNM.

Property : 1.23

Let A be a square TFNM of order n x n,

(i) If A contains a zero row then adj(A). A is a fuzzy null TFNM.

(ii) adj(A′) = (adj(A))′.

Proof

(i) Let $A = (\tilde{A}_{ij})$ be a square TFNM or order n x n, where $\tilde{A}_{ij} = < m_{ij}, \alpha_{ij}, \beta_{ij} >$ and B = adj(A). Then by the definition of adjoint matrix, the ij^{th} element $\tilde{B}_{ij}$ of B is $|A_{ji}|$, where $\tilde{A}_{ij}$ is the sub-matrix obtained from A by suppressing the i^{th} row and j^{th} column. That is, $|A_{ij}|$ is the cofactors of $\tilde{A}_{ij}$ in A. Without loss of generality we assume that the ℓ^{th} row of A be the zero row. Therefore, the elements of the ℓ^{th} row are of the form $\tilde{A}_{\ell j} = < 0, \alpha_{ij}, \beta_{ij} >$ for all j. Then all the elements of adj(A) are of the form $|A_{ij}| = < 0, \alpha_{ij}^*, \beta_{ij}^* >$ except $j \neq \ell$.

Let C = adj(A) . A.

Then the ij^{th} element

$\tilde{C}_{ij}$ of C is $\tilde{C}_{ij} = \sum_{k=1}^{n} |A_{ik}| \tilde{A}_{kj} = \sum_{k \neq \ell} |A_{ik}| \tilde{A}_{kj} + |A_{i\ell}| \tilde{A}_{\ell j}$

Now, all $|A_{ik}|$, $k \neq \ell$ are of the form $< 0, \alpha_{ij}^*, \beta_{ij}^* >$ and $\tilde{A}_{\ell j} = < 0, \alpha_{ij}, \beta_{ij} >$. Hence $\tilde{C}_{ij}$ is of the form $< 0, \gamma_{ij}, \delta_{ij} >$ for all i, j = 1, 2, …, n. Thus C, i.e., adj(A). A is a fuzzy null TFNM.

(ii) The proof is straight forward.

Definition : 1.24 : Singular TFNM

Let A be a square TFNM. Then

(1) **singular** if $|A| = < 0, 0, 0 >$.

(2) **semi-singular** if the value of $|A|$ is of the form $< 0, \alpha, \beta >$.

(3) **constant** if all its rows are equal to each other, i.e., if $< m_{ik}, \alpha_{ik}, \beta_{ik} > = < m_{jk}, \alpha_{jk}, \beta_{jk} >$ for all i, j, k.

Property : 1.25

If $A = (\tilde{A}_{ij})$ is a constant TFNM of order n x n and B is a TFNM of the same order, then A.B is a constant TFNM.

Proof

Suppose A = $(\tilde{A}_{ij})$ is a constant TFNM of order n x n and B = $(\tilde{B}_{ij})$ is a TFNM of the same order, where $\tilde{A}_{ij}$ = < m_{ij}, α_{ij}, β_{ij} > and $\tilde{B}_{ij}$ = < n_{ij}, γ_{ij}, δ_{ij} >. Then $\tilde{A}_{ik}$ = < m_{ik}, α_{ik}, β_{ik} >, are the same for i, j, k ∈ {1, 2, …, n}. Let

$$A.B = \sum_{k=1}^{n} \tilde{A}_{ik}.\tilde{B}_{kj} = D = (\tilde{D}_{ij})$$

Then

$$\tilde{D}_{ij} = < p_{ij}, \mu_{ij}, v_{ij} >, \text{ where } p_{ij} = \sum_{k=1}^{n} m_{ik}\, n_{kj},$$

$$\mu_{ij} = \sum_{k=1}^{n} \{m_{ik}\,\gamma_{kj} + n_{ik}\,\alpha_{kj}\} \text{ and } v_{ij} = \sum_{k=1}^{n} \{m_{ik}\,\delta_{kj} + n_{ik}\,\beta_{kj}\}.$$

A is a constant TFNM, i.e., $m_{ik} = m_{jk}$ for all i, j, k. This implies that $p_{11} = p_{21} = \ldots = p_{n1}$, $p_{12} = p_{22} = \ldots = p_{n2}$, … $p_{1n} = p_{2n} = \ldots = p_{nn}$, i.e., p_{ij} is independent of i, i ∈ {1, 2, …, n}.

Again, μ_{11} = < $m_{11}\,\gamma_{11} + n_{11}\,\alpha_{11}$ > + < $m_{12}\,\gamma_{21} + n_{21}\,\alpha_{12}$ > + … + < $m_{1n}\,\gamma_{n1} + n_{n1}\,\gamma_{1n}$ > = $\mu_{21} = \ldots = \mu_{n1}$, $\mu_{12} = \mu_{22} = \ldots = \mu_{n2}$, …, $\mu_{1n} = \mu_{2n} = \ldots = \mu_{nn}$.

Similarly, $v_{11} = v_{21} = \ldots = v_{n1}$, $v_{12} = v_{22} = \ldots = v_{n2}$, …, $v_{1n} = v_{2n} = \ldots = v_{nn}$.

Therefore, $\tilde{D}_{ij}$ is independent of i, i ∈ {1, 2, …, n} since < m_{ik}, α_{ik}, β_{ik} > = < m_{jk}, α_{jk}, β_{jk} > for all i, j, k.

Hence, A.B is constant.

Property : 1.26

If A is a square constant TFNM then,

(i) A. (adj A) is constant,

(ii) (adj A)′ is constant,

(iii) A. (adj A′) is constant,

(iv) (A′.(adj A))′ is constant,

(v) (adj A)′. A is constant,

(vi) | A | is of the form < 0, α, β >.

Property : 1.27

If A, B be two triangular fuzzy number matrices then det (A) . det (B) may not be equal to det (A . B). If they are distinct, then they differ only by their spreads but the mean will be the same in both.

Proof

If A = $(\tilde{A}_{ij})_{nxn}$ and B = $(\tilde{B}_{ij})_{nxn}$ where $\tilde{A}_{ij}$'s and $\tilde{B}_{ij}$'s are triangular fuzzy numbers, then det (A) = $\sum_{j=1}^{n} \text{Sgn}\,\sigma_j \prod_{i=1}^{n} \tilde{A}_{i\sigma_j(i)}$, where σ_j's are all possible permutation over {1, 2, 3, …, n}.

Clearly, det (A) is the sum of n triangular fuzzy numbers. Let P_j = Sgn $\sigma_j \prod_{i=1}^{n} \tilde{A}_{i\sigma_j(i)}$. Now since Sgn σ_j is either 1 or – 1 then some of the P_j's will be negative so that the sum det (A) contains some subtractions.

Similarly, if B = $(\tilde{B}_{ij})_{nxn}$ then det (B) = $\sum_{j=1}^{n} Q_j$ where Q_j = Sgn $\phi_j \prod_{i=1}^{n} \tilde{B}_{i\phi_j(i)}$

Where ϕ_j runs over all permutations in {1, 2, 3, …, n}. Here also some Q_j's will be negative.

Again A . B = $\left(\sum_{k=1}^{n} \tilde{A}_{ik} . \tilde{B}_{kj}\right)$ = (C_{ij}) (say). Then, det (A . B) = $\sum_{j=1}^{n} \text{Sgn}\,\Psi_j \prod_{i=1}^{n} \tilde{C}_{i\Psi_j(i)}$ where Ψ_j runs over all permutations in {1, 2, 3, …, n}. Thus det (A . B) contains addition and subtraction of only n triangular fuzzy numbers. Since det (A) = $\sum_{j=1}^{n} P_j$ and det (B) = $\sum_{j=1}^{n} Q_j$ then det (A) . det (B) is the product of two expressions each containing n number of triangular fuzzy number among which some are negative.

Since multiplication is not distributive over subtraction in triangular fuzzy numbers, i.e., a . (b − c) ≠ a . b − a . c, in general, for any three triangular fuzzy numbers a, b, c. Thus the proof of theorem follows.

Hence proved.

Definition : 1.28

A TFNM of order m x n is defined as $A = (\tilde{A}_{ij})_{mxn}$, where $\tilde{A}_{ij} = \langle a_{ij}^{\ell}, \tilde{a}_{ij}, a_{ij}^{u} \rangle$ is the ij[th] element of A, $\tilde{A}_{ij}$ is the mean value of A and $a_{ij}^{\ell}, a_{ij}^{u}$ are the left and right spreads of $\tilde{A}_{ij}$ respectively. It is said to be **generalized triangular fuzzy number matrix (GTFNM)** if $a_{ij}^{\ell} \le \tilde{a}_{ij} \le a_{ij}^{u}$.

In this chapter we proposed a method to make a score value by standardizing each element $\tilde{A}_{ij} = \langle a_{ij}^{\ell}, \tilde{a}_{ij}, a_{ij}^{u} \rangle$ of a TFNM A as follows :

Step 1 : Each generalized TFN is standardized as follows :

$$\tilde{a}_{ij} = \left\langle \frac{a_{ij}^{\ell}}{a_{ij}^{u}}, \frac{\tilde{a}_{ij}}{a_{ij}^{u}}, \frac{a_{ij}^{u}}{a_{ij}^{u}} \right\rangle$$

$$= \langle a_{ij}^{\ell*}, \tilde{a}_{ij}^{*}, a_{ij}^{u*} \rangle$$

Step 2 : Calculate the defuzified value $x_{\tilde{a}_{ij}}^{*}$ using the following rule :

$$x_{\tilde{a}_{ij}}^{*} = \frac{a_{ij}^{\ell*} + \tilde{a}_{ij}^{*} + a_{ij}^{u*}}{3}$$

Step 3 : Calculate the spread std $\tilde{a}_{ij}^{*}$ of the $\tilde{a}_{ij}^{*}$ as follows :

$$\text{Std } \tilde{a}_{ij}^{*} = \sqrt{\frac{(a_{ij}^{\ell*} - x_{\tilde{a}_{ij}}^{*})^2 + (\tilde{a}_{ij}^{*} - x_{\tilde{a}_{ij}}^{*})^2 + (a_{ij}^{u*} - x_{\tilde{a}_{ij}}^{*})^2}{3}}$$

Step 4 : Calculate score ($\tilde{a}_{ij}^*$), the score value of the standardized generalized TFN as follows :

$$\text{Score}\,(\tilde{a}_{ij}^*) = x_{\tilde{a}_{ij}}^* \,.\, (1 - \alpha \,.\, \text{std}\ \tilde{a}_{ij}^*).$$

Remark : 1.29

It is noted that score value of any generalized TFN must be a real number and its value belongs to the interval [0, 1].

Also, it is noted that, α is a parameter for adjusting the degree of importance of the spread of a generalized TFN and $\alpha = \{0.5, 1.5\}$.

Now we define two basic distance between TFNMs. The distance δ is a mapping from the set of TFNMs (M) to the set of real numbers (R).

$$\delta : M \times M \rightarrow R.$$

Definition : 1.30

The **score-distance (SD)** between two TFNMs A and B of order m x n is

$$SD\,(A, B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \text{score}\,(\tilde{a}_{ij}^*) - \text{score}\,(\tilde{b}_{ij}^*) \right|$$

It is obvious that $0 \leq SD\,(A, B) \leq m.n$. The score distance $SD : M \times M \rightarrow R$ satisfy the following conditions :

(i) $SD\,(A, B) \geq 0$ for all $A, B \in M$

(ii) $SD\,(A, B) = 0$ iff $A = B$ for all $A, B \in M$

(iii) $SD\,(A, B) = SD\,(B, A)$ for all $A, B \in M$

(iv) $SD\,(A, B) \leq SD\,(A, C) + SD\,(C, B)$ for all $A, B, C \in M$ (triangular property)

Thus the SD is metric on M.

Definition : 1.31

The **normalized score-distance** is defined as :

$$SD^*(A, B) = \frac{SD(A,B)}{m.n}, \text{ where } 0 \le SD^*(A, B) \le 1.$$

Definition : 1.32

The **Euclidian SD** is defined as :

$$E(A, B) = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} [\text{score}(\tilde{a}_{ij}^*) - \text{score}(\tilde{b}_{ij}^*)]^2}, \text{ where } 0 \le SD^*(A, B) \le \sqrt{mn}.$$

The Euclidian distance is also a metric on M.

Definition : 1.33

The **normalized Euclidian SD** is defined as :

$$E^*(A, B) = \frac{E(A,B)}{\sqrt{m.n}}, \text{ where } 0 \le E^*(A, B) \le 1.$$

Example : 1.34

Consider two circulant generalized triangular fuzzy number matrices.

$$A = \begin{bmatrix} <1,3,4> & <1,4,5> & <2,3,5> \\ <2,3,6> & <1,3,5> & <2,3,8> \\ <1,4,5> & <1,3,4> & <2,4,5> \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} <3,5,6> & <1,3,7> & <2,2,5> \\ <2,3,8> & <1,4,5> & <2,4,5> \\ <3,4,7> & <1,3,5> & <2,3,6> \end{bmatrix}$$

Now consider $< 1, 3, 4 > = \left\langle \frac{1}{4}, \frac{3}{4}, 1 \right\rangle$

$= < 0.25, 0.75, 1 >$

i.e., $\tilde{a}_{11} = < 0.25, 0.75, 1 >$

$$x_{11}^{*} = \frac{(0.25 + 0.75 + 1)}{3} = 0.6667$$

$$\text{Std } \tilde{a}_{11}^{*} = \sqrt{\frac{(0.25-0.6667)^2 + (0.75-0.6667)^2 + (1-0.6667)^2}{3}}$$

$$= 0.3118$$

$$\text{Score } (\tilde{a}_{11}^{*}) = 0.6667\ (1 - \alpha\ 0.3118)$$

$$= 0.3549 \text{ (by taking } \alpha = 1.5)$$

Similarly for the other elements we have

$$\text{Score } (\tilde{A}^{*}) = \begin{bmatrix} 0.35 & 0.24 & 0.43 \\ 0.35 & 0.30 & 0.27 \\ 0.32 & 0.35 & 0.47 \end{bmatrix}$$

$$\text{Score } (\tilde{B}^{*}) = \begin{bmatrix} 0.40 & 0.24 & 0.39 \\ 0.27 & 0.32 & 0.48 \\ 0.42 & 0.31 & 0.35 \end{bmatrix}$$

$$\text{Then SD(A, B)} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \text{score} (\tilde{a}_{ij}^{*}) - \text{score} (\tilde{b}_{ij}^{*} \right|$$

$$= 0.66.$$