CHAPTER 4

Fermatean Neutrosophic Approximation Space

(4.1) Fermatean Neutrosophic Rough Set

(4.2) Topology of Fermatean Temporal Neutrosophic Rough Sets

(4.3) Application of Fermatean Temporal Neutrosophic Rough Sets

In this chapter, the mathematical tools, rough set and Fermatean neutrosophic set are integrated to introduce a new class of rough set within an arbitrary Fermatean neutrosophic approximation space. Using a constructive method, the upper and lower Fermatean neutrosophic rough approximation operators are defined and their properties are investigated. Additionally, various notions of level cut sets of Fermatean neutrosophic rough set are established to characterize these approximation operators. The idea of Fermatean temporal neutrosophic rough set and their associated topology is introduced. Also, a working rule to find their core is also provided, along with an example.

4.1 Fermatean Neutrosophic Rough Set

In this section, the Fermatean neutrosophic approximation space and their induced Fermatean neutrosophic approximation operators are introduced. Furthermore, a new type of set called the Fermatean neutrosophic rough set (FNRS) is introduced and some of its properties are investigated.

Definition 4.1.1

Let Q be a non empty set. A Fermatean neutrosophic relation (FN-r) R_{F_N} on Q is a Fermatean neutrosophic subset

$$R_{F_N} = \left\{\left[\langle x,y\rangle, T_{R_{F_N}}(x,y), I_{R_{F_N}}(x,y), F_{R_{F_N}}(x,y)\right] / x,y \in Q \times Q\right\}$$

$$T_{R_{F_N}}: Q \times Q \to [0,1]; I_{R_{F_N}}: Q \times Q \to [0,1]; F_{R_{F_N}}: Q \times Q \to [0,1]$$

satisfies $0 \leq T^3_{R_{F_N}}(x,y) + I^3_{R_{F_N}}(x,y) + F^3_{R_{F_N}}(x,y) \leq 2$ for all $(x,y) \in Q \times Q$, then the family of all FN-r on $Q \times Q$ by $F(Q \times Q)$.

Definition 4.1.2

Let $R_{F_N} \in F(Q \times Q)$, then the relation R_{F_N} is

i. Reflexive

If $T_{R_{F_N}}(x,x) = 1$, $I_{R_{F_N}}(x,x) = 1$, and $F_{R_{F_N}}(x,x) = 0$ for all $x \in Q$.

ii. Symmetric

If for all $(x,y) \in Q \times Q$, $T_{R_{F_N}}(x,y) = T_{R_{F_N}}(y,x)$, $I_{R_{F_N}}(x,y) = I_{R_{F_N}}(y,x)$ and $F_{R_{F_N}}(x,y) = F_{R_{F_N}}(y,x)$.

iii. Transitive

R_{F_N} is transitive if the following conditions are satisfied:

for all $x,y,z \in Q$, $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$

i. $\forall y \in Q, T_{R_{F_N}}(x,y) \geq \lambda_1, T_{R_{F_N}}(y,z) \geq \lambda_1 \Rightarrow T_{R_{F_N}}(x,z) \geq \lambda_1$

ii. $\forall y \in Q, I_{R_{F_N}}(x,y) \geq \lambda_2, I_{R_{F_N}}(y,z) \geq \lambda_2 \Rightarrow I_{R_{F_N}}(x,z) \geq \lambda_2$

iii. $\forall y \in Q, F_{R_{F_N}}(x,y) \geq \lambda_3, F_{R_{F_N}}(y,z) \geq \lambda_3 \Rightarrow F_{R_{F_N}}(x,z) \geq \lambda_3$

Definition 4.1.3

Consider a non empty set Q. An arbitrary FN-r R_{F_N} over $(Q \times Q)$ then the pair (Q, R_{F_N}) is called Fermatean neutrosophic approximation space (FNAS). For any $A \in F(Q)$, of the form

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle / x \in Q\}$$

define the upper and lower approximations with respect to (Q, R_{F_N}), denoted by $\underline{R}_{F_N}(A)$ and $\overline{R}_{F_N}(A)$ respectively.

$$\underline{R}_{F_N}(A) = \left\{\left\langle x, T_{\underline{R}_{F_N}(A)}(x), I_{\underline{R}_{F_N}(A)}(x), F_{\underline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

$$\overline{R}_{F_N}(A) = \left\{\left\langle x, T_{\overline{R}_{F_N}(A)}(x), I_{\overline{R}_{F_N}(A)}(x), F_{\overline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

where,

$$T_{\underline{R}_{F_N}(A)}(x) = \bigwedge_{y \in Q} \left[F_{R_{F_N}}(x, y) \vee T_A(y)\right]$$

$$I_{\underline{R}_{F_N}(A)}(x) = \bigwedge_{y \in Q} \left[1 - I_{R_{F_N}}(x, y) \vee I_A(y)\right]$$

$$F_{\underline{R}_{F_N}(A)}(x) = \bigvee_{y \in Q} \left[T_{R_{F_N}}(x, y) \wedge F_A(y)\right]$$

$$T_{\overline{R}_{F_N}(A)}(x) = \bigvee_{y \in Q} \left[T_{R_{F_N}}(x, y) \wedge T_A(y)\right]$$

$$I_{\overline{R}_{F_N}(A)}(x) = \bigvee_{y \in Q} \left[I_{R_{F_N}}(x, y) \wedge I_A(y)\right]$$

$$F_{\overline{R}_{F_N}(A)}(x) = \bigwedge_{y \in Q} \left[F_{R_{F_N}}(x, y) \vee F_A(y)\right]$$

The pair $\left(\underline{R}_{F_N}(A), \overline{R}_{F_N}(A)\right)$ is called FNRS of A with respect to (Q, R_{F_N}) and $\underline{R}_{F_N}, \overline{R}_{F_N}: F(Q) \rightarrow F(Q)$ is referred to as upper and lower Fermatean neutrosophic rough approximation operators respectively.

Example 4.1.4

Let (Q, R_{F_N}) be a FNRAS where $Q = \{p_1, p_2, p_3, p_4\}$ and $R_{F_N} \in FNRA(Q \times Q)$ is defined as

Table 4.1.1 Fermatean neutrosophic relation

R_{F_N}	p_1	p_2	p_3	p_4
p_1	0.9, 0.7, 0.3	0.7, 0.9, 0.8	0.5, 0.6, 0.9	0.5, 0.9, 0.8
p_2	0.8, 0.7, 0.4	0.6, 0.5, 0.9	0.9, 0.8, 0.6	0.4, 0.8, 0.7
p_3	0.5, 0.8, 0.8	0.8, 0.6, 0.2	0.7, 0.7, 0.8	0.7, 0.8, 0.8
p_4	0.6, 0.4, 0.9	0.5, 0.8, 0.9	0.6, 0.4, 0.9	0.6, 0.9, 0.9

If a FNS

$X = \{\langle p_1, 0.5,0.7,0.6\rangle, \langle p_2, 0.9,0.8,0.4\rangle, \langle p_3, 0.7,0.5,0.8\rangle, \langle p_4, 0.9,0.9,0.6\rangle\}$.

Then,

$$\overline{R}_{F_N} = \{\langle p_1, 0.7,0.9,0.6\rangle, \langle p_2, 0.7,0.8,0.6\rangle, \langle p_3, 0.6,0.9,0.9\rangle, \langle p_4, 0.8,0.8,0.8\rangle\}$$

$$\underline{R}_{F_N} = \{\langle p_1, 0.6, 0.5, 0.6\rangle, \langle p_2, 0.5, 0.5, 0.8\rangle, \langle p_3, 0.8, 0.6, 0.6\rangle, \langle p_4, 0.8,0.5,0.8\rangle\}$$

upper and lower approximations of X respectively.

Remark 4.1.5

If R_{F_N} is an intuitionistic fuzzy relation on a non empty universe Q defined by $R_{F_N} = \left\{\left[\langle x, y\rangle, \mu_{R_{F_N}}(x, y), \vartheta_{R_{F_N}}(x, y)\right] / x, y \in Q \times Q\right\}$ then $\left(Q, R_{F_N}\right)$ is an intuitionistic fuzzy approximation space, then Fermatean neutrosophic rough operators are induced from a intuitionistic fuzzy approximation space as follows, for a Fermatean neutrosophic set $A \in F(Q)$ of the form

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle | x \in Q\}$$

$$\underline{R}_{F_N}(A) = \left\{\left\langle x, T_{\underline{R}_{F_N}(A)}(x), I_{\underline{R}_{F_N}(A)}(x), F_{\underline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

$$\overline{R}_{F_N}(A) = \left\{\left\langle x, T_{\overline{R}_{F_N}(A)}(x), I_{\overline{R}_{F_N}(A)}(x), F_{\overline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

where

$$T_{\underline{R}_{F_N}(A)}(x) = \bigwedge_{y \in Q} \left[\vartheta_{R_{F_N}}(x, y) \vee T_A(y)\right]$$

$$I_{\underline{R}_{F_N}(A)}(x) = \underset{y\in Q}{\wedge}\left[\left(\mu_{R_{F_N}}(x,y) + \vartheta_{R_{F_N}}(x,y)\right) \vee I_A(y)\right]$$

$$F_{\underline{R}_{F_N}(A)}(x) = \underset{y\in Q}{\vee}\left[\mu_{R_{F_N}}(x,y) \wedge F_A(y)\right]$$

$$T_{\overline{R}_{F_N}(A)}(x) = \underset{y\in Q}{\vee}\left[\mu_{R_{F_N}}(x,y) \wedge T_A(y)\right]$$

$$I_{\overline{R}_{F_N}(A)}(x) = \underset{y\in Q}{\vee}\left[1 - \left(\mu_{R_{F_N}}(x,y) + \vartheta_{R_{F_N}}(x,y)\right) \wedge I_A(y)\right]$$

$$F_{\overline{R}_{F_N}(M)}(x) = \underset{y\in Q}{\wedge}\left[\vartheta_{R_{F_N}}(x,y) \vee F_A(y)\right]$$

Remark 4.1.6

Let R_{F_N} be a crisp binary relation on Q and $\left(Q, R_{F_N}\right)$ be a crisp approximation space, then Fermatean neutrosophic rough operators are induced from crisp approximation space, such that $A \in F(Q)$ and $y \in R_S(x)$, where $R_S(x)$, is the successor neighbourhood of x .

$$\underline{R}_{F_N}(A) = \left\{\left\langle x, T_{\underline{R}_{F_N}(A)}(x), I_{\underline{R}_{F_N}(A)}(x), F_{\underline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

$$\overline{R}_{F_N}(A) = \left\{\left\langle x, T_{\overline{R}_{F_N}(A)}(x), I_{\overline{R}_{F_N}(A)}(x), F_{\overline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\}$$

where,

$$T_{\underline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\wedge} T_A(y), \qquad I_{\underline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\wedge} I_A(y),$$

$$F_{\underline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\vee} F_A(y),$$

$$T_{\overline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\vee} T_A(y), \quad I_{\overline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\vee} I_A(y),$$

$$F_{\overline{R}_{F_N}(A)}(x) = \underset{y\in R_S(x)}{\wedge} F_A(y).$$

Theorem 4.1.7

Let (Q, R_{F_N}) be a FNAS. Then the lower and upper Fermatean neutrosophic rough operators generated by (Q, R_{F_N}) meet the following conditions. $\forall A, B \in F(Q)$, $\forall\ \xi, \psi, \zeta \in [0,1]$ with $\xi^3 + \psi^3 + \zeta^3 \leq 2$

$(FL1)$ $\underline{R}_{F_N}(A) = \sim \overline{R}_{F_N}(\sim A)$ $\quad$ $(FU1)$ $\overline{R}_{F_N}(A) = \sim \underline{R}_{F_N}(\sim A)$

$(FL2)$ $\underline{R}_{F_N}\left(\bigcap_{i\in J} A_i\right) = \bigcap_{i\in J} \underline{R}_{F_N}(A_i)$ $\quad$ $(FU2)$ $\overline{R}_{F_N}\left(\bigcup_{i\in J} A_i\right) = \bigcup_{i\in J} \overline{R}_{F_N}(A_i)$

$(FL3)$ $A \subseteq B \Rightarrow \underline{R}_{F_N}(A) \subseteq \underline{R}_{F_N}(B)$ $\quad$ $(FU3)$ $A \subseteq B \Rightarrow \overline{R}_{F_N}(A) \subseteq \overline{R}(B)$

$(FL4)$ $\underline{R}_{F_N}\left(\bigcup_{i\in J} A_i\right) \supseteq \bigcup_{i\in J} \underline{R}_{F_N}(A_i)$ $\quad$ $(FU4)$ $\overline{R}_{F_N}\left(\bigcap_{i\in J} A_i\right) \subseteq \bigcap_{i\in J} \overline{R}_{F_N}(A_i)$

$(FL5)$ $\underline{R}_{F_N}\left(A \cup \left(\overline{\xi, \psi, \zeta}\right)\right) = \underline{R}_{F_N}(A) \cup \left(\overline{\xi, \psi, \zeta}\right)$ $\quad$ $(FU5)$ $\overline{R}_{F_N}\left(A \cap \left(\overline{\xi, \psi, \zeta}\right)\right) = \overline{R}_{F_N}(A) \cap \left(\overline{\xi, \psi, \zeta}\right)$

Proof:

The conditions of lower Fermatean neutrosophic rough operators $\underline{R}_{F_N}(A)$ are shown. The upper Fermatean neutrosophic rough operators $\overline{R}_{F_N}(A)$ can be proved in a comparable way.

$(FL1)$ By Definition 4.1.3,

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle | x \in Q\}$$

$$C(A) = \sim A = \{\langle x, (F_A(x), 1 - I_A(x), T_A(x))\rangle | x \in Q\}$$

$$\underline{R}_{F_N}(\sim A) = \left\{\left\langle x, T_{\underline{R}_{F_N}(\sim A)}(x), I_{\underline{R}_{F_N}(\sim A)}(x), F_{\underline{R}_{F_N}(\sim A)}(x)\right\rangle / x \in Q\right\}$$

$$T_{\underline{R}_{F_N}(A)}(x) = \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee T_{\sim A}(y)\right] \Rightarrow \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee F_A(y)\right]$$

$$I_{\underline{R}_{F_N}(A)}(x) = \bigwedge_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee I_{\sim A}(y)\right]$$

$$\Rightarrow \bigwedge_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee \left(1 - I_A(y)\right)\right]$$

$$F_{\underline{R}_{F_N}(A)}(x) = \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge F_{\sim A}(y)\right] \Rightarrow \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge T_A(y)\right]$$

$$\underline{R}_{F_N}(\sim A) = \begin{bmatrix} \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee F_A(y)\right], \\ \bigwedge_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee \left(1 - I_A(y)\right)\right], \\ \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge T_A(y)\right] \end{bmatrix}$$

$$\sim \underline{R}_{F_N}(\sim A) = \begin{bmatrix} \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge F_A(y)\right], \\ \bigvee_{y\in Q}\left[I_{R_{F_N}}(x,y) \wedge I_A(y)\right], \\ \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee T_A(y)\right] \end{bmatrix} = \underline{R}_{F_N}(A)$$

$$= \left\{\left\langle x, T_{\overline{R}_{F_N}(A)}(x), I_{\overline{R}_{F_N}(A)}(x), F_{\overline{R}_{F_N}(A)}(x)\right\rangle / x \in Q\right\} = \overline{R}_{F_N}(A)$$

$(FL2)$, $\underline{R}_{F_N}\left(\bigcap_{i\in J} A_i\right) = \bigcap_{i\in J} \underline{R}_{F_N}(A_i)$

$$\underline{R}_{F_N}\left(\bigcap_{i\in J} A_i\right) = \left\{\left\langle x, T_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x), I_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x), F_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x)\right\rangle / x \in F_N\right\}$$

$$T_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x) = \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee T_{\cap_{i\in J} A_i}(y)\right]$$

$$\Rightarrow \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee \left[\wedge_{i\in J} T_{A_i}(y)\right]\right] \Rightarrow \bigwedge_{y\in Q}\left\{\begin{matrix} \left[F_{R_{F_N}}(x,y) \vee \left[T_{A_1}(y)\right]\right], \\ \wedge\left[F_{R_{F_N}}(x,y) \vee \left[T_{A_2}(y)\right]\right], \dots, \\ \wedge\left[F_{R_{F_N}}(x,y) \vee \left[T_{A_i}(y)\right]\right] \end{matrix}\right\}$$

$$I_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x) = \bigwedge_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee I_{\cap_{i\in J} A_i}(y)\right]$$

$$= \bigwedge_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee \left[\bigwedge\nolimits_{i\in J} I_{A_i}(y)\right]\right] \Rightarrow \bigwedge_{y\in Q}\left\{\begin{array}{c} \left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_1}(y)\right]\right], \\ \wedge\left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_2}(y)\right]\right], \dots, \\ \wedge\left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_i}(y)\right]\right] \end{array}\right\}$$

$$F_{\underline{R}_{F_N}(\cap_{i\in J} A_i)}(x) = \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge F_{(\cap_{i\in J} A_i)}(y)\right]$$

$$= \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge \left[\bigwedge\nolimits_{i\in J} F_{A_i}(y)\right]\right] = \bigvee_{y\in Q}\left\{\begin{array}{c} \left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_1}(y)\right]\right], \\ \wedge\left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_2}(y)\right]\right], \dots, \\ \wedge\left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_i}(y)\right]\right] \end{array}\right\}$$

$$= \left[\bigwedge_{y\in Q}\left\{\begin{array}{c} \left[F_{R_{F_N}}(x,y) \vee \left[T_{A_1}(y)\right]\right], \\ \left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_1}(y)\right]\right], \\ \left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_1}(y)\right]\right] \end{array}\right\}\right] \bigcap \left[\bigwedge_{y\in Q}\left\{\begin{array}{c} \left[F_{R_{F_N}}(x,y) \vee \left[T_{A_2}(y)\right]\right], \\ \left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_2}(y)\right]\right], \\ \left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_2}(y)\right]\right] \end{array}\right\}\right]$$

$$\dots \bigcap \left[\bigwedge_{n\in Q}\left\{\begin{array}{c} \left[F_{R_{F_N}}(x,y) \vee \left[T_{A_i}(y)\right]\right], \\ \left[1 - I_{R_{F_N}}(x,y) \vee \left[I_{A_i}(y)\right]\right], \\ \left[T_{R_{F_N}}(x,y) \wedge \left[F_{A_i}(y)\right]\right] \end{array}\right\}\right] = \bigcap_{i\in J} \underline{R}_{F_N}(A_i)$$

Hence $\underline{R}_{F_N}(\cap_{i\in J} A_i) = \cap_{i\in J} \underline{R}_{F_N}(A_i)$

$(FL3)$ obvious.

$(FL4)$, It is established that $\cup_{i\in J} A_i \supseteq A_1$, $\cup_{i\in J} A_i \supseteq A_2$ ... $\cup_{i\in J} A_i \supseteq A_i$

$\underline{R}_{F_N}(\cup_{i\in J} A_i \supseteq A_1)$, $\underline{R}_{F_N}(\cup_{i\in J} A_i \supseteq A_2)$,… $\underline{R}_{F_N}(\cup_{i\in J} A_i \supseteq A_i)$

$$\underline{R}_{F_N}\left(\bigcup_{i\in J} A_i\right) \supseteq \bigcup_{i\in J} \underline{R}_{F_N}(A_i)$$

$(FL5)$ Given that a constant Fermatean neutrosophic set on Q is defined as follows:

$\left(\overline{\xi,\psi,\zeta}\right) = \{\langle x,\xi,\psi,\zeta\rangle | x \in Q\}$ where $\forall\ \xi,\psi,\zeta \in [0,1]$ with $\xi^3 + \psi^3 + \zeta^3 \leq 2$.

To prove $\underline{R}_{F_N}\left(A \cup \left(\overline{\xi,\psi,\zeta}\right)\right) = \underline{R}_{F_N}(A) \cup \left(\overline{\xi,\psi,\zeta}\right)$

$$\underline{R}_{F_N}\left(A \cup \left(\overline{\xi,\psi,\zeta}\right)\right) = \left\{\left\langle \begin{matrix} x, \left(T_{\underline{R}_{F_N}(A)} \vee \xi\right)(x), \\ \left(I_{\underline{R}_{F_N}(A)} \vee \psi\right)(x), \\ \left(F_{\underline{R}_{F_N}(A)} \wedge \zeta\right)(x) \end{matrix} \right\rangle / x \in Q\right\}$$

$$\left(T_{\underline{R}_{F_N}(A)} \vee \xi\right)(x) = \bigwedge_{y \in Q}\left[F_{R_{F_N}}(x,y) \vee (T_A \vee \xi)(y)\right]$$

$$= \bigwedge_{y \in Q}\left[F_{R_{F_N}}(x,y) \vee (T_A)(n) \vee \xi(y)\right]$$

$$= \bigwedge_{y \in Q}\left[F_{R_{F_N}}(x,y) \vee (T_A)(n)\right] \vee \xi(y)$$

$$\left(I_{\underline{R}_{F_N}(A)} \vee \psi\right)(x) = \bigwedge_{y \in Q}\left[1 - I_{R_{F_N}}(x,y) \vee (I_A \vee \psi)(y)\right]$$

$$= \bigwedge_{y \in Q}\left[1 - I_{R_{F_N}}(x,y) \vee (I_A)(y) \vee \psi(y)\right]$$

$$= \bigwedge_{y \in Q}\left[1 - I_{R_{F_N}}(x,y) \vee (I_A)(y)\right] \vee \psi(y)$$

$$\left(F_{\underline{R}_{F_N}(A)} \wedge \zeta\right)(x) = \bigvee_{y \in Q}\left[T_{R_{F_N}}(x,y) \wedge (F_A \wedge \zeta)(y)\right]$$

$$= \bigvee_{y \in Q}\left[T_{R_{F_N}}(x,y) \wedge (F_A)(y) \wedge \zeta(y)\right]$$

$$= \bigvee_{y \in Q}\left[T_{R_{F_N}}(x,y) \wedge (F_A)(y)\right] \wedge \zeta(y)$$

$$\Rightarrow \left[\begin{matrix} \bigwedge_{y \in Q}\left[F_{R_{F_N}}(x,y) \vee (T_A)(y)\right] \vee \xi(y), \\ \bigwedge_{y \in Q}\left[1 - I_{R_{F_N}}(x,y) \vee (I_A)(y)\right] \vee \psi(y), \\ \bigvee_{y \in Q}\left[T_{R_{F_N}}(x,y) \wedge (F_A)(y)\right] \wedge \zeta(y) \end{matrix}\right]$$

$$\Rightarrow \begin{bmatrix} \bigwedge\limits_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee (T_A)(y)\right], \\ \bigwedge\limits_{y\in Q}\left[1 - I_{R_{F_N}}(x,y) \vee (I_A)(y)\right], \\ \bigvee\limits_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge (F_A)(y)\right] \end{bmatrix} \cup [\xi(y), \psi(y), \zeta(y)]$$

Hence $\underline{R}_{F_N}\left(A \cup \left(\overline{\xi,\psi,\zeta}\right)\right) = \underline{R}_{F_N}(A) \cup \left(\overline{\xi,\psi,\zeta}\right)$ proved.

Level cuts of Fermateam neutrosophic rough sets

Definition 4.1.9

Let $X \in F(Q)$ and $\xi, \psi, \zeta \in [0,1]$ with $\xi^3 + \psi^3 + \zeta^3 \leq 2$ and (ξ, ψ, ζ) level cut of X denoted by $X^{(\xi\psi\zeta)}$ is defined as

$$X^{(\xi\psi\zeta)} = \{T_X(x) \geq \xi, I_X(x) \geq \psi, F_X(x) \leq \zeta / x \in Q\}$$

Define

$X_\xi = \{T_X(x) \geq \xi / x \in Q\}$ and $X_{\xi+} = \{T_X(x) > \xi / x \in Q\}$ the ξ level cut and strong ξ level cut of truth value function generated by A and

$X\psi = \{I_X(x) \geq \psi / x \in Q\}$ and $X_{\psi+} = \{I_X(x) > \psi / x \in Q\}$ the ψ level cut and strong ψ level cut of truth value function generated by A and

$X^\zeta = \{F_X(x) \leq \zeta / x \in Q\}$ and $X^{\zeta+} = \{F_X(x) < \zeta / x \in Q\}$ the ζ level cut and strong ζ level cut of truth value function generated by X.

Similarly, define the level cuts sets, such as

$X^{(\xi+,\psi+,\zeta+)} = \{T_X(x) > \xi,\ I_X(x) > \psi, F_X(x) < \zeta / x \in Q\}$ is $(\xi+, \psi+, \zeta\ +)$,

$X^{(\xi+,\psi,\zeta)} = \{T_X(x) > \xi, I_X(x) \geq \psi, F_X(x) \leq \zeta / x \in Q\}$ is $(\xi+, \psi, \zeta)$

$X^{(\xi,\psi+,\zeta)} = \{T_X(x) \geq \xi, I_X(x) > \psi, F_X(x) \leq \zeta\ / x \in Q\}$ is $(\xi, \psi+, \zeta)$

$$X^{(\xi,\psi,\zeta+)} = \{T_X(x) \geq \xi, I_X(x) \geq \psi, F_X(x) < \zeta \ / x \in Q\} \text{ is } (\xi, \psi, \zeta +)$$

level cut set of X respectively. Likewise, other level cuts can also be defined.

Theorem 4.1.10

The level cut sets of FNSs meet the following criteria: $\forall A, B \in F(Q)$, $\xi, \psi, \zeta \in [0,1]$ with $\xi_1^3 + \psi_1^3 + \zeta_1^3 \leq 2$, $\xi_1, \psi_1, \zeta_1 \in [0,1]$ with $\xi_1^3 + \psi_1^3 + \zeta_1^3 \leq 2$ and $\xi_2, \psi_2, \zeta_2 \in [0,1]$ with $\xi_2^3 + \psi_2^3 + \zeta_2^3 \leq 2$.

(1) $A^{(\xi,\psi,\zeta)} = A_\xi \cap A\psi \cap A^\zeta$

(2) $(\sim A)_\xi = \sim A^{\xi+}$; $(\sim A)\psi = \sim A(1 - \psi +)$; $(\sim A)^\zeta = \sim A_{\zeta+}$

(3) $\left(\bigcap_{i \in J} A_i\right)_\xi = \bigcap_{i \in J} (A_i)_\xi$

$\left(\bigcap_{i \in J} A_i\right)\psi = \bigcap_{i \in J} (A_i)\psi$

$\left(\bigcap_{i \in J} A_i\right)^\zeta = \bigcap_{i \in J} (A_i)^\zeta$

(4) $\left(\bigcup_{i \in J} A_i\right)_\xi = \bigcup_{i \in J} (A_i)_\xi$

$\left(\bigcup_{i \in J} A_i\right)\psi = \bigcup_{i \in J} (A_i)\psi$

$\left(\bigcup_{i \in J} A_i\right)^\zeta = \bigcup_{i \in J} (A_i)^\zeta$

(5) $\left(\bigcup_{i \in J} A_i\right)^{(\xi,\psi,\zeta)} \supseteq \bigcup_{i \in J} (A_i)^{(\xi,\psi,\zeta)}$

(6) $\left(\bigcap_{i \in J} A_i\right)^{(\xi,\psi,\zeta)} \supseteq \bigcap_{i \in J} (A_i)^{(\xi,\psi,\zeta)}$

(7) For $\xi_1 \geq \xi_2 \psi_1 \geq \psi_2 \zeta_1 \geq \zeta_2$

$A_{\xi_1} \subseteq A_{\xi_2}$; $A\psi_1 \subseteq A\psi_2$; $A^{\zeta_1} \subseteq A^{\zeta_2}$, $A^{(\xi_1,\psi_1,\zeta_1)} \subseteq A^{(\xi_2,\psi_2,\zeta_2)}$

Proof: (1) and (3) follow directly from Definition 4.1.9

(2) Since $\sim A = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle / x \in Q\}$, $(\sim A)_\xi = \{F_A(x) \geq \xi \ / x \in Q\}$

By definition,

$$A^{\xi+} = \{x \in Q / \ F_A(x) < \xi \ \}$$

$$\sim A^{\xi+} = \{x \in Q / \ F_A(x) \geq \xi\}$$

$$\Rightarrow \sim A_\xi = \left(\sim A^{\xi+}\right)$$

Similarly, it can be proved that,

$(\sim A)\psi = \left(\sim A(1 - \psi +)\right)$ and $(\sim A)^\zeta = \sim A_{\zeta+}$.

(4) $\bigcap_{i \in J} A_i = \left\{\left\langle x, \bigwedge_{i \in J} T_{A_i}(x), \bigwedge_{i \in J} I_{A_i}(x), \bigvee_{i \in J} F_{A_i}(x)\right\rangle / x \in Q\right\}$

and

$$\left(\bigcap_{i \in J} A_i\right)_\xi = \left\{x \in Q / \bigwedge_{i \in J} T_{A_i}(x) \geq \xi\right\} = \left\{x \in Q / \ T_{A_i}(x) \geq \xi, \forall i \in J\right\} = \bigcap_{i \in J} (A_i)_\xi$$

Similarly,

$$\left(\bigcap_{i \in J} A_i\right)\psi = \left\{x \in Q / \bigwedge_{i \in J} I_{A_i}(x) \geq \psi\right\} = \left\{x \in Q / \ I_{A_i}(x) \geq \psi, \forall i \in J\right\} = \bigcap_{i \in J} (A_i)\psi$$

$$\left(\bigcap_{i \in J} A_i\right)^\zeta = \left\{x \in Q / \bigvee_{i \in J} F_{A_i}(x) \geq \zeta\right\} = \left\{x \in Q \ / F_{A_i}(x) \leq \zeta, \forall i \in J\right\} = \bigcap_{i \in J} (A_i)^\zeta$$

$$\left(\bigcap_{i \in J} A_i\right)^{\xi,\psi,\zeta} = \left(\bigcap_{i \in J} A_i\right)_\xi \cap \left(\bigcap_{i \in J} A_i\right)\psi \cap \left(\bigcap_{i \in J} A_i\right)^\zeta$$

$$= \bigcap_{i\in J}\left((A_i)_\xi \cap (A_i)\psi \cap (A_i)^\zeta\right) = \bigcap_{i\in J}(A_i)^{(\xi,\psi,\zeta)}$$

(5) Consider $\bigcup_{i\in J}(A_i) = \left\{\left\langle x, \bigvee_{i\in J} T_{A_i}(x), \bigvee_{i\in J} I_{A_i}(x), \bigwedge_{i\in J} F_{A_i}(x)\right\rangle / x \in Q\right\}$ then,

$$\left(\bigcup_{i\in J} A_i\right)_\xi = \left\{x \in Q / \bigvee_{i\in J} T_{A_i}(x) \geq \xi\right\} = \left\{x \in Q / T_{A_i}(x) \geq \xi, \exists i \in J\right\} = \bigcup_{i\in J}(A_i)_\xi$$

Similarly,

$$\left(\bigcup_{i\in J} A_i\right)\psi = \left\{x \in Q / \bigvee_{i\in J} I_{A_i}(x) \geq \psi\right\} = \left\{x \in Q / I_{A_i}(x) \geq \psi, \exists i \in J\right\} = \bigcup_{i\in J}(A_i)\psi$$

$$\left(\bigcup_{i\in J} A_i\right)^\zeta = \left\{x \in Q / \bigwedge_{i\in J} F_{A_i}(x) \geq \zeta\right\} = \left\{x \in Q / F_{A_i}(x) \leq \zeta, \exists i \in J\right\} = \bigcup_{i\in J}(A_i)^\zeta$$

then conclude

$$\left(\bigcup_{i\in J} A_i\right)^{\xi,\psi,\zeta} = \left(\bigcup_{i\in J} A_i\right)_\xi \cup \left(\bigcup_{i\in J} A_i\right)\psi \cup \left(\bigcup_{i\in J} A_i\right)^\zeta$$

$$= \bigcup_{i\in J}(A_i)_\xi \cup (A_i)\psi \cup (A_i)^\zeta = \bigcup_{i\in J}(A_i)^{(\xi,\psi,\zeta)}$$

(6) For any $x \in A_\xi$, according to Definition 4.1.9 implies that if $T_A(x) \geq \xi_1 \geq \xi_2$, then $A_{\xi_1} \subseteq A_{\xi_2}$

Similarly, for $\psi_1 \geq \psi_2$ and $\zeta_1 \geq \zeta_2$ it follows that $A\psi_1 \subseteq A\psi_2$ and $A^{\zeta_1} \subseteq A^{\zeta_2}$.

Hence, $A^{(\xi_1,\psi_1,\zeta_1)} \subseteq A^{(\xi_2,\psi_2,\zeta_2)}$.

Corollary 4.1.11

Assume that R_{F_N} is a FN-r in Q,

$(R_{F_N})_{\xi} = \{(x,y) \in Q \times Q / T_{R_{F_N}}(x,y) \geq \xi\},$

$$(R_{F_N})_{\xi}(x) = \{y \in Q / T_{R_{F_N}}(x,y) \geq \xi\}$$

$(R_{F_N})_{\xi+} = \{(x,y) \in Q \times Q / T_{R_{F_N}}(x,y) > \xi\},$

$$(R_{F_N})_{\xi+}(x) = \{y \in Q / T_{R_{F_N}}(x,y) > \xi\}$$

$(R_{F_N})\psi = \{(x,y) \in Q \times Q / I_{R_{F_N}}(x,y) \geq \psi\},$

$$(R_{F_N})\psi(x) = \{y \in Q / I_{R_{F_N}}(x,y) \geq \psi\}$$

$(R_{F_N})\psi+= \{(x,y) \in Q \times Q / I_{R_{F_N}}(x,y) > \psi\},$

$$(R_{F_N})\psi + (x) = \{y \in Q / I_{R_{F_N}}(x,y) > \psi\}$$

$(R_{F_N})^{\zeta} = \{(x,y) \in Q \times Q / F_{R_{F_N}}(x,y) \leq \zeta\},$

$$(R_{F_N})^{\zeta}(x) = \{y \in Q / F_{R_{F_N}}(x,y) \leq \zeta\}$$

$(R_{F_N})^{\zeta+} = \{(x,y) \in Q \times Q / F_{R_{F_N}}(x,y) < \zeta\},$

$$(R_{F_N})^{\zeta+}(x) = \{y \in Q / F_{R_{F_N}}(x,y) < \zeta\}$$

$$(R_{F_N})^{(\xi,\psi,\zeta)} = \{(x,y) \in Q \times Q / T_{R_{F_N}}(x,y) \geq \xi, I_{R_{F_N}}(x,y) \geq \psi, F_{R_{F_N}}(x,y) \leq \zeta\}$$

$$(R_{F_N})^{(\xi,\psi,\zeta)}(x) = \{y \in Q / T_{R_{F_N}}(x,y) \geq \xi, I_{R_{F_N}}(x,y) \geq \psi, F_{R_{F_N}}(x,y) \leq \zeta\}$$

Then for all

$\left(R_{F_N}\right)_{\xi}, \left(R_{F_N}\right)_{\xi+}, \left(R_{F_N}\right)\psi, \left(R_{F_N}\right)\psi+, \left(R_{F_N}\right)^{\zeta}, \left(R_{F_N}\right)^{\zeta+}, \left(R_{F_N}\right)^{(\xi\psi\zeta)}$ are crisp relation in Q and

1) Let R_{F_N} is reflexive, then the above level cuts are reflexive.

2) If R_{F_N} is symmetric, then the above level cuts are symmetric.

3) If R_{F_N} is transitive, then the above level cuts are transitive.

Theorem 4.1.12

Consider $\left(Q, R_{F_N}\right)$ be a FNAS and $A \in F(Q)$, then the upper Fermatean neutrosophic approximation operator can be shown as below $\forall x \in Q$.

1) $T_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi}\left(A_{\xi}\right)(x)\right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi}\left(A_{\xi+}\right)(x)\right]$

$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi+}\left(A_{\xi}\right)(x)\right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi+}\left(A_{\xi+}\right)(x)\right]$

2) $I_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi(A\xi)(x)\right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi(A\xi +)(x)\right]$

$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi + (A\xi)(x)\right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi + (A\xi +)(x)\right]$

3) $F_{\overline{R}_{F_N}(A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi}\left(A^{\xi}\right)(x)\right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi}\left(A^{\xi+}\right)(x)\right]$

$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi+}\left(A^{\xi}\right)(x)\right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi+}\left(A^{\xi+}\right)(x)\right]$

for any $\xi \in [0,1]$

4) $\left[\overline{R}_{F_N}(A)\right]_{\xi+} \subseteq \overline{R}_{F_N \xi+}\left(A_{\xi+}\right) \subseteq \overline{R}_{F_N \xi}\left(A_{\xi}\right) \subseteq \left[\overline{R}_{F_N}(A)\right]_{\xi}$

5) $\left[\overline{R}_{F_N}(A)\right]\xi+ \subseteq \overline{R}_{F_N}\xi + (A\xi +) \subseteq \overline{R}_{F_N}\xi(A\xi) \subseteq \left[\overline{R}_{F_N}(A)\right]\xi$

6) $\left[\overline{R}_{F_N}(A)\right]^{\xi+} \subseteq \overline{R_{F_N}{}^{\xi+}}\left(A^{\xi+}\right) \subseteq \overline{R_{F_N}{}^{\xi}}\left(A^{\xi}\right) \subseteq \left[\overline{R}_{F_N}(A)\right]^{\xi}$

Proof:

1) For $x \in Q$, the following equality holds:

$$\bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi}\left(A_{\xi}\right)(x)\right] = Sup\left\{\xi \in [0,1]/x \in \left(\overline{R}_{F_N}\right)_{\xi}\left(A_{\xi}\right)\right\}$$

$$= Sup\left\{\xi \in [0,1]/\left(R_{F_N}\right)_{\xi}(x) \cap A_{\xi} \neq \varphi\right\}$$

$$= Sup\left\{\xi \in [0,1]/\exists y \in Q\left(y \in \left(R_{F_N}\right)_{\xi}(x), y \in A_{\xi}\right)\right\}$$

$$= Sup\left\{\xi \in [0,1]/\exists y \in Q\left[T_{R_{F_N}}(x,y) \geq \xi, T_A(y) \geq \xi\right]\right\}$$

$$= \bigvee_{y\in Q}\left[T_{R_{F_N}}(x,y) \wedge T_A(y)\right] = T_{\overline{R}_{F_N}}(x)$$

2) $$\bigvee_{\xi\in[0,1]}\left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi(A\xi)(x)\right] = Sup\left\{\xi \in [0,1]/x \in \left(\overline{R}_{F_N}\right)\xi(A\xi)\right\}$$

$$= Sup\left\{\xi \in [0,1]/\left(R_{F_N}\right)\xi(x) \cap A\xi \neq \varphi\right\}$$

$$= Sup\left\{\xi \in [0,1]/\exists y \in Q\left(y \in \left(R_{F_N}\right)\xi(x), y \in A\xi\right)\right\}$$

$$= Sup\left\{\xi \in [0,1]/\exists y \in Q\left[I_{R_{F_N}}(x,y) \geq \xi, I_A(y) \geq \xi\right]\right\}$$

$$= \bigvee_{y\in Q}\left[I_{R_{F_N}}(x,y) \wedge I_A(y)\right] = I_{\overline{R}_{F_N}}(x)$$

3) $$\bigwedge_{\xi\in[0,1]}\left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi}\left(A^{\xi}\right)(x)\right] = inf\left\{\xi \in [0,1]/x \in \left(\overline{R}_{F_N}\right)^{\xi}\left(A^{\xi}\right)\right\}$$

$$= inf\left\{\xi \in [0,1]/\left(R_{F_N}\right)^{\xi}(x) \cap A^{\xi} \neq \varphi\right\}$$

$$= inf\left\{\xi \in [0,1]/\exists y \in Q\left(y \in \left(R_{F_N}\right)^{\xi}(x), y \in A^{\xi}\right)\right\}$$

$$= inf\left\{\xi \in [0,1]/\exists y \in Q\left[F_{R_{F_N}}(x,y) \leq \xi, F_A(y) \leq \xi\right]\right\}$$

$$= \bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee F_A(y)\right] = F_{\overline{R}_{F_N}}(x)$$

Similarly

$$T_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\xi\in[0,1]}\left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi}\left(A_{\xi+}\right)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]}\left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi+}\left(A_{\xi}\right)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]}\left[\xi \wedge \left(\overline{R}_{F_N}\right)_{\xi+}\left(A_{\xi+}\right)(x)\right]$$

$$I_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\xi\in[0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi(A\xi +)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi + (A\xi)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]} \left[\xi \wedge \left(\overline{R}_{F_N}\right)\xi + (A\xi +)(x)\right]$$

$$F_{\overline{R}_{F_N}(A)}(x) = \bigwedge_{\xi\in[0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi}\left(A^{\xi+}\right)(x)\right]$$

$$= \bigwedge_{\xi\in[0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi+}\left(A^{\xi}\right)(x)\right]$$

$$= \bigwedge_{\xi\in[0,1]} \left[\xi \vee \left(\overline{R}_{F_N}\right)^{\xi+}\left(A^{\xi+}\right)(x)\right]$$

4) Since $\overline{R}_{F_{N_{\xi+}}}\left(A_{\xi+}\right) \subseteq \overline{R}_{F_{N_{\xi+}}}\left(A_{\xi}\right) \subseteq \overline{R}_{F_{N_{\xi}}}\left(A_{\xi}\right)$

To prove only $\left[\overline{R}_{F_N}(A)\right]_{\xi+} \subseteq \overline{R}_{F_{N_{\xi+}}}\left(A_{\xi+}\right)$ and

$\overline{R}_{F_{N_{\xi}}}\left(A_{\xi}\right) \subseteq \left[\overline{R}_{F_N}(A)\right]_{\xi}$ for any $x \in \left[\overline{R}_{F_N}(A)\right]_{\xi+}$, $T_{\overline{R}_{F_N}(A)} > \xi$

$\Rightarrow \bigvee_{y\in Q} \left[T_{R_{F_N}}(x,y) \wedge T_A(y)\right] > \xi \exists$ atleast one $y' \in Q$ such that

$$\left[T_{R_{F_N}}(x,y') \wedge T_{R_{F_N}}(y')\right] > \xi$$

$\Rightarrow y' \in R_{F_{N_{\xi+}}}(x)$ and $y' \in A_{\xi+} \Rightarrow R_{F_{N_{\xi+}}}(x) \cap A_{\xi+} \neq \varphi$

According to the definition of upper crisp approximation operator, $x \in \overline{R}_{F_{N_{\xi+}}}\left(A_{\xi+}\right)$. Hence $\left[\overline{R}_{F_N}(A)\right]_{\xi+} \subseteq \overline{R}_{F_{N_{\xi+}}}\left(A_{\xi+}\right)$

Next, to prove $\overline{R}_{F_{N_{\xi}}}\left(A_{\xi}\right) \subseteq \left[R_{F_N}(A)\right]_{\xi}$

for any $x \in \overline{R}_{F_{N_{\xi}}}\left(A_{\xi}\right), R_{F_{N_{\xi}}}\left(A_{\xi}\right)(x) = 1$, if there exist ψ, then

$$T_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\psi\in[0,1]} \left[\psi \wedge \left(\overline{R}_{F_N}\right)_{\psi}\left(A_{\psi}\right)(x)\right] \geq \xi \wedge \left(\overline{R}_{F_N}\right)_{\psi}\left(A_{\psi}\right)(x) = \xi$$

and $x \in \left[\overline{R}_{F_N}(A)\right]_{\xi}, \overline{R}_{F_{N_{\xi}}}\left(A_{\xi}\right) \subseteq \left[\overline{R}_{F_N}(A)\right]_{\xi}$

5) Similar to (4). It is enough to prove

$$\left[\overline{R_{F_N}\xi +}\,(A\xi +)\right]_{\xi} \subseteq \overline{R_{F_N}\xi +}\,(A\xi) \subseteq \left[\overline{R_{F_N}\xi}(A\xi)\right]$$

Hence,

i. $\left[\overline{R}_{F_N}(A)\right]\xi + \subseteq \overline{R_{F_N}\xi +}\,(A\xi +)$

ii. $\left[\overline{R_{F_N}\xi}(A\xi)\right] \subseteq \left[\overline{R}_{F_N}(A)\right]\xi$

i) For $x \in \left[\overline{R}_{F_N}(A)\right]\xi +, I_{\overline{R}_{F_N}(A)}(x) > \xi \Rightarrow \bigvee_{y \in Q}\left[I_{R_{F_N}}(x,y) \wedge I_A(y)\right] > \xi\ \exists y' \in Q$ such that $I_{R_{PF}}(x,y') \wedge I_A(y') > \xi$, (i.e.,)

$I_{R_{F_N}}(x,y')\xi$ and $I_A(y')\xi \Rightarrow y' \in R_{F_N}\xi + (x)$ and $y' \in A\xi +, y' \in R_{F_N}(x) \cap A\xi +$
$\Rightarrow R_{F_N}\xi + (x) \cap A\xi + \neq \varphi$.

According to the definition of crisp approximation operator, $x \in \overline{R_{F_N}\xi +}\,(A\xi +)$ therefore $\left[\overline{R}_{F_N}(A)\right]\xi + \subseteq \overline{R_{F_N}\xi +}\,(A\xi +)$.

For any $x \in \left[\overline{R_{F_N}\xi}(A\xi)\right], \left[\overline{R}_{F_N}\xi(A\xi)\right](x) = 1$.

If there exists ψ then

$$T_{\overline{R}_{F_N}(A)}(x) = \bigvee_{\psi \in [0,1]}\left[\psi \wedge \overline{R}_{F_N\psi}(A\psi)(x)\right] \geq \xi \wedge \overline{R}_{F_N\xi}(A\xi)(x) = \xi.$$

And $x \in \left[\overline{R}_{F_N}(A)\right]\xi$, therefore $\overline{R_{F_N}\xi}(A\xi) \subseteq \left[\overline{R}_{F_N}(A)\right]\xi$.

6) The proof of (6) is similar to (4) and (5), hence, it is sufficient to prove the following inclusions:

$$\left[\overline{R}_{F_N}(A)\right]^{\xi+} \subseteq \overline{R_{F_N}{}^{\xi+}}\left(A^{\xi+}\right) \text{ and } \left[\overline{R_{F_N}{}^{\xi}}\left(A^{\xi}\right)\right] \subseteq \left[\overline{R}_{F_N}(A)\right]^{\xi}.$$

For any $x \in \left[\overline{R}_{F_N}(A)\right]^{\xi+}, F_{\overline{R}_{F_N}}(x) < \xi$ (i.e.,) $\bigwedge_{y \in Q}\left[F_{R_{F_N}}(x,y) \vee F_A(y)\right] < \xi$ and $\exists y' \in Q \ni F_{R_{F_N}}(x,y') \vee F_A(y') < \xi$.

Hence $F_{R_{F_N}}(x, y') < \xi, T_A(y') < \xi$ (i.e.,) $y' \in R^{\xi+}(x)$and $y' \in A^{\xi+}$. ${R_{F_N}}^{\xi+}(x) \cap A^{\xi+} \neq \phi$. Therefore, $x \in \overline{{R_{F_N}}^{\xi+}}(A^{\xi+})$ and $\left[\bar{R}_{F_N}(A)\right]^{\xi+} \subseteq \overline{{R_{F_N}}^{\xi+}}(A^{\xi+})$.

Next for any $x \in \overline{{R_{F_N}}^{\xi}}(A^{\xi})$ note $\overline{{R_{F_N}}^{\xi}}(A^{\xi})(x) = 1$ then

$$F_{\bar{R}_{F_N}(A)}(x) = \bigwedge_{\psi \in [0,1]} \left[\psi \vee \overline{{R_{F_N}}^{\psi}}(A^{\psi})(x)\right] \leq \xi \vee \overline{{R_{F_N}}^{\xi}}(A^{\xi})(x) = \xi.$$

Thus $x \in \left[\overline{R}_{F_N}(A)\right]^{\xi}$. Hence $\overline{{R_{F_N}}^{\xi}}(A^{\xi}) \subseteq \left[\overline{R}_{F_N}(A)\right]^{\xi}$.

Theorem 4.1.13

Let (Q, R_{F_N}) be a FNAS and $A \in F(Q)$ then $\forall x \in Q$.

1) $T_{\underline{R}_{F_N}(A)}(x)$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(1 - \underline{{R_{F_N}}^{\xi}}\right)(A_{\xi+})(x)\right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - R_{F_N})^{\xi}(A_{\xi})(x)\right]$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(1 - \underline{{R_{F_N}}^{\xi+}}\right)(A_{\xi+})(x)\right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - R_{F_N})^{\xi+}(A_{\xi})(x)\right]$$

2) $I_{\underline{R}_{F_N}(A)}(x)$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(1 - \underline{R_{F_N}(1 - \xi)}\right)(A\xi +)(x)\right]$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - R_{F_N})(1 - \xi)(A\xi)(x)\right]$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(1 - \underline{R_{F_N}(1 - \xi +)}\right)(A\xi +)(x)\right]$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee \left(1 - R_{F_N}(1 - \xi +)(A\xi)(x)\right)\right]$$

3) $F_{\underline{R}_{F_N}(A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(1 - \underline{R_{F_N \xi}}(A^{\xi+})(x)\right)\right]$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - R_{F_N})_{\xi}(A^{\xi})(x)\right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \underline{R}_{F_N})_{\xi+}(A^{\xi+})(x)\right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge \left(1 - \underline{R_{F_N \xi+}}\right)(A^{\xi})(x)\right]$$

and for $\xi \in [0,1]$

4) $\left[\underline{R}_{F_N}(A)\right]_{\xi+} \subseteq \underline{R}_{F_N}{}^{\xi}\left(A_{\xi+}\right) \subseteq \underline{R_{F_N}{}^{\xi+}}\left(A_{\xi+}\right) \subseteq \underline{R_{F_N}{}^{\xi+}}\left(A_{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]_{\xi}$

5) $\left[\underline{R}_{F_N}(A)\right]\xi{+}\subseteq R_{F_N}(1-\xi)(A\xi+) \subseteq R_{F_N}(1-\xi+)(A\xi+)$
$\subseteq R_{F_N}(1-\xi+)(A\xi) \subseteq \left[\underline{R}_{F_N}(A)\right]\xi$

6) $\left[\underline{R}_{F_N}(A)\right]^{\xi+} \subseteq \underline{R}_{F_N}\left(A^{\xi+}\right) \subseteq \underline{R_{F_N}{}_{\xi+}}\left(A^{\xi+}\right) \subseteq \underline{R_{F_N}{}_{\xi+}}\left(A^{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]^{\xi}$

7) $\left[\underline{R}_{F_N}(A)\right]_{\xi} \subseteq \underline{R}_{F_N}{}^{\xi}\left(A_{\xi+}\right) \subseteq \underline{R_{F_N}{}^{\xi}}\left(A_{\xi}\right) \subseteq \underline{R_{F_N}{}^{\xi+}}\left(A_{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]_{\xi}$

8) $\left[\underline{R}_{F_N}(A)\right]\xi{+}\subseteq \underline{R_{F_N}(1-\xi)}(A\xi+) \subseteq \underline{R_{F_N}\xi}(A\xi) \subseteq \underline{R_{F_N}(1-\xi+)}(A\xi)$
$\subseteq \left[\underline{R}_{F_N}(A)\right]^{\xi}$

9) $\left[\underline{R}_{F_N}(A)\right]^{\xi+} \subseteq \underline{R}_{F_N}\left(A^{\xi+}\right) \subseteq \underline{R_{F_N}{}_{\xi}}\left(A^{\xi}\right) \subseteq \underline{R_{F_N}{}_{\xi+}}\left(A^{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]^{\xi}$

Proof:

(1) and (2). For any $x \in Q$ by the duality of upper and lower crisp approximation operators and in terms of theorems 4.1.7,4.1.10 and 4.1.12

$$T_{\overline{R}_{F_N}(\sim A)}(x) = \bigvee_{\xi\in[0,1]} \left[\xi \wedge \overline{\left(R_{F_N}\right)_{\xi}}\left(\sim A_{\xi}\right)(x)\right],$$

$$I_{\overline{R}_{F_N}(\sim A)}(x) = \bigvee_{\xi\in[0,1]} \left[\xi \wedge \overline{\left(R_{F_N}\right)\xi}\left(\sim A_{\xi}\right)(x)\right],$$

$$F_{\overline{R}_{F_N}(\sim A)}(x) = \bigwedge_{\xi\in[0,1]} \left[\xi \vee \overline{\left(R_{F_N}\right)^{\xi}}\left(\sim A_{\xi}\right)(x)\right],$$

$$T_{\overline{R}_{F_N}(\sim A)}(p) = \bigvee_{\xi\in[0,1]} \left[\xi \wedge \overline{\left(R_{F_N}\right)_{\xi}}\left(\sim A_{\xi}\right)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]} \left[\xi \wedge \overline{\left(R_{F_N}\right)_{\xi}}\left(\sim A^{\xi+}\right)(x)\right]$$

$$= \bigvee_{\xi\in[0,1]} \left[\xi \wedge \underline{\left(\sim R_{F_N}\right)_{\xi}}\left(A^{\xi+}\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge\underline{\left(1-R_{F_N}\right)_{\xi}}\left(\sim A^{\xi+}\right)(x)\right]$$

$$I_{\overline{R}_{F_N}(\sim A)}(x)=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge\overline{\left(R_{F_N}\right)\xi}\left(\sim A_{\xi}\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge\overline{\left(R_{F_N}\right)\xi}\left(\sim A(1-\xi+)\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge\sim\underline{\left(R_{F_N}\right)\xi}\left(A(1-\xi+)\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge 1-\underline{\left(R_{F_N}\right)\xi}\left(\sim A(1-\xi+)\right)(x)\right]$$

$$F_{\overline{R}_{F_N}(\sim A)}(x)=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee\overline{\left(R_{F_N}\right)^{\xi}}\left(\sim A^{\xi}\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee\overline{\left(R_{F_N}\right)^{\xi}}\left(\sim A_{\xi+}\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee\sim\underline{\left(R_{F_N}\right)^{\xi}}\left(A_{\xi+}\right)(x)\right]$$

$$=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee 1-\underline{\left(R_{F_N}\right)^{\xi}}\left(A_{\xi+}\right)(x)\right]$$

Thus, by fixing $\underline{R}_{F_N}(A)=\sim\overline{R}_{F_N}(\sim A)$,

Hence $T_{\underline{R}_{F_N}(A)}(x)=F_{\overline{R}_{F_N}(\sim A)}(x)=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee 1-\left(\underline{R}_{F_N}\right)^{\xi}\left(A^{\xi+}\right)(x)\right]$

$$I_{\underline{R}_{F_N}(A)}(x)=1-I_{\overline{R}_{F_N}(\sim A)}(x)=_{\xi\in[0,1]}^{\wedge}\left[\xi\vee 1-\underline{\left(R_{F_N}\right)\xi}\left(A(1-\xi+)\right)(x)\right]$$

$$F_{\underline{R}_{F_N}(A)}(x)=T_{\overline{R}_{F_N}(\sim A)}(x)=_{\xi\in[0,1]}^{\vee}\left[\xi\wedge 1-\left(\underline{R}_{F_N}\right)_{\xi}\left(A_{\xi+}\right)(x)\right]$$

Likewise, It is easy to prove that $\underline{R_{F_N}{}^{\xi}}\left(A_{\xi+}\right)\subseteq\underline{R_{F_N}{}^{\xi+}}\left(A_{\xi+}\right)\subseteq\underline{R_{F_N}{}^{\xi+}}\left(A_{\xi}\right)$.

To show that $\left[\underline{R}_{F_N}(A)\right]_{\xi+} \subseteq \underline{R_{F_N}}^{\xi}\left(A_{\xi+}\right)$, $\underline{R_{F_N}}^{\xi+}\left(A_{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]_{\xi}$ for any $x \in \underline{R_{F_N}}^{\xi}\left(A_{\xi+}\right)$, if $T_{\underline{R}_{F_N}}(A)(x) > \xi$ then, $\bigwedge_{y\in Q}\left[F_{R_{F_N}}(x,y) \vee T_A(y)\right] > \xi$ and $\left[F_{R_{F_N}}(x,y) \vee T_A(y)\right] > \xi$ for any $y \in Q$, that is if $F_{R_{F_N}}(x,y) \leq \xi$, then $T_A(y) > \xi$.

Alternatively, for any $y \in Q$, if $y \in R^{\xi}_{F_N}(x)$ then $y \in A_{\xi+}$. Therefore, $R^{\xi}_{F_N}(x) \subseteq A_{\xi+}$, then by the definition of lower approximation operator it follows that $x \in \underline{R_{F_N}}^{\xi}\left(A_{\xi+}\right)$. Therefore, $\left[\underline{R}_{F_N}(A)\right]_{\xi+} \subseteq \underline{R_{F_N}}^{\xi}\left(A_{\xi+}\right)$. Also, for any $x \in \underline{R_{F_N}}^{\xi+}\left(A_{\xi}\right)$, and $\underline{R_{F_N}}^{\xi+}\left(A_{\xi}\right) = 1$. Then,

$$
\begin{aligned}
T_{\underline{R}_{F_N}}(x) &= \bigwedge_{\xi'\in[0,1]}\left[\xi' \vee \underline{R_{F_N}}^{\xi'+}\left(A_{\xi'}\right)(x)\right] \\
&= \bigvee_{\xi'\in[0,1]}\left[\xi' \wedge \underline{R_{F_N}}^{\xi'+}\left(A_{\xi'}\right)(x)\right] \\
&\geq \xi \wedge \underline{R_{F_N}}^{\xi'+}\left(A_{\xi}\right)(x) = \xi
\end{aligned}
$$

Hence $x \in \left[\underline{R}_{F_N}(A)\right]_{\xi} \subseteq \underline{R_{F_N}}^{\xi+}\left(A_{\xi}\right)$ and $\underline{R_{F_N}}^{\xi+}\left(A_{\xi}\right) \subseteq \left[\underline{R}_{F_N}(A)\right]_{\xi}$.

Similarly, to prove (5) and (6) and hence (7), (8) and (9) can be concluded.

4.2 Topology of Fermatean Temporal Neutrosophic Rough Sets

This section introduces the concept of the Fermatean temporal neutrosophic rough set and its associated topology, and discusses some of its properties.

Fermatean Temporal Neutrosophic Rough Set

Definition 4.2.1

Let Q be a universe and T be a non-empty set and the elements of T be time moments. Let a nonempty set $A(T)$ be a Fermatean temporal neutrosophic set. For an arbitrary Fermatean neutrosophic relation R_{F_N} over $(Q \times Q)$, the pair $\left(Q, R_{F_N}\right)_t$ is called Fermatean temporal neutrosophic approximation space (FTNAS) at time t, (where $t \in T$ can be discrete or continuous). For any $A(T) \in F(Q)$ of the form

$$A(T) = \{\langle x, T_A(x,t), I_A(x,t), F_A(x,t)\rangle : (x,t) \in Q \times T\}$$

the upper and lower approximations of $A(T)$ with respect to $\left(Q, R_{F_N}\right)$, denoted by $\underline{R}_{F_N}(A(T)\,)$ and $\overline{R}_{F_N}(A(T)\,)$ respectively defined as

$$\underline{R}_{F_N}(A(T)\,) = \left\{\left\langle x, T_{\underline{R}_{F_N}(A)}(x,t), I_{\underline{R}_{F_N}(A)}(x,t), F_{\underline{R}_{F_N}(A)}(x,t)\right\rangle / (x,t) \in Q \times T\right\}$$

$$\overline{R}_{F_N}(A(T)\,) = \left\{\left\langle x, T_{\overline{R}_{F_N}(A)}(x,t), I_{\overline{R}_{F_N}(A)}(x,t), F_{\overline{R}_{F_N}(A)}(x,t)\right\rangle / (x,t) \in Q \times T\right\}$$

where,

$$T_{\underline{R}_{F_N}(A)}(x,t) = \underset{y \in Q}{\wedge} \left[\left(F_{R_{F_N}}(x,y), t\right) \vee T_A(y,t)\right]$$

$$I_{\underline{R}_{F_N}(A)}(x,t) = \underset{y \in Q}{\wedge} \left[\left(\left(1 - I_{R_{F_N}}(x,y)\right), t\right) \vee I_A(x,t)\right]$$

$$F_{\underline{R}_{F_N}(A)}(x,t) = \underset{y \in Q}{\vee} \left[\left(T_{R_{F_N}}(x,y), t\right) \wedge F_A(y,t)\right]$$

$$T_{\overline{R}_{F_N}(A)}(x,t) = \bigvee_{y\in Q}\left[\left(T_{R_{F_N}}(x,y),t\right) \wedge T_A(y,t)\right]$$

$$I_{\overline{R}_{F_N}(A)}(x,t) = \bigvee_{y\in Q}\left[\left(I_{R_{F_N}}(x,y),t\right) \wedge I_A(y,t)\right]$$

$$F_{\overline{R}_{F_N}(A)}(x,t) = \bigwedge_{y\in Q}\left[\left(F_{R_{F_N}}(x,y),t\right) \vee F_A(y,t)\right]$$

The pair $\left(\underline{R}_{F_N}(A(T)\,), \overline{R}_{F_N}(A(T)\,)\right)$ is called Fermatean temporal neutrosophic rough set (FTNRS) of $A(T)$ with respect to $\left(Q, R_{F_N}\right)_t$ and $\underline{R}_{F_N}, \overline{R}_{F_N}: F(Q) \to F(Q)$ and referred to as upper and lower Fermatean temporal neutrosophic rough approximation operators respectively.

Let Q be an approximation space, and let $A(T) = \left(\underline{R}_{F_N}(A(T)\,), \overline{R}_{F_N}(A(T)\,)\right) \in Q$ be a FTNRS with $\underline{R}_{F_N}\big(A(T)\big) \subset \overline{R}_{F_N}\big(A(T)\big)$. Let C_X be the collection of all $FTNRSs$ in $A(T)$. $0^t_{FNR} = \langle x, 0,0,1\rangle : (x,t) \in X \times T$ and $1^t_{FNR} = \langle x, 1,0,0\rangle : (x,t) \in X \times T$ are respectively called null FTNRS and whole FTNRS in X. Clearly $C(0^t_{FNR}) = 1^t_{FNR}$ and $C(1^t_{FNR}) = 0^t_{FNR}$

Definition 4.2.2

Let $X = \left(\underline{R}_{F_N}\big(A(T)\big), \overline{R}_{F_N}\big(A(T)\big)\right)$ be a Fermatean temporal neutrosophic rough set and τ_t be a family of $FTNRSs$ satisfies the following axioms for fixed time moment $t \in T$ in $A(T)$ such that

i. $0^t_{FNR}, 1^t_{FNR} \in \tau_t$.
ii. For each $A_1, A_2 \in \tau_t$, there exist a $F \in \tau_t$ such that $T_F(x,t) = T_{A_1 \cap_t A_2}(x,t)$ $I_F(x,t) = I_{A_1 \cap_t A_2}(x,t)$, $F_F(x,t) = F_{A_1 \cap_t A_2}(x,t)$ for each $(x,t) \in X \times \{t\}$.
iii. For any arbitrary family $\{A_i; i \in J\} \in \tau_t$, there exist a $D \in \tau_t$ such that $T_D(x,t) = T_{\cup^t_{i\in J} A_i}(x,t)$, $I_D(x,t) = I_{\cup^t_{i\in J} A_i}(x,t)$, $F_D(x,t) = F_{\cup^t_{i\in J} A_i}(x,t)$ for each $(x,t) \in A \times \{t\}$.

Then τ_t is called a topology of $FTNRSs$ in $A(T)$ and the pair $(A(T), \tau_t)$ is called a topological space of $FTNRSs$ in $A(T)$. Every member of τ_t is called open $FTNRS$.

Theorem 4.2.3

Let F_t denote the collection of all closed $FTNRSs$ in $(A(T), \tau_t)$ Then the collection F_t of all closed $FTNRSs$ fulfills the following axioms:

a) $0_{FNR}^{t}, 1_{FNR}^{t} \in F_t$.

b) $A_1(t), A_2(t) \in F \Rightarrow A_1(t) \cup A_2(t) \in F_t$.

c) $A_i(t) \in F, i \in J \Rightarrow \cap_{i \in J} A_i(t) \in F_t$.

Proof. The proof is straightforward.

Definition 4.2.4

Consider $A(T)$ be a $FTNRS$ in X**.** The union of all open $FTNRSs$ in (X, τ_t) contained in $A(T)$ is called the interior of $A(T)$ in (X, τ_t) for fixed time moment $t \in T$, and is denoted by $Int_{\tau_t}A(T)$. Clearly $Int_{\tau_t}A(T)$ is the largest open $FTNRS$ contained in $A(T)$ and $A(T)$ is open iff $A(t) = Int_{\tau_t}A(t)$.

Definition 4.2.5

Let $A(T)$ be a $FTNRS$ in X. The closure of $A(T)$ in (X, τ_t) denoted $Cl_T A(T)$, is defined by the intersection of all closed $FTNRSs$ in (X, τ_t) containing $A(T)$ for fixed time moment $t \in T$. Clearly $Cl_{\tau_t}A(T)$ is the smallest closed $FTNRS$ containing and $A(T)$ is closed iff $A(t) = Cl_{\tau_t}A(t)$.

Theorem 4.2.6

Let $A(t)$ and $B(t)$ be a $FTNRS$s in $X(T)$.

i) $cl_{\tau_t}(0_{FTN}) = 0_{FTN}$,

ii) $cl_{\tau_t}\left(cl_{\tau_t}(A(t))\right) = cl_{\tau_t}(A(t))$.

iii) $A(t) \subset B(t) \Rightarrow cl_{\tau_t}(A(t)) \subset cl_{\tau_t}(B(t))$.

iv) $cl_{\tau_t}(A(t) \cup B(t)) = cl_{\tau_t}(A(t)) \cup cl_{\tau(t)}(B(t))$.

v) $cl_{\tau_t}\big(A(t) \cap B(t)\big) \subset cl_{\tau_t}\big(A(t)\big) \cap cl_{\tau_t}\big(B(t)\big)$.

Proof. The proof is straightforward.

Definition 4.2.7

Let Q and R be two universes and $f: Q \to R$ be a mapping. Let $A(t) = \big(T_{A(t)}, I_{A(t)}, F_{A(t)}\big)$ be an $FTNRS$ in $X(t) = \left(\underline{R}_{F_N}(A,t), \overline{R}_{F_N}(A,t)\right)$ and $Y(t) = f\big(X(t)\big) \in V \times V, t \in T$ for fixed time moment $t \in T$, where $\underline{R}_{F_N}\big(Y(t)\big) = f\left(\underline{R}_{F_N}(A,t)\right)$ and $\overline{R}_{F_N}\big(Y(t)\big) = f\left(\overline{R}_{F_N}(A,t)\right)$. Then, image of $A(t)$ under f by

$$f\big(A(t)\big) = \begin{bmatrix} \left(f\left(T_{\underline{R}_{F_N}(A(t))}\right), f\left(T_{\bar{R}_{F_N}(A(t))}\right)\right), \\ \left(f\left(I_{\underline{R}_{F_N}(A(t))}\right), f\left(I_{\bar{R}_{F_N}(A(t))}\right)\right), \\ \left(f\left(F_{\underline{R}_{F_N}(A(t))}\right), f\left(F_{\bar{R}_{F_N}(A(t))}\right)\right) \end{bmatrix}$$

where,

$$f\begin{pmatrix} T_{\underline{R}_{F_N}(A(t))}(y), \\ I_{\underline{R}_{F_N}(A(t))}(y), \\ F_{\underline{R}_{F_N}(A(t))}(y) \end{pmatrix} = \begin{Bmatrix} \vee\left[T_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \vee\left[I_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \wedge\left[F_{\underline{R}_{F_N}(A(t))}(x)\right] \end{Bmatrix}, x \in \underline{R}_{F_N}\big(A(t)\big) \cap f^{-1}(y), \forall y \in \underline{R}_{F_N}\big(Y(t)\big)$$

$$f\begin{pmatrix} T_{\overline{R}_{F_N}(A(t))}(y), \\ I_{\overline{R}_{F_N}(A(t))}(y), \\ F_{\overline{R}_{F_N}(A(t))}(y) \end{pmatrix} = \begin{Bmatrix} \vee\left[T_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \vee\left[I_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \wedge\left[F_{\underline{R}_{F_N}(A(t))}(x)\right] \end{Bmatrix} x \in \overline{R}_{F_N}\big(A(t)\big) \cap f^{-1}(y), \forall y \in \underline{R}_{F_N}\big(Y(t)\big)$$

$$= \begin{Bmatrix} \vee\left[T_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \vee\left[I_{\underline{R}_{F_N}(A(t))}(x)\right], \\ \wedge\left[F_{\underline{R}_{F_N}(A(t))}(x)\right] \end{Bmatrix} x \in \overline{R}_{F_N}\big(A(t)\big) \cap f^{-1}(y), \forall y \in x \in \overline{R}_{F_N}\big(Y(t)\big)\underline{R}_{F_N}\big(Y(t)\big)$$

Definition 4.2.8

Let $f: Q \rightarrow R$ be a mapping. Let $X(t) = \left(\underline{R}_{F_N}(X,t), \overline{R}_{F_N}(X,t)\right) \subseteq Q$ and $Y(t) = \left(\underline{R}_{F_N}(Y,t), \overline{R}_{F_N}(Y,t)\right) \subseteq R$ are $FTNRS$s in Q and R respectively. Let $A(t) = \left(T_{A(t)}, I_{A(t)}, F_{A(t)}\right)$ be an $FTNRS$ in $Y(t)$ for fixed time moment $t \in T$. Then $X(t) = f^{-1}\left(Y(t)\right) \in Q$,where $\underline{R}_{F_N}(X,t) = f^{-1}\underline{R}_{F_N}(Y,t)$ and $\overline{R}_{F_N}(X,t) = f^{-1}\overline{R}_{F_N}(Y,t)$. Then the inverse image $f^{-1}A(t)$ of $A(t)$ under f, is defined by

$$f^{-1}A(t) = \left(f^{-1}\left(T_{A(t)}\right), f^{-1}\left(I_{A(t)}\right), f^{-1}\left(F_{A(t)}\right)\right).$$

Theorem 4.2.9

Let $f: Q \rightarrow R$ be a mapping. Then for all $FTNRSs$ $A(t)$ and $B(t)$, the following properties hold

(i) $A(t) \subset B(t) \Rightarrow f\left(A(t)\right) \subset f\left(B(t)\right)$.

(ii) $f\left(A(t) \cup B(t)\right) = f\left(A(t)\right) \cup f\left(B(t)\right)$.

(iii) $f\left(A(t) \cap B(t)\right) \subset f\left(A(t)\right) \cap f\left(B(t)\right)$.

Definition 4.2.10

Let $(X(t), \tau_t)$ and $(Y(t), \sigma_t)$ be two topological spaces of $FTNRSs$ and $f: Q \rightarrow R$ be a mapping. Then $f: (X(t), \tau_t) \rightarrow (Y(t), \sigma_t)$ is said to be $FTNR$ continuous if $f^{-1}\left(A(t)\right) \in \tau_t, \forall \tilde{A} \in \sigma_t$.

Theorem 4.2.11

The following statements are equivalent:

i. $f: \left(\hat{X}(t), \tau_t\right) \rightarrow \left(\hat{Y}(t), \sigma_t\right)$ is $FTNR$ continuous.

ii. $f^{-1}\left(\tilde{A}(t)\right)$ is closed $FTNRS$ in $(\hat{X}, T)$, for every closed $FTNRS$ $\tilde{A}(t)$ in $\left(\hat{Y}(t), \sigma_t\right)$

iii. $f\left(cl_{\tau_t}\tilde{P}(t)\right) \subset cl_{\sigma_t}\left(f\left(\tilde{P}(t)\right)\right)$, for every $FTNRS$ $\tilde{P}(t)$ in $\hat{X}(t)$.

Proof. (i) ⇔ (ii)

Suppose, $f\colon \left(\hat{X}(t), \tau_t\right) \to \left(\hat{Y}(t), \sigma_t\right)$ is $FTNR$ continuous and let $\tilde{A}(t) \in \hat{Y}(t)$ be a closed set. Since $\left(\hat{Y}(t) - \tilde{A}(t)\right)$ is open in $\hat{Y}(t)$ and by continuity $f^{-1}\left(\hat{Y}(t) - \tilde{A}(t)\right)$ is open in $\hat{X}(t) \Rightarrow \hat{X}(t) - f^{-1}\left(\hat{Y}(t) - \tilde{A}(t)\right) = f^{-1}\left(\tilde{A}(t)\right)$ is closed.

Therefore $f^{-1}\left(\tilde{A}(t)\right)$ is $FTNR$ closed.

Conversely, Let $\tilde{A}(t) \in \hat{Y}(t)$ be an open set, $\hat{Y}(t) - \tilde{A}(t)$ is closed in $\hat{Y}(t) \Rightarrow f^{-1}\left(\hat{Y}(t) - \tilde{A}(t)\right)$ is closed in $\left(\hat{X}(t), \tau(t)\right)$ $\hat{X}(t) - f^{-1}\left(\hat{Y}(t) - \tilde{A}(t)\right)$ is an open $FTNRS$ in $\left(\hat{X}(t), \tau_t\right) \Rightarrow f^{-1}\left(\tilde{A}(t)\right)$ is open.

Hence f is continuous.

(i) ⇔ (iii)

Let $\tilde{P}(t) \in \hat{X}(t)$ be a closed set. Then $cl\left(\tilde{P}(t)\right) = \left(\bigcap_{j=1}^{n} \tilde{P}_j(t)\right)$, such that $\tilde{P}_j(t) \in T$ implies $f\left(cl\left(\tilde{P}(t)\right)\right) = f\left(\bigcap_{j=1}^{n} \tilde{P}^j(t)\right) \subset \bigcap_{j\in J} f\left(\tilde{P}_j(t)\right)$

[by Theorem 4.2.9] = $cl\left(f\left(\tilde{P}(t)\right)\right)$. Converse is straightforward.

4.3. Application of Fermatean Temporal Neutrosophic Rough Sets

In many data analysis and decision-making tasks, datasets often contain redundant or irrelevant attributes that increase complexity and reduce efficiency. Attribute reduction aims to identify the smallest possible subset of attributes that preserves the dataset's classification or decision-making capability.

Using our defined Fermatean neutrosophic rough set**,** Fermatean temporal neutrosophic rough set, and Fermatean temporal neutrosophic rough topology**,** an attribute reduction method that removes redundant or irrelevant attributes has been developed by keeping the dataset's original decision-making ability. This method considers both uncertainty and time-based changes, ensuring the selected attributes remain accurate and consistent.

Let $S = ((Q, A)_t)$ be an Information System (IS) with time t, (where $t \in T$ can be discrete or continuous). where Q is a set of non-empty objects called universe, A is a set of attributes. The element of Q can be called object, case, instance, or observation. The attributes can be called features, variables, or characteristic conditions. If an attribute a is given, then $a: Q \to V_a$ for $a \in A$. V_a is called the set of Fermatean neutrosophic values of a at time t.

Definition 4.3.1

If $a \in A$, $H \subseteq A$, then an indiscernibility relation $IND(H_t)$ at fixed time $t \in T$ can be defined as:

$$IND(H_t) = \{((m,n)_t) | (m,n) \in U \times U, t \in T: \text{for all } a \in \mathrm{H}, a(m_t) = a(n_t)\}$$

or in the statement that the two objects are said to be indiscernible at fixed time t when the two objects are indistinguishable since they do not have sufficient differences in the set of attributes called H.

Definition 4.3.2

Let $S = ((Q, A)_t)$ be an information system, $H \subset A$, and let $a \in H, t \in T$ is fixed. It can be said that a is dispensable in H at time t, if $IND_S(H_t) = IND_S(H - a)_t$ otherwise

$IND_S(H_t) \neq IND_S(H - a)_t$ is indispensable in H at t. A set H is called independent if all of its attributes are indispensable.

Any subset H' of H is called a reduct of H if H' is independent at fixed time $t \in T$ and

$$IND_T(H'_t) = IND_T(H_t).$$

Therefore, reduct is the minimal set of attributes without changing the classification results when using all attributes. In other words, the attributes not in reduct are considered redundant and have no effect on classification.

Definition 4.3.3

Let P be a subset of A. The core of P is the set of all indispensable attributes of P at fixed time $t \in T$.

$$Core(P_t) = \cap\, Red(P_t)$$

where $Red(P_t)$ is the set of all reducts of P_t.

Working rule for attribute reduction:

This section developes an algorithm to find the deciding factors or core to pick the minimum number of attributes necessary for the classification of objects.

Let $S = \big((Q, A), t\big)$ be an information system. Let Q be the universe, $R_{F_N}(t)$ be the Fermatean neutrosophic relation on $A(t)$ and $(A(t), \tau_t)$ be a Fermatean temporal neutrosophic rough topology in S at fixed time $t \in T$.

Let $P_t \subset R_{F_N}(t)$, and $r \in P$. Then by definition 4.3.2, $IND_S(P_t) = IND_S(P - r)_t$.

A subset P_t, the set of attributes is known as the core of $R_{F_N}(t)$ if

$$IND_S(P_t) \neq IND_S(P - r)_t.$$

That is, a core of $R_{F_N}(t)$ is a subset of attributes which is in a way that none of its elements can be removed without affecting the classification power of attributes.

The procedure can be put in the form of an algorithm as follows:

Step 1: Let Q be the universe of discourse, and $A(t)$ be the set of attributes. An FN-r $R_{F_N}(t)$ with respect to fixed time t is defined on Q . The elements of the universe are represented by the columns, the attributes by the rows, and the values of the attributes are given by the table entries with Fermaten neutrosophic values. Apply the Fematean neutrosophic level level-cut to each value to produce crisp values (or classes) per attribute. Build the equivalence relation on Q: two objects are equivalent iff their level-cut values match for all attributes in A at fixed time t.

Step 2: Fix a target set $W(t) \subseteq Q$ at time t. Find the lower approximation, upper approximation and the boundary region of $W(t)$ with respect to $R_{F_N}(t)$.

Step 3: Generate the topologies $(A(t), \tau_t)$on U and also $(W(t), \tau_t)$.

Step 4: Remove an attribute x from $A(t)$ and find the lower and upper approximations and the boundary region of $W(t)$ with respect to the equivalence relation on $A(t) - \{x\}$.

Step 5: Generate the Fermatean temporal neutrosophic topology $(A(t) - \{x\}, \tau_t)$ on Q.

Step 6: Repeat steps 3 and 4 for all attributes in $A(t)$.

Step 7: Those attributes in $A(t)$ for which $(A(t), \tau_t) \neq (A(t) - \{x\}, \tau_t)$ form the core of $R_{F_N}(t)$.

Illustration.

Consider a universe $Q = \{X_1, X_2, X_3, X_4, X_5\}$, set of attributes and $A = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ criteria. The relation between set of attributes and criteria given in the Table 4.3.1 using Fermaten neutrosophic values at fixed time $t = 1$. In this example fix two target sets such as

CASE 1. $W_1(1) = \{X_1, X_3\}$

CASE 2. $W_2(1) = \{X_2, X_4, X_5\}$

Table 4.3.1

$Q/R_{F_N}(1)$	X_1	X_2	X_3	X_4	X_5
C_1	(0.6,0.7,0.5)	(0.8,0.9,0.5)	(0.75,0.85,0.5)	(0.6,0.65,0.4)	(0.8,0.6,0.3)
C_2	(0.8,0.8,0.4)	(0.4,0.6,0.9)	(0.5,0.7,0.8)	(0.8,0.75,0.3)	(0.3,0.7,0.85)
C_3	(0.3,0.7,0.8)	(0.8,0.75,0.5)	(0.8,0.8,0.4)	(0.4,0.7,0.8)	(0.5,0.6,0.9)
C_4	(0.95,0.8,0.5)	(0.8,0.95,0.7)	(0.8,0.9,0.4)	(0.9,0.8,0.4)	(0.95,0.85,0.4)
C_5	(0.9,0.75,0.4)	(0.4,0.65,0.8)	(0.3,0.7,0.9)	(0.9,0.85,0.3)	(0.95,0.8,0.3)
C_6	(0.8,0.9,0.4)	(0.95,0.8,0.3)	(0.9,0.75,0.5)	(0.8,0.85,0.4)	(0.7,0.8,0.4)
C_7	(0.9,0.8,0.4)	(0.8,0.6,0.5)	(0.7,0.8,0.3)	(0.9,0.8,0.4)	(0.95,0.8,0.2)
C_8	(0.8,0.9,0.4)	(0.3,0.6,0.8)	(0.4,0.6,0.7)	(0.8,0.9,0.5)	(0.8,0.85,0.3)

Using Fematean neutrosophic level-cut values such as $\xi \in (0.8,1]$, $\psi \in (0.7,0.8]$ and $\zeta \in (0.5,0.6]$ the equivalence class is given by

$$Q/R_{F_N(\xi,\psi,\zeta)} = \{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$$

CASE 1.

Let $Q = \{X_1,X_2,X_3,X_4,X_5\}$ and $W_1(1) = \{X_1,X_3\}$. The set of equivalence classes corresponding to $R_{F_N(\xi,\psi,\zeta)}(1)$ is $Q/R_{F_N(\xi,\psi,\zeta)} = \{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$ then the Fermatean temporal neutrosophic topology $(W(1),\tau_1)$ on Q is given by

$$\tau_{R_{F_N(\xi,\psi,\zeta)}}(W_1(1)) = \{Q,\phi,\{X_3\},\{X_1,X_4\},\{X_1,X_3,X_4\}\}.$$

If C_1 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_1\} = \{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_1\}}(W_1(1)) = \{Q,\phi,\{X_3\},\{X_1,X_4\},\{X_1,X_3,X_4\}\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_1(1))$.

If C_2 is removed from set of condition attributes,

$$Q/R_{F_N}(1) - \{C_2\} = \{\{X_1,X_4,X_5\},\{X_2\},\{X_3\}\}$$

Then

$$\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_2\}}(W_1(1)) = \{Q,\phi,\{X_3\},\{X_1,X_4,X_5\},\{X_1,X_3,X_4,X_5\}\}$$

$\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_2\}}\big(W_1(1)\big) \neq \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

If C_3 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_3\} = \big\{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_3\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\big\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

If C_4 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_4\} = \big\{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_4\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\big\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

If C_5 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_5\} = \big\{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_5\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\big\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

If C_6 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_6\} = \big\{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_6\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\big\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$

If C_7 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_7\} = \big\{\{X_1, X_4\}, \{X_2, X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_7\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_2, X_4\}, \{X_1, X_2, X_3, X_4\}\big\} \neq \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

If C_8 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_8\} = \big\{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\big\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_8\}}\big(W_1(1)\big) = \big\{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\big\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}\big(W_1(1)\big)$.

From CASE 1, the $Core\left(R_{F_N(\xi,\psi,\zeta)}(1)\right) = \{C_2, C_7\}$.

CASE 2.

Let $Q = \{X_1, X_2, X_3, X_4, X_5\}$ and $W_2(1) = \{X_2, X_4, X_5\}$. Using Fematean neutrosophic level-cut values such as $\xi \in (0.8,1]$, $\psi \in (0.7,0.8]$ and $\zeta \in (0.5,0.6]$ the equivalence class is given by

$$Q/R_{F_N(\xi,\psi,\zeta)} = \{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\},$$

then the Fermatean temporal neutrosophic topology $(W(1), \tau_1)$ on Q is given by

$$\tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1)) = \{Q, \phi, \{X_2, X_5\}, \{X_1, X_4\}, \{X_1, X_2, X_4, X_5\}\}.$$

If C_1 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_1\} = \{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_1\}}(W_2(1)) = \{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1))$.

If C_2 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_2\} = \{\{X_1, X_4, X_5\}, \{X_2\}, \{X_3\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_2\}}(W_2(1)) = \{Q, \phi, \{X_2\}, \{X_1, X_4, X_5\}, \{X_1, X_2, X_4, X_5\}\} \neq \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1))$.

If C_3 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_3\} = \{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_3\}}(W_2(1)) = \{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1))$.

If C_4 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1) - \{C_4\} = \{\{X_1, X_4\}, \{X_2\}, \{X_3\}, \{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_4\}}(W_2(1)) = \{Q, \phi, \{X_3\}, \{X_1, X_4\}, \{X_1, X_3, X_4\}\} = \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1))$.

If C_5 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1)-\{C_5\}=\{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_5\}}(W_2(1))=\{Q,\phi,\{X_3\},\{X_1,X_4\},\{X_1,X_3,X_4\}\}=\tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(1))$.

If C_6 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(1)-\{C_6\}=\{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_6\}}(W_2(t))=\{Q,\phi,\{X_3\},\{X_1,X_4\},\{X_1,X_3,X_4\}\}=\tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(t))$

If C_7 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(t)-\{C_7\}=\{\{X_1,X_4\},\{X_2,X_3\},\{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_7\}}(W_2(t))=\{Q,\phi,\{X_5\},\{X_1,X_2,X_4\},\{X_1,X_2,X_4,X_5\}\}\neq \tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(t))$.

If C_8 is removed from set of condition attributes,

$$Q/R_{F_N(\xi,\psi,\zeta)}(t)-\{C_8\}=\{\{X_1,X_4\},\{X_2\},\{X_3\},\{X_5\}\}$$

Then $\tau_{R_{F_N(\xi,\psi,\zeta)}-\{C_8\}}(W_2(t))=\{Q,\phi,\{X_3\},\{X_1,X_4\},\{X_1,X_3,X_4\}\}=\tau_{R_{F_N(\xi,\psi,\zeta)}}(W_2(t))$.

From CASE 2, the $Core\left(R_{F_N(\xi,\psi,\zeta)}(t)\right)=\{C_2,C_7\}$.

Observation.

From the Core of $R_{F_N(\xi,\psi,\zeta)}(t)$, it is concluded that $\{C_2,C_7\}$ attributes are necessary.