CHAPTER 2

CHAPTER 2

INTUITIONISTIC FUZZY SETS

In this chapter fundamentals of intuitionistic fuzzy topological spaces, Intuitionistic fuzzy topological spaces, intuitionistic fuzzy Continuous mappings, intuitionistic fuzzy Compact spaces and intuitionistic fuzzy Hausdorff spaces due to Dogan Coker [22] are studied.

Section 2.1

Preliminary Definitions and Results of Intuitionstic Fuzzy Sets

Definition: 2.1.1

(Antanassov [6]). Let X be a nonempty fixed set. An intuitionstic fuzzy set (IFS in short) A is an object having the form $A = \{x, \mu_A(x), \gamma_A(x)): x \in X\}$, where the functions $\mu_A: X \rightarrow I$ and $\gamma_A: X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \varepsilon X$.

Remark: 2.1.2

An intutionistic fuzzy set $A = \{x, \mu_A(x), \gamma_A(x)): x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$ or to an element in $(I \times I)^X$.

Remark: 2.1.3

For the sake of simplicity, we shall use the symbol $A = (x, \mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Example: 2.1.4

Every fuzzy set A on a nonempty set X is obviously an IFS having the form $A = \{(x, \mu_A(x), 1- \mu_A(x)) : x \in X\}$.

Definition: 2.1.5

Let X be a nonempty set, and the IFSs A and B be $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, $B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}$. Then

a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

c) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)): x \in X\}$;

d) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)): x \in X\}$;

e) $A \cup B = \{x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)): x \in X\}$;

f) $[\,]\, A = \{x, \mu_A(x), 1 - \mu_A(x)): x \in X\}$;

g) $\langle\,\rangle\, A = \{(x, 1-\gamma_A(x), \gamma_A(x)): x \in X\}$.

Definition: 2.1.6

Let $\{A_i: i \in J\}$ be an arbitrary family of IFSs in X. Then

a) $\cap A_i = \{(x, \wedge \mu_{A_i}(x), \vee \gamma_{A_i}(x)) : x \in X\}$;

b) $\bigcup A_i = \{(x, \vee \mu_{A_i}(x), \wedge \gamma_{A_i}(x)) : x \in X\}$;

Definition: 2.1.7

$0\sim = \{(x,0,1) : x \in X\}$ and $1\sim = \{(x,1,0) : x \in X\}$.

Basic properties of inclusion and complementation:

Corollary: 2.1.8

Let A, B, C be IFSs in X. Then

a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$,

b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \cap C$,

c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$,

d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$,

e) $\overline{A \cup B} = \bar{A} \cap \overline{B}$, (f) $\bar{A}\, \overline{\cap B} = \bar{A} \cup \overline{B}$

g) $A \subseteq B \Rightarrow B\, \overline{\subseteq}\, \bar{A}$, (h) $(\bar{A}) = A$.

i) $\bar{1}\sim = 0\sim$, (j) $0\overline{\sim} = 1\sim$

Proof

We shall only prove (e), and the others are obvious.

e) Let $A = (x, \mu_A, \gamma_A)$ and $B = (x, \mu_B, \gamma_B)$. Then we have

$A \cup B = (x, \mu_A, \vee \mu_B, \gamma_A \wedge \gamma_B)$

$\Rightarrow \overline{A \cup B} = (x, \gamma_A, \vee \gamma_B, \mu_A \wedge \mu_B)$ and

$\bar{A} = (x, \gamma_A, \mu_A), \overline{B} = (x, \gamma_B, \mu_B)$

$\Rightarrow \bar{A} \cap \overline{B} = (x, \gamma_A \wedge \gamma_B, \mu_A \vee \mu_B)$

i.e. $\overline{A \cup B} = \bar{A} \cap \overline{B}$ follows.

Definition: 2.1.9

(a) If $B = \{(y, \mu_B(y), \gamma_B(y)) : y \in Y\}$ is an IFS in Y, then the preimage of B under f, denoted by f^{-1} (B), is the IFS in X defined by

$f^{-1} (B) = \{(x, f^{-1} (\mu_B(x), f^{-1}(\gamma_B)(x)) : x \in X\}$.

b) If $A = \{x, \lambda_A(x), \vartheta_A(x)) : x \in X\}$ is an IFS in X, then the image of A under f, denoted by f(A), is the IFS in Y defined by

$f(A) = \{(y, f(\lambda_A)(y), (1-f(1-\vartheta_A))(y)) : y \in Y\}$

where $f(\lambda_A)(y) = \left\{ \begin{matrix} \inf\limits_{x \varepsilon f^{-1}(y)} & \vartheta_A(x) \quad iff_{-1}(y) \neq \theta \\ 1 & otherwise \end{matrix} \right\}$

For the sake of simplicity, let us use the symbol $f - (\vartheta_A)$ for $1-f(1-\vartheta_A)$.

Corollary: 2.1.10

Let A, A_i ($i \varepsilon J$) be IFSs in X, B, B_j ($j \varepsilon K$) IFSs in Y and f: X → Y a function. Then

a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

b) $B_1 \subseteq B2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

c) $A \subseteq f^{-1}(A))$

[If f is injective, then $A = f^{-1}(f(A))$]

d) $f(f^{-1}(B) \subseteq B$

[If f is surjective, then $f(f^{-1}(B)) = B$]

e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

f) $f^{-1}(\cap f^{-1}(B_j) = \cap f^{-1}(B_j)$

g) $f(\bigcup A_i) = \bigcup f(A_i)$

h) $f(\cap A_i) \subseteq \cap f(A_i)$

[If f is injective, then $f(\cap A_i) = \cap f(A_i)$]

i) $f^{-1}(1\sim) = 1\sim$, (j) $f^{-1}(0\sim) = 0\sim$,

j) $f(1\sim) = 1\sim$,if f is surjective,

k) $f(0_\sim) = 0_\sim$

l) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective,

m) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Section 2.2

Intuitionstic Fuzzy Topological Spaces

Intuitionistic fuzzy topological spaces where introduced by Cooker [22] which is the generalization of fuzzy topological spaces introduced by Chang [20].

Definition: 2.2.1

An intutionistic fuzzy topology (IFT for short) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(T_1) $0_\sim, 1_\sim \in \tau$,

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(T_3) $\cup G_i \in \tau$ for any arbitrary family $(G_i;\ i \in J) \subseteq \tau$.

In this case the pair (X,τ) is called an intutionistic fuzzy topological space (IFTS for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS for short) in X.

Example: 2.2.2

Any fuzzy topological space (X, τ_0) in the sense of Chang is obviously an IFTS in the form $\tau = \{A : \mu_A \in \tau_0)$ whenever we identify a fuzzy set in X whose membership function is μ_A with its counter-part $A = \{(x, \mu_A(x), 1 - \mu_A(x)\}$ as in Example 2.1.4.

Example: 2.2.3

Let X = {a,b,c} and

$$A = \left\langle x, \left(\frac{a}{05}, \frac{b}{0.5}, \frac{c}{0.4}\right), \left(\frac{a}{02}, \frac{b}{0.4}, \frac{c}{0.4}\right)\right\rangle,$$

$$B = \left\langle x, \left(\frac{a}{04}, \frac{b}{0.6}, \frac{c}{0.2}\right), \left(\frac{a}{05}, \frac{b}{0.3}, \frac{c}{0.3}\right)\right\rangle,$$

C = $\left\langle x, \left(\frac{a}{05}, \frac{b}{0.6}, \frac{c}{0.4}\right), \left(\frac{a}{02}, \frac{b}{0.3}, \frac{c}{0.3}\right)\right\rangle$,

D = $\left\langle x, \left(\frac{a}{04}, \frac{b}{0.5}, \frac{c}{0.2}\right), \left(\frac{a}{05}, \frac{b}{0.4}, \frac{c}{0.4}\right)\right\rangle$,

Then the family τ = {0~, 1~, A, B, C, D} of IFSs in X is an IFT on X.

Example: 2.2.4

Let X = {1,2} and define the IFSs G_n as follows ($n \in N^+$):

$$G_n = \left\langle x, \left(\frac{1}{\frac{n}{n+1}}, \frac{1}{\frac{n+1}{n+2}}\right), \frac{1}{\frac{n}{n+2}}, \frac{2}{\frac{n+1}{n+3}}\right\rangle$$

In this case the family τ = {0~, 1~) $\cup$ {G_n : $n \in N^+$} is an IFT on X.

Example: 2.2.5

Let (X, τ_0) be a fuzzy topological space in Chang' sense such that τ_0 is not indiscrete. Let τ_0 = {0, 1} $\cup$ {ν_i:$i \in J$}.

a) τ^1 = {0~,1~) $\cup$ {(x, ν_{I} , 0) : $i \in J$},

b) τ^2 = {0~,1~} $\cup$ {(x,0,1 - ν_i) : $i \in J$} are two IFTs on X.

Proposition: 2.2.6

Let (X,τ) be an IFTs on X. Then several IFTs on X can be constructed in the following way.

a) $\tau_{0,1}$ = {[ ] : G $\in \tau$}

b) $\tau_{0,2}$ = {[ ] : G $\in \tau$}

Proof

From definition 2.2.1 (a) (T_1) and (T_2) are obvious

(T_3) Let {[ ] G_i:$i \in J$, $G_i \in \tau$} $\subseteq \tau_{0,1}$. Since $\bigcup$ G_i = (x,$\vee$ μ_{Gi}, $\wedge$ γ_{Gi})$\in \tau$,

we have $\cup$ ([ ] G_i) = [x, $\vee$ μ_{Gi}, $\wedge$ (1- μ_{Gi})]

= [x, $\vee$ μ_{Gi}, $\wedge$ (1- μ_{Gi})] $\in \tau_{0,1}$

b) This is similar to (a)

Remark: 2.2.7

Let (X,τ) be an IFTS.

a) $\tau_1 = \{\mu_{G:}\ G \in \tau\}$ is a fuzzy topological space on X in Chang's sense.

b) $\tau_2{}^* = \{\gamma_G : G \in \tau)$ is the family of all fuzzy closed sets of the fuzzy topological space $\tau_2 = \{1-\gamma_G : G \in \tau)$ on X in Chang's sense.

c) Since $0 \le \mu_G(x) + \gamma_G(x) \le 1$ for each $x \in X$ and each $G \in \tau$, we obtain $\mu_G \le 1-\gamma_G$

d) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bifuzzy topological space.

Definition: 2.2.8

Let $(X,\tau_1,)$, (X,τ_2) be two IFTSs on X. Then τ_1 is said to be contained in τ_2 (in symbols, $\tau_1 \subseteq \tau_2$) if $G \in \tau_2$ for each $G \in \tau_1$. In this case, we also say that τ_1 is coarser than τ_2.

Proposition: 2.2.9.

Let $\{\tau_{i\,:}\ i \in J\}$ be a family of IFTs on X. Then $\cap\tau_i$ is an IFT on X. Also, $\cap\tau_i$ is the coarsest IFT on X containing all τ_i's.

Definition: 2.2.10

An intuitionistic fuzzy topological space in the sense of Lowen is a pair (X,τ) where (X,τ) is an IFTS and each IFS in the form $C_{\alpha,\beta} = \{(x, \alpha, \beta): x \in X\}$, where $\alpha, \beta \in I$ are arbitrary and $\alpha+\beta \le 1$, belongs to τ.

Example: 2.2.11

If (X,τ) is an IFTS in the sense of Lowen, then (X, τ_l) and (X, τ_2) [Remark 2.2.7] are also fuzzy topological spaces in the sense of Lowen.

Definition: 2.2.12

The complement $\bar{A}$ of an IFOS A in an IFTS (X,τ) is called an intutionistic fuzzy closed set (IFCS for short) in X.

Definition: 2.2.13

Let (X,τ) be an IFTS and $A = (x, \mu_A, \gamma_A)$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are defined by

$cl(A) = \cap \{K : K$ is an IFCS in X and $A \subseteq K\}$.

$int(A) = \cup \{G : G$ is an IFOS in X and $G \subseteq A\}$.

Remark: 2.2.14

cl(A) is an IFCS and int(A) is an IFOS in X, and

a) A is an IFCS in X iff cl(A) = A;

b) A is an IFOS in X iff int(A) = A.

Example: 2.2.15

Consider the IFTS (X, τ) in Example 3.3 If

$$F = \left\langle x, \left(\frac{a}{0.55}, \frac{b}{0.55}, \frac{c}{0.45}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.3}\right)\right\rangle,$$

then

$$int(F) = \left\langle x, \left(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}\right), \left(\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4}\right)\right\rangle,$$

and cl(F) = 1~.

Proposition: 2.2.16

For any IFS A in (X,τ) we have (a) $cl(\bar{A}) = \overline{int(A)}$, (b) $int(\bar{A}) = \overline{cl(A)}$.

Proof

(a) Let $A = (x, \mu_A, \gamma_A)$ and suppose that the family of IFOS's contained in A are indexed by the family $\{x, \mu_{Gi}, \gamma_{Gi}): i \in J\}$. Then $int(A) = \{x, \vee \mu_{Gi}, \wedge \gamma_{Gi})$ and hence $\overline{int(A)} = (x, \wedge \gamma_{Gi}, \vee \mu_{Gi})$. Since $\overline{A} = (x, \gamma_A, \mu_A)$ and $\mu_{Gi} \leq \mu_A$, $\gamma_{Gi} \geq \gamma_A$ for each $i \in J$, $\{(x, \gamma_{Gi}, \mu_{Gi}): i \in J\}$ is the family of IFSC's containing A, i.e. $cl(\bar{A}) = \{x, \wedge \gamma_{Gi}, \vee \mu_{Gi})$. Hence $cl(\bar{A}) = \overline{int(A)}$ follows immediately.

b) This is analogous to (a)

Proposition: 2.2.17

Let (X, τ) be an IFTS and A, B be IFSs in X. Then the following properties hold:

a) (i) $\mathrm{int}(A) \subseteq A$,

(ii) $A \subseteq \mathrm{cl}(A)$,

b) (i) $A \subseteq B \Rightarrow \mathrm{int}(A) \subseteq \mathrm{int}(B)$,

(ii) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$

c) (i) $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$

(ii) $\mathrm{cl}(A)) = \mathrm{cl}(A)$,

d) (i) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$

(ii) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$

e) (i) $\mathrm{int}(1\sim) = 1\sim$

(ii) $\mathrm{cl}(0\sim) = 0\sim$

Proposition: 2.2.18

Let (X,τ) be an IFTS. If $A = (x,\mu_A, \gamma_A)$ is an IFS in X, then we have

(i) $\mathrm{int}(A) \subseteq (x, \mathrm{int}_{\tau_1}(\mu_A), \mathrm{cl}_{\tau_2}(\gamma_A)) \subseteq A$

(ii) $A \subseteq (x,\mathrm{cl}_{\tau_2}(\mu_A), \mathrm{int}_{\tau_1}(\gamma_A)) \subseteq \mathrm{cl}(A)$,

Where τ_1 and τ_2 are the fuzzy topological spaces on X defined in Remark 2.2.7.

Proof

(i) Let $A = (x, \mu_A, \gamma_A)$ and suppose that the family of IFOSs contained in A are indexed by the family $\{(x,\mu_{Gi}, \gamma_{Gi}) : i \in J\}$. Then $\mathrm{int}(A) = (x, \vee\, \mu_{Gi} \wedge \gamma_{Gi})$. Each member of the family of fuzzy open sets $\{\mu_{Gi} : i \in J\}$ in τ_1 is contained in μ_A and hence $\vee \{\mu_{Gi} : i \in J\} \leq \mathrm{int}\,\tau_1(\mu_A)$. Similarly $\wedge \{\gamma_{Gi}: i \in J\} \geq \mathrm{cl}\tau_2 (\gamma_A)$. Thus we get $\mathrm{int}(A) \subseteq (x, \mathrm{int}\,\tau_1 (\mu_A), \mathrm{cl}\,\tau_2 (\gamma_A)) \subseteq A$.

(ii) is obvious

Corollary: 2.2.19

Let $A = (x, \mu_A, \gamma_A)$ be an IFS in (X, τ). Then

(a) If A is an IFCS, then μ_A is fuzzy closed in (X, τ_2) and γ_A is fuzzy open in $X, \tau_1)$;

(b) If A is an IFCS, then μ_A is fuzzy open in $(X\ \tau_1)$ and γ_A is fuzzy closed in (X, τ_2).

Example: 2.2.20

Consider again the IFTS (X, τ) in Examples 2.2.3 and 2.2.14 now we obtain

int$_{,1}$ (μ_F) = $\left(\frac{a}{05}, \frac{b}{0.5}, \frac{c}{0.4}\right)$,

cl$_{,2}$ (γ_F) = $\left(\frac{a}{05}, \frac{b}{0.4}, \frac{c}{0.4}\right)$,

which justifies the nonequality of the inclusion in Proposition 2.2.17.

Section 2.3

Intuitionstic Fuzzy Continuity

Definition: 2.3.1

Let (X,τ) and (Y,θ) be two IFTSs and let f: X → Y be a function. Then f is said to be fuzzy continuous if the preimage of each IFS in θ is an IFS in τ

Definition: 2.3.2

Let (X,τ) and (y,θ) be two IFTs and let f: X → Y be a function. Then f is said to be fuzzy open iff the image of each IFS in τ is an IFS inθ.

Example: 2.3.3

Let (X, τ_0), (Y, θ_0) be two fuzzy topological spaces in the sense of Chang.

(a) If f: X → Y is fuzzy continuous in the usual sense, then in this case, f is fuzzy continuous in the sense of Definition 4.1, too. Here we consider the IFTs on X and Y respectively, as follows:

$\tau = \{x, \mu_G, 1- \mu_G) : \mu_G \in \tau_0\}$ and

$\theta = \{(y, \lambda_H, 1-\lambda_H) : \lambda_H \in \theta_0\}$

In this case we have, for each $(y, \lambda_H, 1-\lambda_H) \in \mu_H \in \theta_0$,

$f^{-1} ((y, \lambda_{H,})) = (x, f^{-1}(\lambda_H), f^{-1}(1-\lambda_H)) = (x, f^{-1} (\lambda_H)\ 1 - f^{-1}(\lambda_H)) \in \tau$.

b) Let f: X → Y be a fuzzy open function in the usual sense. Then f is also fuzzy open in the sense of Definition 4.2.

Example: 2.3.4

Let (X,τ) be an IFTs in the sense of Lowen, (y,θ) an IFTS and $c_0 \in \gamma$. Then the constant function $c: X \to Y$, $c(x) - c_0$ is obviously fuzzy continuous.

Here we obtain some characterizations of fuzzy continuity:

Proposition: 2.3.5

$f: (X, \tau) \to (Y, \theta)$ is fuzzy continuous iff the preimage of each IFCS in θ is an IFCS in τ.

Proof

This is obvious if we make use of Corollary 2.1.10.

Proposition: 2.3.6.

The following are equivalent to each other.

(a) $f: (X,\tau) \to (Y, \theta)$ is fuzzy continuous.

(b) $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$ for each IFS B in Y.

(c) $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for each IFS B in Y.

Example: 2.3.7

Let (Y,θ) be an IFTS, X a nonempty set and $f: X \to Y$ a function. In this case $\tau = \{f^{-1}(H) : H \in \theta)$ is an IFT on X. Indeed, τ is the coarsest IFT on X which makes the function f: X which makes the function $f: X \to Y$ fuzzy continuous. One may call the IFT τ on X the initial intuitionistic fuzzy topology with respect to f.

Proposition: 2.3.8

Let $f: (X,\tau) \to (Y,\theta)$ be a fuzzy continuous function. Then the functions

(a) $f: (X, \tau_1) \to (Y,\theta_1)$, (b) $f : (X, \tau_2) \to (Y,\theta_2)$ are also fuzzy continuous where τ_1, θ_2 are the fuzzy topological spaces defined in Remark 3.7. (In other words, $\tau_1 =\{\mu_{Gi}: \in \tau\}$, $\theta_1 = \{\lambda_H; H_j = \in \theta)$, $\tau_2 = \{1-\gamma_{Gi}: G_j \in \tau\}$, $\theta_2 = \{\vartheta_{Hi}: H_j \in \theta\}$, if $\tau = \{G_i: i \in J\}$, $\theta = \{H_j: j \in K\}$, $G_i = \{x, \mu_{Gi}, \gamma_{Gi})$ and $H_j = (y,\lambda_{Hi}, \vartheta_{Hi})\}$.

Proof

These follow from definitions, immediately.

Section 2.4

Intuitionstic Fuzzy Compactness

Definition: 2.4.1

Let (X,τ) be an IFTS.

(i) If a family $\{(x, \mu_{Gi}, \gamma_{Gi}) : i \in J\}$ of IFOSs in X satisfies the condition $\cup \{(x,\mu_{Gi}, \gamma_{Gi}) : i \in J\} = 1\sim$, then it is called a fuzzy open cover of X.

(ii) A finite subfamily of a fuzzy open cover $\{(x,\mu_{Gi}, \gamma_{Gi}) : i \in J\}$ of X, which is also a fuzzy open cover of X, is called a finite sub cover of $\{(x, \mu_{Gi}, \gamma_{Gi}) : i \in J\}$.

(iii) A family $\{(x, \mu_{ki}, \gamma_{ki}): i \in J\}$ of IFCs in X satisfies the finite intersection property (FIP for short) iff every finite subfamily $\{(x, \mu_{ki}, \gamma_{ki}): i = 1,2,....,n\}$ of the family satisfies the condition $\cap_{i=1}^{n} \{(x, \mu_{ki}, \gamma_{ki})\} \neq 0\sim$

Definition: 2.4.2

An IFTS (X,τ) is called fuzzy compact iff every fuzzy open cover of X has a finite subcover.

Example: 2.4.3

The IFTS (X,τ) defined in Example 2.2.4 is not fuzzy compact. This is because the fuzzy open cover $(G_n: n \in N+)$ has no finite subcover.

Remark: 2.4.4

Fuzzy compactness in (X,τ) is indeed identical to fuzzy compactness in (X, τ_0) is indeed identical to fuzzy compactness $(X, \tau_{0,1})$

Proposition: 2.4.5

Let (X,τ) be an IFTS on X. then, (X, τ) is fuzzy compact iff the IFTS $(X, \tau_{0,1})$ is fuzzy compact.

Proof

(⇒) Let (X,τ) be fuzzy compact, and consider a fuzzy open cover $\{[\ ]G_j : j \in K\}$ of X in $(X, \tau_{0,1})$. Since $\bigcup([\ ]G_j) = 1\sim$ we obtain $\vee\, \mu_G = 1$, and hence, by $\gamma_{Gi} \leq 1 - \mu_{Gj} \Rightarrow \wedge\, \gamma_{Gi} \leq 1 - \vee\, \mu_{Gi} = 1 - 1 = 0 \Rightarrow \wedge\, \gamma_{Gj} = 0$. Since (X,τ) is fuzzy compact, there exists G_1, G2...., G_n such that $\bigcup_{i=1}^{n} G_i = 1\ G_i = 1\sim$, from which we obtain $\bigvee_{i=1}^{n} \mu_{Gi} = 1$ and $\bigwedge_{i=1}^{n} (1-\mu_{Gi}) = 0$, i.e. $(X,\tau_{0,1})$ is fuzzy compact.

(⇒) Suppose that $(X, \tau_{0,1})$ is fuzzy compact and consider a fuzzy open cover {Gj: $j \in K$} of X in (X,τ). Since $\bigcup G_j = 1\sim$, we obtain $\vee \mu_{Gj} = 1$ and $\wedge\, (1-\mu_{Gj}) = 0$. Since $(X\ \tau_{0,1})$ is fuzzy compact, there exists $G_1, G_2, \ldots, G_n$ such that $\bigcup_{i=1}^{n}([\ ]G_i) = 1\sim$,

i.e. $\bigvee_{i=1}^{n} \mu_{Gi} = 1$ and $\bigwedge_{i=1}^{n} (1-\mu_{Gi}) = 0$. Hence

$$\mu_{Gi} \leq 1 - \gamma_{Gi} \Rightarrow 1 = \bigvee_{i=1}^{n} \mu_{Gi} \leq 1 - \bigvee_{i=1}^{n} \gamma_{Gi}$$

$$\Rightarrow \bigwedge_{i=1}^{n} \gamma\mu_{Gi}) = 0$$

Hence $\bigcup_{i=1}^{n} G_i = 1\sim$ follows, i.e. (X,τ) is fuzzy compact.

Corollary: 2.4.6

An IFTS (X, τ) is fuzzy compact iff every family $\{(x,\mu_{ki},\gamma_{ki}) : i \in J\}$ of IFCSs in X having the FIP has the non empty intersection.

Corollary: 2.4.7

Let (X, τ), (Y,θ) be IFTSs and f: X → Y a fuzzy continuous surjection. If (X, τ) is fuzzy compact, then so is (Y,θ). Since fuzzy compactness of an IFTS (X,τ) is identical to the fuzzy compactness of the fuzzy topological space (X,τ_0).

Definition: 2.4.8

Let (X, τ) be an IFTS and A an IFS in X.

(a) If a family $\{(x,\mu_{Gi},\gamma_{Gi}) : i\in J\}$ of IFOSs in X satisfies the condition $A \subseteq \bigcup\{(x, \mu_{Gi},\gamma_{Gi}): i\in J\}$, then it is called a fuzzy open cover of A. A finite subfamily of the fuzzy open cover $\{(x, \mu_{Gi},\gamma_{Gi}): i\in J\}$ of A, which is also a fuzzy open cover of A, is called a finite sub cover of $\{(x, \mu_{Gi},\gamma_{Gi}): i\in J\}$.

(b) An IFS $A = (x, \mu_A,\gamma_A)$ in an IFTS (X, τ) is fuzzy compact iff every fuzzy open cover of A has a finite subcover.

Corollary: 2.4.9

An IFS $A=(x, \mu_A,\gamma_A)$ in an IFTS (X, τ) is fuzzy compact iff for each family $G= \{G_i : i\in J\}$, where $G_i =\{ (x, \mu_{Gi},\gamma_{Gi}): i\in J\}$, of IFOSs in X with properties

$$\mu_A \le \bigvee_{i\varepsilon J} \mu_{Gi} \text{ and } 1 - \gamma_A \le \bigvee_{i\varepsilon J} (1-\gamma_{Gi}),$$

there exists a finite subfamily $\{G_i, I = 1,2,\ldots,n)$ of such that

$$\mu_A \le \bigvee_{i=1} \mu_{Gi} \text{ and } 1 - \gamma_A \le \bigvee_{i=1} (1-\gamma_{Gi}),$$

Example: 2.4.10

Let (X,τ_0) be a fuzzy topological space in Chang's sense and $\mu_A \in I^X$ a fuzzy compact set in X. We can construct an IFTS τ on X as in Example 2.2.2. Now the IFS $A = (x,\mu_A, 1- \mu_A)$ is also fuzzy compact in (X,τ).

Example: 2.4.11

Let X = 1 and consider the IFSs $(G_n)_{n\in Z_2}$ as follows.

First we define the IFSs $G_n = (x, \mu_{Gi},\gamma_{Gi})$ by

$$\mu_{Gn}(x) = \begin{Bmatrix} 0.8, & if = 0, \\ nx, & if\, 0 < x \leq 1/n, \\ 1, & if\, x = 0, \end{Bmatrix}$$

$$\gamma_{Gn}(x) = \begin{Bmatrix} 0.1, & if = 0, \\ nx, & if\, 0 < x \leq 1/n, \\ 0, & if\, 1/n < x \leq 1, \end{Bmatrix}$$

$$\mu_G\ (x) = \begin{Bmatrix} 0.8, & if\ x = 0 \\ 1, & otherwise \end{Bmatrix}$$

$$\gamma_G\ (x) = \begin{Bmatrix} 0.1, & if\ x = 0 \\ 0, & otherwise \end{Bmatrix}$$

Then τ = {0~, 1~G} ∪ {G_n : n ε Z_2} is an IFT on X, and consider that IFSs C_{x},β (see Definition 3.10) in (X,τ). Then the IFSs $C_{0.85,\ 0.05}$, $C_{0.85,\ 0.15}$, $C_{0.75,\ 0.05}$ are all fuzzy compact, but the IFS $C_{0.75,0.15}$ is not fuzzy compact.

Corollary: 2.4.12

Let (X,τ), (Y,θ) be IFTSs and f : X → Y a fuzzy continuous function. If A is fuzzy compact in (X,τ) then so is f(A) in (Y,θ).

Proof

Let B = {G_i: i ε J}, where G_i = (y, μ_{Gi},γ_{Gi}), i ε J, be a fuzzy open cover of f(A). Then, by Definition 2.4.1 and Corollary 2.1.10 A = { f^{-1}(G_i): i ε J} is a fuzzy open cover of A, too. Since A is fuzzy compact, there exists a finite subcover of A, i.e., ∃ G_i (i=1,2,…,n) such that A ⊆ $\left(\bigcup_{i=1}^{n} f^{-1}(G_i)\right)$. Hence

f(A) ⊆ f $\left(\bigcup_{i=1}^{n} f^{-1}(G_i)\right)$ = $\bigcup_{i=1}^{n} f(f^{-1}(G_i)) \subseteq \bigcup_{i=1}^{n} G_i$ follows. Therefore, f (A) is also fuzzy compact.

Section 2.5

Intuitionistic Fuzzy C_5-Connectedness

Intuitionistic fuzzy C_5-disconnectedness and fuzzy C_5-connectdness introduced by Coker[25] is an extension of Chaudhuri and Das [26].

Definition: 2.5.1

Let (X,τ) be an IFTS.

(a) X is said to be fuzzy C_5-disconnected if there exists an intuitionistic fuzzy open and fuzzy closed set G such that G ≠ 1~ and G ≠ 0~

(b) X is said to be fuzzy C_5-connected if it is not fuzzy C_5-disconnected.

Example: 2.5.2

Let (X, τ_0) be a fuzzy topological space in Chang's sense. If X is fuzzy C_5-connected in this sense, then X is also fuzzy C_5-connected with respect to the IFT as is constructed in Example 2.2.2. (Suppose, on the contrary, that an IFOS G = {x,μ_G, 1-μ_G) in X which is also an IFCS such that 0~ ≠ G ≠ 1~ in this case X is fuzzy C_5-disconnected in (X, τ_0), an obvious contradiction).

Example: 2.5.3

Let X = {a,b} and

$$A = \left\langle x, \left(\frac{a}{0.4}, \frac{b}{0.2}\right), \left(\frac{a}{0.3}, \frac{b}{0.7}\right)\right\rangle,$$

$$B = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.7}\right), \left(\frac{a}{0.4}, \frac{b}{0.2}\right)\right\rangle,$$

$$C = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.2}\right), \left(\frac{a}{0.4}, \frac{b}{0.7}\right)\right\rangle,$$

$$D = \left\langle x, \left(\frac{a}{0.4}, \frac{b}{0.7}\right), \left(\frac{a}{0.3}, \frac{b}{0.2}\right)\right\rangle,$$

Then the family τ = {0~, 1~, A, B, C, D} is an IFT on X, and (X,τ) is fuzzy C_5-disconnected, since A is a proper nonzero IFOS and IFCS in X.

Example: 2.5.4

Let X = {1, 2} and define the IFSs $G_{i,j}$ and H_i as follows ($i,j \in Z_2$):

$$G_{ij} = \left\{x, \left(\frac{1}{\frac{1}{1}}, \frac{2}{\frac{1}{1}}\right), \left(\frac{1}{\frac{1}{1}}, \frac{2}{\frac{1}{1}}\right)\right\} \text{ and}$$

$$H_i = \left\{x, \left(\frac{1}{\frac{1}{1}}, \frac{2}{\frac{1}{1}}\right), \left(\frac{1}{0}, \frac{2}{0}\right)\right\}$$

In this case the family $\tau = \{0\sim, 1\sim) \cup \{G_{i,j}, H_i : i,j \in Z_2\}$ is an IFT on X, and it is fuzzy C_5-disconnected, too.

Now we shall obtain a characterization of fuzzy C_5-connectedness inspired by a paper of Ali and Srivatsava [2]. For this purpose we shall construct an IFTS on I as follows. Let I_D denote the unit interval I with the IFT τ_D generated by the IFSs B = (I,id, I – id) and $\overline{B}$ = (i, I – id, id), in other words. Let $\tau_D = \{0\sim, 1\sim, B, \overline{B}, B \cap \overline{B}, B \cup \overline{B}\}$ where id denotes the identity map on I.

Proposition: 2.5.5

An IFTS (X,τ) is fuzzy C_5-disconnected iff there exists a fuzzy continuous function $f : (X,\tau) \rightarrow (I_D, \tau_D)$ with $f \neq 0$ and $f \neq 1$.

Proof

This is obviouis if we notice that f is fuzzy continuous iff the IFSs.

$f^{-1}(B) = (x, f^{-1}(id),, f^{-1}(1\text{-}id) = (x, f, 1\text{-}f)$

$f^{-1}(B) = (x, f^{-1}(1\text{-}id), f^{-1}(id) = (x, 1\text{-}f, f)$ are IFOSs in τ (notice that $f^{-1}\overline{(B)} = \overline{f^{-1}(B)}$) iff $f^{-1}(B)$ is both an IFOS and IFCS in X.

Corollary: 2.5.6

An IFTS (X,τ) is fuzzy C_5-connected iff there exists no fuzzy continuous function $f : (X,\tau) \rightarrow (I_D, \tau_D)$ with $f \neq 0$ and $f \neq 1$.

Here we state that fuzzy C_5-connectedness is preserved under a fuzzy continuous surjection.

Proposition: 2.5.7

Let (X,τ), (Y,θ) be IFTSs and f : X → Y a fuzzy continuous surjection. If (X,τ) is fuzzy C_5-connected, then so is (Y,θ).

Proof

On the contrary, suppose that (Y,θ) is fuzzy C_5-disconnected. Then there exists an intuitionistic fuzzy open and fuzzy closed set G such that G ≠ 1~ and G ≠ 0~. Since f is fuzzy continuous, f^{-1} (G) is both an IFOS and IFCS by Proposition 2.3.5. The equalities $f^{-1}(G) = 1\sim$ or f^{-1} (G) = 0~ cannot hold. (Because, otherwise we have $G = f(f^{-1}(G)) = f(1\sim) = 1\sim$ and $G = f(f^{-1}(G)) = f(0\sim) = 0\sim$ by Corollary 2.1.10(d), (k), (l)). Hence (Y,θ) is fuzzy C_5-connected.

Section 2.6

Intuitionistic Fuzzy Hausdorff Spaces

Definition: 2.6.1

An IFTS (X,τ) is called Hausdorff iff $x_1,x_2 \in X$ amd $x_1 \neq x_2$ imply that there exist $G_1 = \{(x,\mu_{Gi},\gamma_{Gi})$, $G2 = \{(x,\mu_{Gi},\gamma_{Gi}) \in \tau$ with

$\mu_{G1}(x_1) = 1$, $\gamma_{G1}(x_1) = 0$,

$\mu_{G2}(x_2) = 1$, $\gamma_{G2}(x_2) = 0$ and $G_1 \cap G_2 = 0\sim$

Example: 2.6.2

Let (X,τ) be a fuzzy topological space in Chang's sense. If X is a fuzzy Hausdorff space in this sense, then X is also fuzzy Hausdorff with respect to the IFTS (X,τ) as is constructed in Example 2.2.2.