6. Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Non-Persistent Customers, Vacation and Orbital Search

Batch arrival retrial G-queue with non-persistent customers is considered. If an arriving batch finds the server available, then one of the customers from the batch receives the service immediately and others join the orbit. Otherwise all the arriving customers join the orbit. The server provides services in two phases. The first phase of essential service is needed to all the arriving customers. The second phase consists of M stages of sequential services. Stage i (i=1,2,3,....,M) includes k_i heterogeneous optional services. The customer may leave the system after service completion from any stage in phase 2. Customers may balk and renege at particular times. The arrival of the negative customer eliminates the positive customer being in service and makes the down. The repair of the failed server starts immediately. The idle server may take a single vacation with certain probability or remain idle in the system with the complementary probability. At the vacation completion epoch, the server may search for customers in the orbit. By applying the supplementary variable technique, the steady state queue size distribution of the number of customers in the orbit, expected queue length and expected system size are derived. Reliability measures of the system are obtained. Special cases are discussed. Numerical results are presented to demonstrate the influence of various parameters on the system behaviour.

6.1 Model Description

Infinite capacity single server queueing system with positive and negative customers is considered. The positive customers arrive in batches according to Poisson process with rate λ^+. It is assumed that at every arrival epoch a batch of k positive customers arrives with probability C_k. The probability generating function of the sequence $\{C_k\}$ is $C(k)=\sum_{k=1}^{\infty} C_k$ with first two moments m_1 and m_2. Negative customers arrive in single according to Poisson process with λ^-.

If the server is free upon the arrival of a batch, then one of the arriving customers receives service immediately and others join the orbit. If the server is not available, then all the customers join the orbit with probability b or balk the system with probability $\bar{b}\,(=1-b)$. The customers joining the orbit attempt to take the service after some random amount of time.

If a primary customer arrives earlier as compared to retrial customer, then the retrial customer cancels its attempt for service and either returns to its initial position with probability r or reneges the system with probability $\bar{r}\,(=1-r)$. The retrial times are arbitrarily distributed with distribution function A(x), density function a(x) and Laplace-Stieltjes transform $A^{*}(s)$. There are two phases of service, first phase of essential service and the second phase of multistages of service having multi-options in each stage. The essential service is provided to all the arriving customers. After completing the first phase of essential service, the customer may proceed to the first stage in second phase and opt any one of the k_1 heterogeneous optional services with probability p_{j_1} $(1 \le j_1 \le k_1)$ or quit the system with probability q_0. After the completion of first stage service, the customer opts j_2^{th} $(1 \le j_2 \le k_2)$ option in second stage with probability p_{j_2} or quits the system with probability q_1. In general, after completion of i^{th} stage (i=1,2,….,M-1) service, the customer may opt j_{i+1}^{th} $(1 \le j_{1+1} \le k_{i+1})$ option in $(i+1)^{th}$ stage with probability $p_{j_{i+1}}$ or leave the system with probability q_i. The customer in the final stage (M^{th} Stage) leaves the system after the service completion. The service times of first phase essential service and i^{th} stage (i=1,2,….,M), j_i^{th} (j_i=1,2,....,k_i) optional service are arbitrarily distributed with distribution function $B_0(x)$ and $B_{i,j_i}(x)$, Laplace-Stieltjes transform $B_0^{*}(s)$ and $B_{i,j_i}^{*}(s)$ and n^{th} factorial moment $\mu_0^{(n)}$ and $\mu_{i,j_i}^{(n)}$.

The occurance of the negative customer vanishes the positive customer being in service and makes the server down. The repair work of the failed server starts immediately. The repair time of the server failed during the essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^{*}(s)$ and n^{th} factorial moment $\beta_0^{(n)}$. The repair time of the server failed during i^{th}

(i=1,2,.....,M) stage j_i^{th} (j_i=1,2,......,k_i) optional service follows general distribution with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R^*_{i,j_i}(s)$ and n^{th} factorial moment $\beta^{(n)}_{i,j_i}$.

At the service completion epoch, the server may take a single vacation with probability ν or wait for the customers in the system with probability $\bar{\nu}\,(=1-\nu)$. The vacation time is generally distributed with distribution function $V(x)$, Laplace-Stieltjes transform $V^*(s)$ and n^{th} factorial moment γ_n.

After vacation completion the server searches for customers in the orbit with probability θ or remains idle with probability $\bar{\theta} = (1-\theta)$.

The diagrammatic representation of the model under consideration is shown in Fig. 6.1.

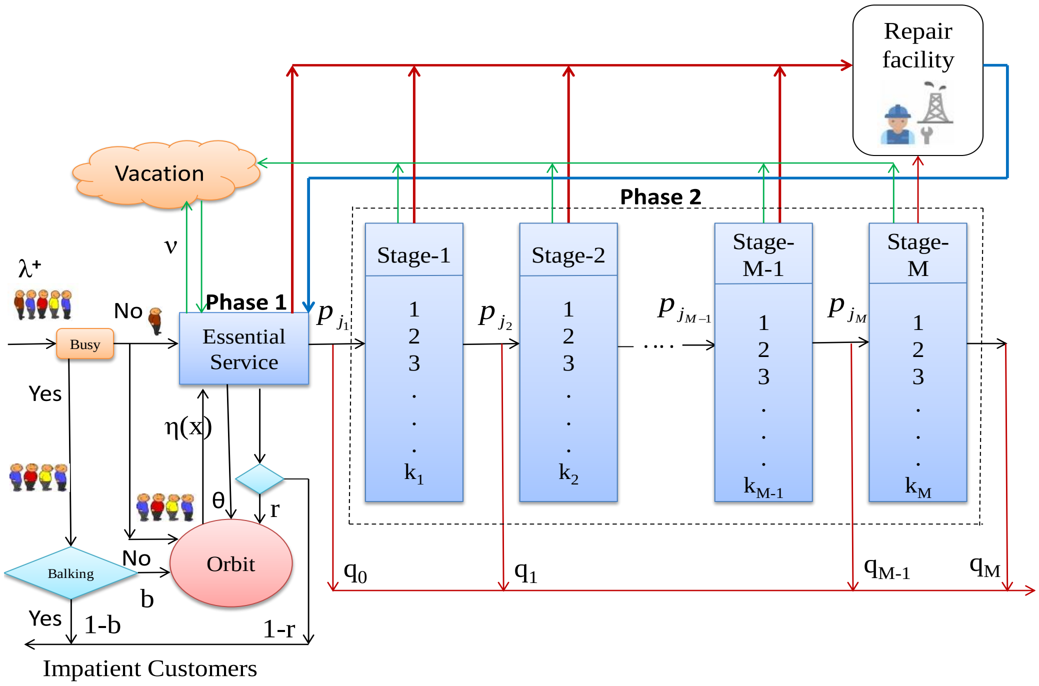


Fig. 6.1 Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Non-Persistent Customers, Vacation and Orbital Search

6.2 Definitions and Notations

At time t, the state of the system can be described by the Markov process $\{ X(t), t \geq 0 \} = \{ S(t), N(t) \}$, where S(t) denote the server state 0, 1, 2, 3, 4 or 5 according as the server being idle, busy in essential service, busy in i^{th} stage j_i^{th} optional service, under repair in essential service, repair in i^{th} stage j_i^{th} optional service or on vacation. N(t) corresponds to the number of customers in the orbit at time t.

Let the functions $\eta(x)$, $\mu_0(x)$, $\mu_{i,j_i}(x)$, $\beta_0(x)$, $\beta_{i,j_i}(x)$ and $\gamma(x)$ be the conditional completion rates for repeated attempts, essential service, i^{th} stage j_i^{th} optional service, repair in essential service, repair in i^{th} stage j_i^{th} optional service and server vacation respectively.

Define the following steady state probabilities

$I_0(t)$ is the probability that the server is idle in the empty system.

$I_n(x,t)dx$ is the probability that at time t the server is idle, there are n (≥ 1) customers in the orbit and the elapsed retrial time is between x and x+dx.

$P_{0,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in first phase of essential service and the elapsed service time is between x and x+dx.

$P_{i,j_i,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in i^{th} stage j_i^{th} optional service and the elapsed service time is between x and x+dx, $i = 1,2,....,M, j_i = 1,2,....,k_i$

$R_{0,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server in essential service is under repair and the elapsed repair time is between x and x+dx.

$R_{i,j_i,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server in i^{th} stage j_i^{th} optional service is under repair and the elapsed repair time is between x and x+dx, $i = 1,2,....,M, j_i = 1,2,....,k_i$

$V(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is on vacation and the elapsed vacation time is between x and x+dx.

6.3 Governing Equations

The system of steady state equations that governs the model under consideration is given below.

$$\lambda^{+} I_0 = \bar{\nu}\left[q_0 \int_0^\infty P_{0,0}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)dx \right]$$

$$+ \int_0^\infty R_{0,0}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx + \int_0^\infty V_0(x)\gamma(x)dx \qquad (6.1)$$

$$\frac{d}{dx} I_n(x) = -(\lambda^{+} + \eta(x))\, I_n(x), \; n \ge 1 \qquad (6.2)$$

$$\frac{d}{dx} P_{0,n}(x) = -(b\lambda^{+} + \lambda^{-} + \mu_0(x))\, P_{0,n}(x) + b\lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{0,n-k}(x), \quad n \ge 0 \qquad (6.3)$$

$$\frac{d}{dx} P_{i,j_i,n}(x) = -(b\,\lambda^{+} + \lambda^{-} + \mu_{i,j_i}(x))\, P_{i,j_i,n}(x) + b\,\lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{i,j_i,n-k}(x),$$
$$n \ge 0,\; i = 1,2,\ldots,M,\; j_i = 1,2,\ldots,k_i \qquad (6.4)$$

$$\frac{d}{dx} R_{0,n}(x) = -(b\lambda^{+} + \beta_0(x))\, R_{0,n}(x) + b\,\lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{0,n-k}(x), \; n \ge 0 \qquad (6.5)$$

$$\frac{d}{dx} R_{i,j_i,n}(x) = -(b\lambda^{+} + \beta_{i,j_i}(x))\, R_{i,j_i,n}(x) + b\lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{i,j_i,n-k}(x), n \ge 0,$$
$$i = 1,2,\ldots,M,\; j_i = 1,2,\ldots,k_i \qquad (6.6)$$

$$\frac{d}{dx} V_n(x) = -(b\lambda^{+} + \gamma(x))\, V_n(x) + b\,\lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k V_{n-k}(x), \; n \ge 0 \qquad (6.7)$$

with boundary conditions

$$I_n(0) = \bar{\nu}\left[q_0 \int_0^\infty P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx \right] + \int_0^\infty R_{0,n}(x)\beta_0(x)dx$$

$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx + \bar{\theta}\int_0^\infty V_n(x)\gamma(x)dx, \quad n \ge 1 \qquad (6.8)$$

$$P_{0,0}(0) = \lambda^{+} C_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx + \theta\int_0^\infty V_n(x)\gamma(x)dx \qquad (6.9)$$

$$P_{0,n}(0) = \lambda^{+} C_{n+1} I_0 + \lambda^{+}\, r \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)dx + \int_0^\infty I_{n+1}(x)\eta(x)dx$$
$$+ \lambda^{+}(1-r)\sum_{k=1}^{n+1} C_k \int_0^\infty I_{n-k+2}(x)dx + \theta\int_0^\infty V_{n+1}(x)\gamma(x)dx, \; n \ge 1 \qquad (6.10)$$

$$P_{1,j_1,n}(0) = p_{j_1}\int_0^{\infty} P_{0,n}(x)\mu_0(x)dx,\ n \ge 0,\ j_1 = 1,2,....,k_1 \tag{6.11}$$

$$P_{i,j_i,n}(0) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^{\infty} P_{i-1,j_{i-1},n}(x)\mu_{i,j_i}(x)dx,\quad n \ge 0,\ i = 2,3,......,M,\ j_i = 1,2,....,k_i \tag{6.12}$$

$$R_{0,n}(0) = \lambda^{-}\int_0^{\infty} P_{0,n}(x)dx\ ,\ n \ge 0 \tag{6.13}$$

$$R_{i,j_i,n}(0) = \lambda^{-}\int_0^{\infty} P_{i,j_i,n}(x)dx\ ,\ n \ge 0,\ i = 2,3,......,M,\ j_i = 1,2,....,k_i \tag{6.14}$$

$$V_n(0) = \nu\left[q_0\int_0^{\infty} P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i\sum_{j_i=1}^{k_i}\int_0^{\infty} P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx\right],\quad n \ge 0 \tag{6.15}$$

6.4 System Size Distribution at Random Epoch

Define the following probability generating functions

$$\left.\begin{array}{ll} I(x,z) = \sum_{n=1}^{\infty} I_n(x)z^n\ ; & P_0(x,z) = \sum_{n=0}^{\infty} P_{0,n}(x)z^n \\ P_{i,j_i}(x,z) = \sum_{n=0}^{\infty} P_{i,j_i,n}(x)z^n\ ; & R_0(x,z) = \sum_{n=0}^{\infty} R_{0,n}(x)z^n \\ R_{i,j_i}(x,z) = \sum_{n=0}^{\infty} R_{i,j_i,n}(x)z^n\ \text{and} & V(x,z) = \sum_{n=1}^{\infty} V_n(x)z^n \end{array}\right\} \tag{6.16}$$

Theorem 6.1

The probability generating functions for the server being in the idle state, in busy state of providing first and second phase services, under repair state while broken down at first and second phase, in vacation at random epoch respectively are given by

$$I(z) = \frac{I_0\left(1 - A^*(\lambda^+)\right)\left[zC(z)T_1(z) + \theta z\nu\sum_{i=0}^{M} q_i\Lambda_i^*(g(z))B_0^*(g(z))V^*(h(z)) - z^2\right]}{D(z)} \tag{6.17}$$

$$P_0(z) = \frac{\lambda^+ I_0 T_2(z)\left(1 - B_0^*(g(z))\right)/g(z)}{D(z)} \tag{6.18}$$

$$P(z) = \frac{\lambda^+ I_0 T_2(z)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))B_0^*(g(z))\left(1 - B_{i,j_i}^*(g(z))\right)/g(z)}{D(z)} \tag{6.19}$$

$$R_0(z) = \frac{\lambda^- \lambda^+ I_0 T_2(z)\left((1-B_0^*(g(z)))/g(z)\right)\left(1-R_0^*(h(z))\right)}{h(z)\ D(z)} \tag{6.20}$$

$$R(z) = \frac{\lambda^- \lambda^+ I_0 T_2(z) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z)) B_0^*(g(z)) \left((1-B_{i,j_i}^*(g(z)))/g(z)\right)(1-R_{i,j_i}^*(g(z)))}{h(z)\ D(z)} \tag{6.21}$$

$$V(z) = \frac{\lambda^+ I_0 \nu T_2(z) \sum_{i=0}^{M} q_i \Lambda_i^*(g(z)) B_0^*(g(z)) \left(1-V^*(h(z))\right)}{h(z)\ D(z)} \tag{6.22}$$

where

$$\begin{aligned} T_1(z) \quad = \quad & \bar{\nu} \sum_{i=0}^{M} q_i\ \Lambda_i^*(g(z))\ B_0^*(g(z)) + \lambda^- \left((1-B_0^*(g(z)))/g(z)\right) R_0^*(h(z)) \\ & + \lambda^- \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z)) \left((1-B_{i,j_i}^*(g(z)))/g(z)\right) R_{i,j_i}^*(h(z)) B_0^*(g(z)) \\ & + \bar{\theta}\, \nu \sum_{i=0}^{M} q_i\ \Lambda_i^*(g(z))\ B_0^*(g(z)) V^*(h(z)) \end{aligned}$$

$$T_2(z) \quad = \quad z C(z) - z A^*(\lambda^+) - C(z)\left(1-A^*(\lambda^+)\right)\left(\bar{r} + r z\right)$$

$$D(z) = z^2 - T_1(z)\left(z A^*(\lambda^+) - C(z)\left(1-A^*(\lambda^+)\right)\left(\bar{r}+rz\right)\right) - \theta z \nu \sum_{i=0}^{M} q_i\ \Lambda_i^*(g(z))\ B_0^*(g(z)) V^*(h(z))$$

$$\Lambda_0^*(g(z)) = 1\ , \qquad \Lambda_i^*(g(z)) = \prod_{l=1}^{i} \sum_{j_l=1}^{k_l} p_{j_l} B_{l,j_l}^*(g(z))$$

$$g(z) = b\lambda^+(1-C(z)) + \lambda^- \quad \text{and} \quad h(z) = b\lambda^+(1-C(z))$$

Proof.

Multiplying the equations (6.2) to (6.15) by z^n and summing over all values of n and including equation (6.1), we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x)\right) I(x,z) = 0 \tag{6.23}$$

$$\left(\frac{d}{dx} + b\lambda^+(1-C(z)) + \lambda^- + \mu_0(x)\right) P_0(x,z) = 0 \tag{6.24}$$

$$\left(\frac{d}{dx} + b\lambda^+(1-C(z)) + \lambda^- + \mu_{i,j_i}(x)\right) P_{i,j_i}(x,z) = 0, \quad i = 1,2,.....,M, \quad j_i = 1,2,.......,k_i \tag{6.25}$$

$$\left(\frac{d}{dx} + b\lambda^+(1-C(z)) + \beta_0(x)\right) R_0(x,z) = 0 \tag{6.26}$$

$$\left(\frac{d}{dx}+b\lambda^{+}(1-C(z))+\beta_{i,j_i}(x)\right)R_{i,j_i}(x,z)=0\ ,\ i=1,2,.....,M,\ \ j_i=1,2,........,k_i \tag{6.27}$$

$$\left(\frac{d}{dx}+b\lambda^{+}(1-C(z))+\gamma(x)\right)V(x,z)=0 \tag{6.28}$$

$$I(0,z)=\bar{\nu}\left(q_0\int_0^\infty P_0(x,z)\,\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx\right)+\int_0^\infty R_0(x,z)\,\beta_0(x)dx$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx+\bar{\theta}\int_0^\infty V(x,z)\,\gamma(x)\,dx-\lambda^{+}I_0 \tag{6.29}$$

$$P_0(0,z)=\frac{1}{z}[\lambda^{+}C(z)I_0+\int_0^\infty I(x,z)\eta(x)dx\ +\lambda^{+}b\,C(z)\int_0^\infty I(x,z)dx$$

$$+\frac{\lambda^{+}\bar{r}}{z}C(z)\int_0^\infty I(x,z)dx+\theta\int_0^\infty V(x,z)\,\gamma(x)\,dx\,] \tag{6.30}$$

$$P_{1,j_1}(0,z)=p_{j_1}\int_0^\infty P_0(x,z)\mu_0(x)dx,\quad j_1=1,2,......,k_1 \tag{6.31}$$

$$P_{i,j_i}(0,z)=p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx,\quad i=2,3,......,M,\ \ j_i=1,2,......,k_i \tag{6.32}$$

$$R_0(0,z)=\lambda^{-}\int_0^\infty P_0(x,z)dx \tag{6.33}$$

$$R_{i,j_i}(0,z)=\lambda^{-}\int_0^\infty P_{i,j_i}(x,z)dx,\quad i=1,2,3,......,M,\ \ j_i=1,2,......,k_i \tag{6.34}$$

$$V(0,z)=\nu\left(q_0\int_0^\infty P_0(x,z)\,\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx\right) \tag{6.35}$$

With usual procedure, the solutions of the partial differential equations (6.23) to (6.28) are obtained as

$$I(x,z)=I(0,z)\,e^{-\lambda^{+}x}(1-A(x)) \tag{6.36}$$

$$P_0(x,z)=P_0(0,z)e^{-g(z)\,x}(1-B_0(x)) \tag{6.37}$$

$$P_{i,j_i}(x,z)=P_{i,j_i}(0,z)e^{-g(z)\,x}(1-B_{i,j_i}(x)),\ i=1,2,......,M,\ j_i=1,2,.....,k_i \tag{6.38}$$

$$R_0(x,z)=R_0(0,z)e^{-h(z)\,x}(1-R_0(x)) \tag{6.39}$$

$$R_{i,j_i}(x,z)=R_{i,j_i}(0,z)e^{-h(z)\,x}(1-R_{i,j_i}(x)),\ i=1,2,......,M,\ j_i=1,2,.....,k_i \tag{6.40}$$

$$V(x,z)=V(0,z)e^{-h(z)\,x}(1-V(x)) \tag{6.41}$$

Using the equations (6.36) to (6.41), equation (6.29) yields

$$I(0,z) = \bar{\nu}\left(q_0\, P_0(0,z)\, B_0^*(g(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\, P_{i,j_i}(0,z)\, B_{i,j_i}^*(g(z))\right) + R_0(0,z)\, R_0^*(h(z))$$

$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z) R_{i,j_i}^*(h(z)) + \bar{\theta}\; V(0,z) V^*(h(z)) - \lambda^+ I_0 \qquad (6.42)$$

Using the equations (6.36) and (6.41), the equation (6.30) becomes

$$P_0(0,z) = \frac{1}{z}\lambda^+ C(z) I_0 + \frac{1}{z^2}\left(z A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))(\bar{r} + r z)\right) I(0,z)$$

$$+ \frac{\theta}{z} V(0,z) V^*(h(z)) \qquad (6.43)$$

Substituting the results in equations (6.37) and (6.38) in the equations (6.31) to (6.35), we obtain respectively

$$P_{1,j_1}(0,z) = p_{j_1}\, P_0(0,z)\, B_0^*(g(z)), \qquad j_1 = 1,2,\ldots\ldots,k_1 \qquad (6.44)$$

$$P_{i,j_i}(0,z) = p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} P_{i-1,j_{i-1}}(0,z)\, B_{i-1,j_{i-1}}^*(g(z)), \quad i = 2,3,\ldots\ldots,M, \quad j_i = 1,2,\ldots\ldots,k_i \qquad (6.45)$$

$$R_0(0,z) = \lambda^-\; P_0(0,z)(1 - B_0^*(g(z)))\big/ g(z) \qquad (6.46)$$

$$R_{i,j_i}(0,z) = \lambda^-\; P_{i,j_i}(0,z)\,(1 - B_{i,j_i}^*(g(z)))\big/ g(z), \; i = 1,2,3,\ldots\ldots,M, \; j_i = 1,2,\ldots\ldots,k_i \qquad (6.47)$$

$$V(0,z) = \nu\left(q_0\, P_0(0,z)\, B_0^*(g(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\, P_{i,j_i}(0,z)\, B_{i,j_i}^*(g(z))\right) \qquad (6.48)$$

By iterative method, the equation (6.45) gives

$$P_{i.j_i}(0,z) = p_{j_i} \Lambda_{i-1}^*(g(z)) B_0^*(g(z))\; P_0(0,z)\;, \; i = 2,3,\ldots\ldots,M, \; j_i = 1,2,\ldots\ldots,k_i \qquad (6.49)$$

Substituting the expression (6.49) in equations (6.47) and (6.48), we obtain

$$R_{i,j_i}(0,z) = \lambda^-\; p_{j_i} \Lambda_{i-1}^*(g(z)) B_0^*(g(z))\; P_0(0,z)\; (1 - B_{i,j_i}^*(g(z)))\big/ g(z),$$
$$i = 1,2,3,\ldots\ldots,M, \quad j_i = 1,2,\ldots\ldots,k_i \qquad (6.50)$$

$$V(0,z) = \nu \sum_{i=0}^{M} q_i \Lambda_i^*(g(z)) B_0^*(g(z))\; P_0(0,z) \qquad (6.51)$$

Using equations (6.44), (6.46), (6.49), (6.50) and (6.51), the equation (6.42) yields

$$I(0,z) = T_1(z)\ P_0(0,z) - \lambda^+ I_0 \tag{6.52}$$

By substituting the equations (6.51) and (6.52) in equation (6.43) and on simplifying, we have

$$P_0(0,z) = \lambda^+ I_0 \left(z\,C(z) - z\,A^*(\lambda^+) - C(z)(1 - A^*(\lambda^+))\left(\bar{r} + r\,z\right)\right) / \ D(z) \tag{6.53}$$

Using the expression of $P_0(0,z)$, equations (6.44), (6.46) and (6.49) to (6.52) yield

$$I(0,z) = \lambda^+ I_0 [z\,C(z) T_1(z) + \theta z \nu \sum_{i=0}^{M} q_i\ \Lambda_i^*(g(z))\ B_0^*(g(z)))\ V^*(h(z)) - z^2] / D(z) \tag{6.54}$$

$$\begin{aligned} P_{i,j_i}(0,z) = {} & p_{j_i} \lambda^+ I_0\ \Lambda_{i-1}^*(g(z)) B_0^*(g(z))[z\,C(z) - z\,A^*(\lambda^+) \\ & - C(z)(1 - A^*(\lambda^+))\left(\bar{r} + r\,z\right)]\ /\ D(z),\ i = 1,2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \end{aligned} \tag{6.55}$$

$$\begin{aligned} R_0(0,z) = {} & \lambda^- \lambda^+ I_0 [z\,C(z) - z\,A^*(\lambda^+) - C(z)(1 - A^*(\lambda^+))\left(\bar{r} + r\,z\right)] \\ & \left((1 - B_0^*(g(z)))/g(z)\right) /\ D(z) \end{aligned} \tag{6.56}$$

$$\begin{aligned} R_{i,j_i}(0,z) = {} & \lambda^- \lambda^+ I_0\ [z C(z) - z A^*(\lambda^+) - C(z)(1 - A^*(\lambda^+))\left(\bar{r} + r z\right)] p_{j_i} \Lambda_{i-1}^*(g(z)) \\ & ((1 - B_{i,j_i}^*(g(z)))/g(z)) B_0^*(g(z)),\ i = 1,2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \end{aligned} \tag{6.57}$$

$$\begin{aligned} V(0,z) = {} & \lambda^+ I_0\,\nu\,[z\,C(z) - z\,A^*(\lambda^+) - C(z)(1 - A^*(\lambda^+))\left(\bar{r} + r\,z\right)] \\ & \sum_{i=0}^{M} q_i\,\Lambda_i^*(g(z)) B_0^*(g(z))\ /\ D(z) \end{aligned} \tag{6.58}$$

By substituting the expressions (6.54) to (6.58) in equations (6.36) to (6.41) and integrating the resultant expression with respect to x from 0 to ∞, we get the probability generating functions for different states of the server given in equations (6.17) to (6.22).

Result: Using the normalizing condition, we get

$$I_0 = \frac{2 - T_1(1)[A^*(\lambda^+) + (1 - A^*(\lambda^+))(m_1 + r)] - T_1'(1) - T_3}{\bar{r}(1 - A^*(\lambda^+))T_7 - A^*(\lambda^+)[T_1(1) + T_1'(1) + T_3 - 2] + \lambda^+ T_2'(1)[T_5 + T_6 + \nu \sum_{i=0}^{M} q_0\,\Lambda_i^*(\lambda^-)\,B_0^*(\lambda^-)\gamma_1]}$$

where

$$T_1(1) = (1-\theta\nu)B_0^*(\lambda^-)\sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) - \lambda^- T_5$$

$$T_1'(1) = b\lambda^+ m_1 [T_5 + T_6 + \bar{\theta}\nu B_0^*(\lambda^-)\sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-)\gamma_1] - f_0^{(1)} + T_4$$
$$-(1-\theta\nu)[q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i (M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)})]$$

$$T_2'(1) = m_1 A^*(\lambda^+) + \bar{r}(1 - A^*(\lambda^+))$$

$$T_3 = \theta\nu[(1 + b\lambda^+ m_1 \gamma_1) B_0^*(\lambda^-)\sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) + q_0 f_0^{(1)}$$
$$+ \sum_{i=1}^{M} q_i (M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)})]$$

$$T_4 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-)) - B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)}$$

$$T_5 = (1/\lambda^-)/[1 - B_0^*(\lambda^-) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))]$$

$$T_6 = (1 - B_0^*(\lambda^-)))\beta_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$

$$f_0^{(1)} = \lambda^+ m_1 \bar{\delta} \int_0^{\infty} x e^{-\lambda^- x} b_0(x) dx, \qquad f_{i,j_i}^{(1)} = \lambda^+ m_1 \bar{\delta} \int_0^{\infty} x e^{-\lambda^- x} b_{i,j_i}(x) dx$$

Corollary 6.1

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 \{ b\lambda^+ (1 - C(z))[T_1(z) T_2(z) - A^*(\lambda^+)(z^2 - z C(z) T_1(z) - \theta z \nu \sum_{i=0}^{M} q_i \Lambda_i^*(g(z))$$
$$B_0^*(g(z)) V^*(h(z))) + \lambda^+ T_2(z)[((1 - B_0^*(g(z)))/g(z))(1 + \lambda^-(1 - R_0^*(h(z))))$$
$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z)) B_0^*(g(z))((1 - B_{i,j_i}^*(g(z)))/g(z))(1 + \lambda^-(1 - R_{i,j_i}^*(h(z))))]$$
$$+ \nu \sum_{i=0}^{M} q_i \Lambda_i^*(g(z)) B_0^*(g(z))(1 - V^*(h(z)))]\}/h(z)D(z) \qquad (6.59)$$

Proof.

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + V(z)$$

By substituting equations (6.17) to (6.22) and simplifying we get equation (6.59).

Corollary 6.2

The probability generating function of the number of customers in the system is

$$P_s(z) = I_0\{b\lambda^+(1-C(z))[T_1(z)T_2(z) - A^*(\lambda^+)(z^2 - zC(z)T_1(z) - \theta z\nu\sum_{i=0}^{M} q_i\Lambda_i^*(g(z))$$
$$B_0^*(g(z))V^*(h(z))) + \lambda^+T_2(z)[((1-B_0^*(g(z)))/g(z))(z+\lambda^-(1-R_0^*(h(z))))$$
$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))B_0^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))(z+\lambda^-(1-R_{i,j_i}^*(h(z))))]$$
$$+\nu\sum_{i=0}^{M} q_i\Lambda_i^*(g(z))B_0^*(g(z))(1-V^*(h(z)))]\}/h(z)D(z) \qquad (6.60)$$

Proof.

The probability generating function of the number of customers in the system is

$$P_s(z) = I_0 + I(z) + zP_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + V(z)$$

Using the equations (6.17) to (6.22) and by direct substitution we get equation (6.60).

6.5 Stability Condition

The inequality

$$T_1'(1) - \theta\nu[A^*(\lambda^+) + (1-A^*(\lambda^+))(m_1+r)]\sum_{i=0}^{M} q_i\Lambda_i^*(\lambda^-) - T_3 + (1-A^*(\lambda^+))m_1 < 1 + (1-A^*(\lambda^+))\bar{r}$$

is the necessary and sufficient condition for the system to be stable.

6.6 Performance Measures

Let $N_I(z), N_{P_0}(z), N_P(z), N_{R_0}(z)$ and $N_R(z)$ denotes the numerators of $I(z)$, $P_0(z), P(z), R_0(z)$ and $R(z)$ respectively.

- The probability that the server is idle in the non empty system is given by

$$I = \lim_{z\to 1} I(z)$$
$$= I_0(1-A^*(\lambda^+))[(1+m_1)T_1(1) + T_1'(1) + T_3 - 2] \,/\, D' \qquad (6.61)$$

- Mean number of customers in the orbit when the server is idle in the non-empty system is

$$L_I = \lim_{z\to 1} \frac{d}{dz} I(z)$$

$$= \frac{D' N_I'' - N_I' D''}{3D'^2} \qquad (6.62)$$

where

$$N_I' = I_0(1 - A^*(\lambda^+))[(1+m_1)T_1(1) + T_1'(1) + T_3 - 2]$$

$$N_I'' = I_0(1 - A^*(\lambda^+))[(2m_1 + m_2)T_1(1) + 2(1+m_1)T_1'(1) + T_1''(1)$$
$$+ \theta\nu(b^2(\lambda^+)^2 m_1^2\gamma_2 + b\lambda^+ m_2\gamma_1 + 2b\lambda^+ m_1\gamma_1)B_0^*(\lambda^-)\sum_{i=0}^{M} q_i\Lambda_i^*(\lambda^-)$$
$$+ 2\theta\nu(1 + b\lambda^+ m_1\gamma_1)(q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i(M_i^{(1)}B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-)f_0^{(1)}))$$
$$+ \theta\nu(q_0 f_0^{(2)} + \sum_{i=1}^{M} q_i(M_i^{(2)}B_0^*(\lambda^-) + 2M_i^{(1)}f_0^{(1)} + \Lambda_i^*(\lambda^-)f_0^{(2)})) - 2]$$

$$D' = 2 - T_1(1)[A^*(\lambda^+) + (1 - A^*(\lambda^+))(m_1 + r)] - T_1'(1) - T_3$$

$$D'' = 2 - 2T_1'(1)[A^*(\lambda^+) + (1 - A^*(\lambda^+))(m_1 + r)] - T_1(1)(1 - A^*(\lambda^+))(m_2 + r)$$
$$- T_1''(1) - \theta\nu(b^2(\lambda^+)^2 m_1^2\gamma_2 + b\lambda^+ m_2\gamma_1 + 2b\lambda^+ m_1\gamma_1)\sum_{i=0}^{M} q_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)$$
$$- 2\theta\nu(1 + b\lambda^+ m_1\gamma_1)(q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i(M_i^{(1)}B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-)f_0^{(1)})$$
$$- \theta\nu(q_0 f_0^{(2)} + \sum_{i=1}^{M} q_i(M_i^{(2)}B_0^*(\lambda^-) + 2M_i^{(1)}f_0^{(1)} + \Lambda_i^*(\lambda^-)f_0^{(2)}))$$

$$T_1''(1) = (1 - \theta\nu)(q_0 f_0^{(2)} + \sum_{i=1}^{M} q_i(M_i^{(2)}B_0^*(\lambda^-) + 2M_i^{(1)}f_0^{(1)} + \Lambda_i^*(\lambda^-)f_0^{(2)}) + b^2(\lambda^+)^2 m_1^2 T_7$$
$$+ b\lambda^+ m_2(T_5 + T_6) + (b^2(\lambda^+)^2 m_1^2\gamma_2 + b\lambda^+ m_2\gamma_1)\bar{\theta}\nu B_0^*(\lambda^-)\sum_{i=0}^{M} q_i\Lambda_i^*(\lambda^-)$$
$$- 2b\lambda^+ m_1[f_0^{(1)}\beta_0^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(M_{i-1}^{(1)}B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-)f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$
$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)} - \bar{\theta}\nu(q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i(M_i^{(1)}B_0^*(\lambda^-)$$
$$+ \Lambda_i^*(\lambda^-)f_0^{(1)})\gamma_1)] + 2((b\lambda^+ m_1)^2/\lambda^-)(T_5 + T_6) - f_0^{(2)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)$$
$$B_0^*(\lambda^-)f_{i,j_i}^{(2)} - 2(b\lambda^+ m_1/\lambda^-)(f_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)f_{i,j_i}^{(1)})$$

$$T_7 = (1-B_0^*(\lambda^-))\beta_0^{(2)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)}$$

$$f_0^{(2)} = (b\lambda^+ m_1)^2\int_0^\infty x^2 e^{-\lambda^- x} b_0(x)dx + b\lambda^+ m_2\int_0^\infty x e^{-\lambda^- x} b_0(x)dx \quad ,$$

$$f_{i,j_i}^{(2)} = (b\lambda^+ m_1)^2\int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx + b\lambda^+ m_2\int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

- The probability that the server is busy in providing essential service is given by

$$P_0 = \lim_{z\to 1} P_0(z)$$

$$= I_0(\lambda^+/\lambda^-)T_2'(1)\,(1-B_0^*(\lambda^-))\Big/D' \qquad (6.63)$$

- Mean number of customers in the orbit when the server busy in providing essential service is

$$L_{P_0} = \lim_{z\to 1}\frac{d}{dz}P_0(z)$$

$$= \frac{D'\,N_{P_0}'' - N_{P_0}'\,D''}{3D'^{\,2}} \qquad (6.64)$$

where

$$N_{P_0}' = I_0(\lambda^+/\lambda^-)\,T_2'(1)\,(1-B_0^*(\lambda^-))$$

$$N_{P_0}'' = I_0(\lambda^+/\lambda^-)\{\,T_2''(1)(1-B_0^*(\lambda^-)) - 2\,T_2'(1)[f_0^{(1)} - (\lambda^+ m_1/\lambda^-)\;(1-B_0^*(\lambda^-))]\}$$

$$T_2''(1) = m_2 + 2m_1 - (1-A^*(\lambda^+))(2m_1 r + m_2)$$

- The probability that the server is busy in providing optional services is given by

$$P = \lim_{z\to 1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}P_{i,j_i}(z)$$

$$= I_0(\lambda^+/\lambda^-)T_2'(1)B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\Big/D' \qquad (6.65)$$

- Mean number of customers in the orbit when the server busy in providing optional services is

$$L_P = \lim_{z\to 1} \frac{d}{dz} P(z)$$

$$= \frac{D' N_P'' - N_P' D''}{3D'^{\,2}} \qquad (6.66)$$

where

$$N_P' = I_0 (\lambda^+ / \lambda^-)\, T_2'(1) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))$$

$$N_P'' = I_0 (\lambda^+ / \lambda^-) \{ T_2''(1) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) + 2T_2'(1)[\, T_4$$

$$+ (b\lambda^+ m_1 / \lambda^-)\, B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)\,(1 - B_{i,j_i}^*(\lambda^-))]]\}$$

- The probability that the server is under repair during essential service is given by

$$R_0 = \lim_{z\to 1} R_0(z)$$

$$= I_0 \lambda^+ T_2'(1)(1 - B_0^*(\lambda^-))\, \beta_0^{(1)} \big/ D' \qquad (6.67)$$

- Mean number of customers in the orbit when the server is under repair during essential service is

$$L_{R_0} = \lim_{z\to 1} \frac{d}{dz} R_0(z)$$

$$= \frac{D_1'' N_{R_0}''' - N_{R_0}'' D_1'''}{3D_1''^{\,2}} \qquad (6.68)$$

where

$$N_{R_0}'' = -2\lambda^+ m_1 I_0 \lambda^+ T_2'(1)(1 - B_0^*(\lambda^-))\beta_0^{(1)}$$

$$N_{R_0}''' = \lambda^+ I_0 \{ -3b\lambda^+ m_1 T_2''(1)\ (1 - B_0^*(\lambda^-))\beta_0^{(1)} - T_2'(1)[3((b\lambda^+ m_1)^2 \beta_0^{(2)} + 3b\lambda^+ m_2 \beta_0^{(1)})$$

$$(1 - B_0^*(\lambda^-)) + 4b\lambda^+ m_1 f_0^{(1)} \beta_0^{(1)} + 4((\lambda^+ m_1)^2 / \lambda^-)(1 - B_0^*(\lambda^-))\ \beta_0^{(1)}]\}$$

$$D_1'' = -2b\lambda^+ m_1 D'$$

$$\begin{aligned} D_1''' = & -3\{b\lambda^+ m_2 D' + b\lambda^+ m_1 [2 - 2T_1'(1)[A^*(\lambda^+) + (1 - A^*(\lambda^+))(m_1 + r)] \\ & - T_1(1)(1 - A^*(\lambda^+))(m_2 + r) - T_1''(1) - \theta\nu(b^2 (\lambda^+)^2 m_1^2 \gamma_2 + b\lambda^+ m_2 \gamma_1 \\ & + 2b\lambda^+ m_1 \gamma_1) \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) - 2\theta\nu(1 + b\lambda^+ m_1 \gamma_1)(q_0 f_0^{(1)} \\ & + \sum_{i=1}^{M} q_i (M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}) - \theta\nu(q_0 f_0^{(2)} + \sum_{i=1}^{M} q_i (M_i^{(2)} B_0^*(\lambda^-) \\ & + 2M_i^{(1)} f_0^{(1)} + \Lambda_i^*(\lambda^-) f_0^{(2)})]\} \end{aligned}$$

- The probability that the server is under repair during optional services is given by

$$R = \lim_{z\to 1} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= I_0 \lambda^+ T_2'(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)} \Big/ D' \qquad (6.69)$$

- Mean number of customers in the orbit when the server is under repair during optional services is

$$L_R = \lim_{z\to 1} \frac{d}{dz} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{D_1'' N_R''' - N_R'' D_1'''}{3 D_1''^2} \qquad (6.70)$$

where

$$N_R'' = -2\lambda^+ m_1 I_0 \lambda^+ T_2'(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)}$$

$$\begin{aligned} N_R''' = & \lambda^+ I_0 \{-3b\lambda^+ m_1 T_2''(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)} \\ & - T_2'(1)[\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-)(3(b\lambda^+ m_1)^2 \beta_{i,j_i}^{(2)} + 3b\lambda^+ m_2 \beta_{i,j_i}^{(1)} \\ & + 4((b\lambda^+ m_1)^2/\lambda^-) \beta_{i,j_i}^{(1)}) + 4b\lambda^+ m_1 \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)}) \\ & (1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} - 4b\lambda^+ m_1 \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) f_{i,j_i}^{(1)} \beta_{i,j_i}^{(1)}]\} \end{aligned}$$

- The probability that the server is on vacation is given by

$$V = \lim_{z\to 1} V(z)$$

$$= 2\lambda^+ I_0 \nu T_2'(1)\, b\lambda^+ m_1\gamma_1 \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) \Big/ D' \qquad (6.71)$$

- Mean number of customers in the orbit when the server is on vacation is

$$L_V = \lim_{z\to 1} \frac{d}{dz} V(z)$$

$$= \frac{D_1'' N_V''' - N_V'' D_1'''}{3 D_1''^2} \qquad (6.72)$$

where

$$N_V'' = -2\lambda^+ I_0 \nu T_2'(1)\, b\lambda^+ m_1\gamma_1 \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-)$$

$$N_V''' = \lambda^+ I_0 \nu\{-3b\lambda^+ m_1 T_2''(1) \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-)\gamma_1 - T_2'(1)[3((b\lambda^+ m_1)^2 \gamma_2 + b\lambda^+ m_2\gamma_1)$$
$$\sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) - 4b\lambda^+ m_1 (q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i (M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}))\gamma_1]\}$$

Let $N_q(z)$ be the numerator of $P_q(z)$.

- Mean number of customers in the orbit is

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{D_1'' N_q''' - N_q'' D_1'''}{3 D_1''^2} \qquad (6.73)$$

where

$$N_q''(1) = I_0 \{-2b\lambda^+ m_1 [(T_1(1) + \lambda^+ T_5) T_2'(1) + A^*(\lambda^+)(2 - (1+m_1)T_1(1) - T_1'(1)$$
$$-\theta\nu(1 + b\lambda^+ m_1\gamma_1) \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) - \theta\nu(q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i (M_i^{(1)} B_0^*(\lambda^-)$$
$$+ \Lambda_i^*(\lambda^-) f_0^{(1)})))] - 2\lambda^+ T_2'(1)[b\lambda^+ m_1 T_6 + \nu b\lambda^+ m_1\gamma_1 \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-)]\}$$

$$N_q'''(1) = I_0\{-3b\lambda^+m_1[(T_1(1)+\lambda^- T_5)T_2''(1)+T_2'(1)[2T_1'(1)+2(\lambda^+/\lambda^-)(-f_0^{(1)}$$
$$+b\lambda^+m_1T_5+T_3)]+A^*(\lambda^+)T_8]-3\,b\lambda^+m_2\,T_9+3\lambda^+T_2''(1)[-b\lambda^+m_1T_6$$
$$-\nu\, b\lambda^+m_1\gamma_1\sum_{i=0}^{M}q_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)]+3\lambda^+T_2'(1)[-2(b\lambda^+m_1/\lambda^-)T_6+2b\lambda^+m_1$$
$$f_0^{(1)}\beta_0^{(1)}-(b\lambda^+m_1)^2T_7-b\lambda^+m_2T_6-2b\lambda^+m_1\,\nu\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}(M_{i-1}^{(1)}B_0^*(\lambda^-)$$
$$+\Lambda_{i-1}^*(\lambda^-)f_0^{(1)})(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}+2b\lambda^+m_1\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)$$
$$B_0^*(\lambda^-)\,f_{i,j_i}^{(1)}\,\beta_{i,j_i}^{(1)}-2b\lambda^+m_1\nu(q_0f_0^{(1)}+\sum_{i=1}^{M}q_i(M_i^{(1)}B_0^*(\lambda^-)+\Lambda_i^*(\lambda^-)f_0^{(1)}))$$
$$-((b\lambda^+m_1)^2\gamma_2+b\lambda^+m_2\gamma_1)\sum_{i=0}^{M}q_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)]\}$$

$$T_8 = 2-(2m_1+m_2)T_1(1)-T_1''(1)-2(1+m_1)T_1'(1)-\theta\nu(2b\lambda^+m_1\gamma_1$$
$$+(b\lambda^+m_1)^2\gamma_2+b\lambda^+m_2\gamma_1)\sum_{i=0}^{M}q_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)-2\theta\nu(1+b\lambda^+m_1\gamma_1)$$
$$(q_0f_0^{(1)}+\sum_{i=1}^{M}q_i(M_i^{(1)}B_0^*(\lambda^-)+\Lambda_i^*(\lambda^-)f_0^{(1)}))-\theta\nu(q_0f_0^{(2)}$$
$$+\sum_{i=1}^{M}q_i(M_i^{(2)}B_0^*(\lambda^-)+2M_i^{(1)}f_0^{(1)}+\Lambda_i^*(\lambda^-)f_0^{(2)}))$$

$$T_9 = (T_1(1)+\lambda^+T_5)T_2'(1)+A^*(\lambda^+)[2-(1+m_1)T_1(1)-T_1'(1)T_3]$$

- Mean number of customers in the system is

$$L_s = \lim_{z\to 1}\frac{d}{dz}P_s(z)$$
$$= L_q+P_0+P \qquad (6.74)$$

6.7 Reliability Analysis

Theorem 6.2

The steady state availability of the server is given by

$$\mathcal{A} = \frac{A^*(\lambda^+)[2-T_1(1)-T_1'(1)-T_4]+(A^*(\lambda^+)+(1-A^*(\lambda^+))\bar{r})T_1(1)+\lambda^+T_2'(1)T_5}{D'} \qquad (6.75)$$

Proof.

$$\mathcal{A} = I_0 + \lim_{z\to 1}\left[\int_0^{\infty} I(x,z)\,dx + \int_0^{\infty} P_0(x,z)\,dx + \int_0^{\infty} P(x,z)\,dx\right]$$

$$= I_0 + \lim_{z\to 1}[\,I(z) + P_0(z) + P(z)\,]$$

$$= I_0 + I + P_0 + P \qquad (6.76)$$

Equation (6.75) can be obtained by using equations (6.61), (6.63) and (6.65).

Theorem 6.3

The steady state failure frequency of the server is given by

$$\mathcal{F} = \frac{\lambda^{+} T_2{}'(1)\,[(1 - B_0^{*}(\lambda^{-}) + B_0^{*}(\lambda^{-})\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\,\Lambda_{i-1}^{*}(\lambda^{-})(1 - B_{i,j_i}^{*}(\lambda^{-}))]}{D'} \qquad (6.77)$$

Proof.

$$\mathcal{F} = \lambda^{-}\lim_{z\to 1}\left[\int_0^{\infty} P_0(x,z)\,dx + \int_0^{\infty} P(x,z)\,dx\right]$$

$$= \lambda^{-}\lim_{z\to 1}[\,P_0(z) + P(z)\,]$$

$$= \lambda^{-}\left(P_0 + P\right) \qquad (6.78)$$

Result in equation (6.77) can be obtained by using equations (6.63) and (6.65).

6.8 Special Cases

Case (i) If $\lambda^{-} = 0$, $k_1 = k_2 = ... = k_M = 1$, b=1, r=1, then the system reduces to multistage retrial queueing system with Bernoulli vacation. The probability generating functions of the server states are given by

$$I(z) = \frac{I_0\left(1 - A^{*}(\lambda^{+})\right)\left[\sum_{i=1}^{M} q_i \Lambda_i^{*}(g(z))\left(\bar{v}C(z) + v(\bar{\theta}C(z) + \theta)V^{*}(\lambda^{+}(1 - C(z)))\right) - z\right]}{D(z)}$$

$$P_i(z) = \frac{\lambda^{+} I_0\, z A^{*}(\lambda^{+})(C(z) - 1)\, p_i \Lambda_{i-1}^{*}(g(z))\left(1 - B_i^{*}(\lambda^{+}(1 - C(z)))\right) / \lambda^{+}(1 - C(z))}{D(z)}$$

$$V(z)=\frac{I_0\,\nu\,A^*(\lambda^+)\sum_{i=1}^{M}q_i\Lambda_i^*(\lambda^+(1-C(z)))\,\left(1-V^*(\lambda^+(1-C(z)))\right)}{D(z)}$$

where

$$D(z)=z-\left(A^*(\lambda^+)-C(z)\left(1-A^*(\lambda^+)\right)\right)\left(\bar{\nu}+\bar{\theta}\nu V^*(\lambda^+(1-C(z)))\sum_{i=1}^{M}q_i\,\Lambda_i^*(\lambda^+(1-C(z)))\right.$$
$$\left.-\theta\nu\sum_{i=1}^{M}q_i\,\Lambda_i^*(g(z))\,V^*(\lambda^+(1-C(z)))\right)$$

$$\Lambda_0^*=1\,,\qquad \Lambda_i^*=\prod_{l=1}^{i}p_l B_l^*$$

The above results coincide with Radha et. al (2017a) without impatient customers and orbital search.

Case (ii) If $\lambda^-=0$, M = 1, θ = ν = 0, then the model reduces to $M^X/G/1$ retrial queue with second optional services and impatient customers. The probability generating functions of the server states are given by

$$I(z)=\frac{I_0(1-A^*(\lambda^+))z\left(1-q_0\,B_0^*(h(z))-\sum_{i=1}^{M}q_i\Lambda_i^*(h(z))B_0^*(h(z))\right)}{D(z)}$$

$$P_0(z)=\frac{\lambda^+I_0(1-r+rA^*(\lambda^+))(1-z)(1-B_0^*(h(z)))}{h(z)\,D(z)}$$

$$P_{j_1}(z)=\frac{\lambda^+I_0\,p_{j_1}(1-r+rA^*(\lambda^+))B_0^*(h(z))(1-B_{j_1}^*(h(z)))}{h(z)\,D(z)}\,,\ j_1=1,2,\ldots,k_1$$

where

$$D(z)=\left(q_0\,B_0^*(h(z))+\sum_{i=1}^{M}q_i\Lambda_i^*(h(z))B_0^*(h(z))\right)\left(1+r(z-1)(1-A^*(\lambda^+))\right)-z$$

$$\Lambda_0^*=1\,,\qquad \Lambda_i^*=\prod_{l=1}^{i}p_l B_l^*$$

$$h(z)=\lambda^+(1-C(z))$$

which furnish the results obtained in Bhagat and Jain (2013) without breakdown.

6.9 Practical Justification of the Model

A novel coronavirus, COVID-19 is identified as the root cause for such a deadly outbreak causing respiratory illness. Due to its widespread impact on the global community, stringent measures were implemented to reduce the effect of the outbreak; self-precautionary measures like using face-masks, hand hygiene, and self-quarantine; environmental measures were insisted to follow and also surface cleaning and community measures to prevent the widespread.

The COVID-19 vaccines are expected to provide at least some protection against new virus variants and are effective at preventing serious illness and death. That's because these vaccines create a broad immune response, and any virus changes or mutations should not make vaccines completely ineffective. If any of these vaccines become less effective against one or more variants, it will be possible to change the composition of the vaccines to protect against these variants. The COVID-19 vaccines are safe for most people 18 years and older, including those with pre-existing conditions of any kind, including auto-immune disorders. These conditions include: hypertension, diabetes, asthma, pulmonary, liver and kidney disease, as well as chronic infections that are stable and controlled. Thus, to win corona virus all should get vaccinated.

In India, the third phase of the COVID-19 vaccine drive starts on May 1, 2021. The registration process for vaccination is available via cowin.gov.in, Aarogya Setu App & UMANG App. Our model has its potential application in getting the vaccine through these apps. People from various states in India (Positive Customers) are booking for the vaccine. If the server line is free, an arriving customer registers for vaccine using his mobile number (Essential Service) and the others try after some time. Otherwise, all the customers try their request after some random time (Retrial). On finding the busy server, the customer becomes impatient and leaves the website forever (Non-Persistent Customers). After registration, the customer may search for the vaccine by choosing the state where he is situated (Multi-Optional Stage 1). After selecting the state, the customer has the option to choose the district (Multi-Optional Stage 2). Then, the customer chooses the nearby hospital where the vaccine is available (Multi-Optional Stage 3). After finding the hospital, he opts any one type of vaccine from Sputnik V, Covishield and Covaxin (Multi-Optional stage 4). These are the vaccines which are approved for use in India. The customer may leave the website at any stage.

The server experiences a failure due to software issues (Negative arrival) at any time during the working period and therefore need to be repaired. The unavailability of the vaccine for some random amount of time can be considered as server vacation here. Once if the vaccine is available, a message will be send to the person who has registered (Orbital Search).

6.10 Numerical Results

In this section, numerical illustrations are carried out to exhibit the effect of various parameters on the system performance. Assume that the retrial time, first and second phase service time, repair time during both the phases and vacation time follow (a) Exponential distribution (b) Erlang-2 stage distribution and (c) Hyper Exponential distribution with respective rates η, μ_0, μ_{i,j_i}, β_0, β_{i,j_i} and γ, where i=1,2,...,M and j_i = 1,2,....,k_i . We set the default parameters as $\lambda^+ = 4$, $\lambda^- = 0.2$, $\eta = 50$, M = 3, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, p_{j_1} =[0.4 0.3], p_{j_2} = [0.2 0.3 0.1], p_{j_3} =[0.4 0.2], b = 0.4, r = 0.6, $q_0 = 0.3$, q = [0.4 0.4 1], $\nu = 0.2$, $\mu_0 = 20$, $\beta_0 = 2$, μ_1 = [25 30], μ_2 = [62 42 52], μ_3 = [85 73], β_1 = [1 3], β_2 = [5 7 10], β_3 = [12 14], $\gamma = 10$, $\theta = 0.3$, $c_1 = 0.5$, $c_2 = 0.5$, u = 0.4.

Table 6.1 – 6.6 give the computed values of various system characteristics such as expected number of customers in the system (L_s), probability that the server is idle (I_0), probability that the server is idle during retrial time (I), probability that the server is on vacation (V), availability of the server ($\mathcal{A}$) and the failure frequency of the server ($\mathcal{F}$) corresponding to the selected distributions.

Table 6.1 to 6.4 depict the influence of positive arrival rate (λ^+), retrial rate (η), joining probability (b) and complement of reneging probability (r) on I_0 , I and L_s . It is observed that exponential distribution is given better result and

- increase in λ^+ , b and r increases I and L_s and decreases I_0
- increase in η increases I_0 and decreases I and L_s

Table 6.5 shows the effect of vacation probability (ν) on I_0 , V and L_s . It is observed that increase in ν decreases I_0 and increases V and L_s .

Table 6.6 shows the effect of negative arrival rate (λ^-) on I_0 , $\mathcal{A}$ and $\mathcal{F}$. It is clear that increase in λ^- decreases I_0 and $\mathcal{A}$ and increases $\mathcal{F}$.

Fig. 6.2 to 6.4 exhibit the trend for L_I , L_P , L_R and L_V with respect to λ^+ , b and r. As expected L_I , L_P , L_R and L_V increases with increase in λ^+ , b and r.

Table 6.1 Effect of λ^+ on I_0 , I and L_s

λ^+	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	I	L_s	I_0	I	L_s	I_0	I	L_s
1	0.7645	0.0083	0.2831	0.5712	0.0171	0.5630	0.6908	0.0118	0.3786
2	0.5993	0.0156	0.5426	0.3448	0.0317	1.2032	0.4948	0.0223	0.7267
3	0.4766	0.0224	0.8078	0.2032	0.0453	2.2492	0.3587	0.0322	1.1159
4	0.3817	0.0288	1.1024	0.1051	0.0582	4.6282	0.2581	0.0416	1.6216
5	0.3059	0.0348	1.4518	0.0323	0.0708	15.8892	0.1802	0.0507	2.3732

Table 6.2 Effect of η on I_0 , I and L_s

η	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	I	L_s	I_0	I	L_s	I_0	I	L_s
20	0.3479	0.0689	1.1622	0.0413	0.1407	11.3419	0.2097	0.1012	1.8927
25	0.3589	0.0559	1.1419	0.0620	0.1139	7.6532	0.2255	0.0817	1.7923
30	0.3663	0.0470	1.1285	0.0762	0.0956	6.2869	0.2362	0.0685	1.7313
35	0.3718	0.0406	1.1191	0.0864	0.0824	5.5746	0.2440	0.0590	1.6904
40	0.3759	0.0357	1.1121	0.0941	0.0724	5.1373	0.2498	0.0518	1.6610

Table 6.3 Effect of b on I_0 , I and L_s

b	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	I	L_s	I_0	I	L_s	I_0	I	L_s
0.2	0.4233	0.0223	0.7797	0.1971	0.0387	1.5184	0.3214	0.0300	0.9795
0.4	0.3817	0.0288	1.1024	0.1051	0.0582	4.6282	0.2581	0.0416	1.6216
0.6	0.3345	0.0361	1.6241	0.0784	0.0861	9.2156	0.1838	0.0552	3.0929
0.8	0.2804	0.0445	2.5281	0.0422	0.1054	17.5698	0.0955	0.0713	8.2412
1	0.2179	0.0543	4.2881	0.0151	0.1125	28.1210	0.0342	0.0936	14.1214

Table 6.4 Effect of r on I_0 , I and L_s

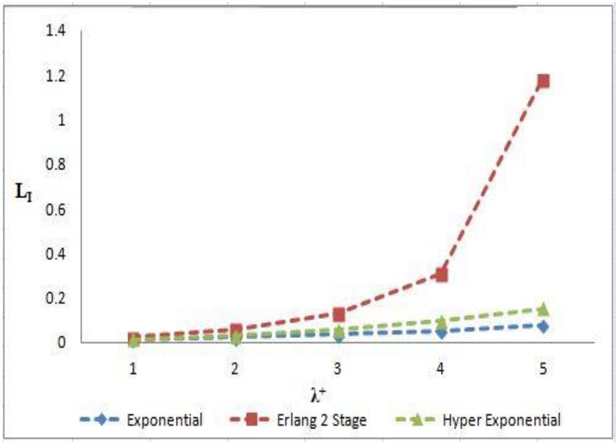
r	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	**I**	**L_s**	**I_0**	**I**	**L_s**	**I_0**	**I**	**L_s**
0.4	0.3843	0.0283	1.0982	0.1128	0.0566	4.3231	0.2626	0.0408	1.6099
0.5	0.3830	0.0285	1.1003	0.1090	0.0574	4.4680	0.2604	0.0412	1.6156
0.6	0.3817	0.0288	1.1024	0.1051	0.0582	4.6282	0.2581	0.0416	1.6216
0.7	0.3804	0.0290	1.1044	0.1011	0.0591	4.8063	0.2557	0.0420	1.6279
0.8	0.3791	0.0292	1.1064	0.0970	0.0599	5.0055	0.2534	0.0425	1.6345

Table 6.5 Effect of ν on I_0 , V and L_s

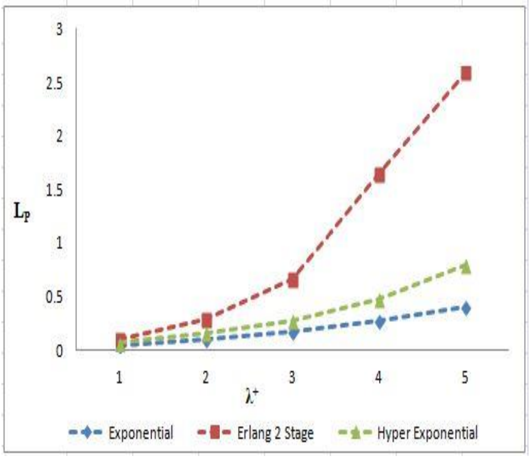
ν	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	**V**	**L_s**	**I_0**	**V**	**L_s**	**I_0**	**V**	**L_s**
0.2	0.3817	0.0635	1.1024	0.1117	0.0821	4.4305	0.2581	0.0734	1.6216
0.4	0.3654	0.1189	1.1243	0.1104	0.1510	4.4695	0.2521	0.1355	1.6436
0.6	0.3511	0.1678	1.1471	0.1089	0.2097	4.5143	0.2470	0.1889	1.6647
0.8	0.3384	0.2111	1.1703	0.1072	0.2602	4.5666	0.2425	0.2352	1.6849
1.0	0.3270	0.2497	1.1937	0.1051	0.3041	4.6282	0.2386	0.2758	1.7040

Table 6.6 Effect of λ^- on I_0 , $\mathcal{A}$ and $\mathcal{F}$

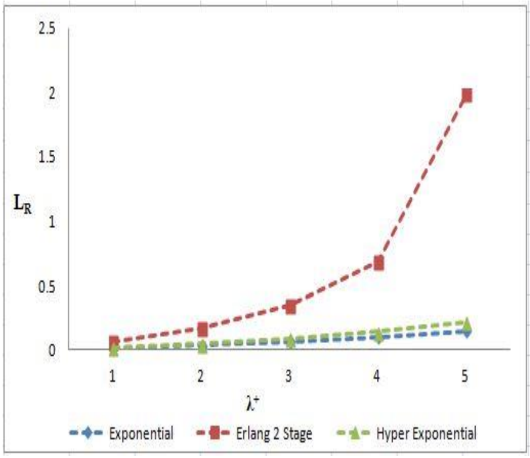
λ^-	Exponential			Erlang-2 Stage			Hyper Exponential		
	I_0	$\mathcal{A}$	$\mathcal{F}$	**I_0**	$\mathcal{A}$	$\mathcal{F}$	**I_0**	$\mathcal{A}$	$\mathcal{F}$
0.1	0.4004	0.9079	0.0480	0.1460	0.8405	0.0641	0.2809	0.8891	0.0569
0.2	0.3817	0.8824	0.0944	0.1051	0.7772	0.1228	0.2581	0.8566	0.1114
0.3	0.3636	0.8578	0.1393	0.0676	0.7191	0.1767	0.2362	0.8254	0.1637
0.4	0.3461	0.8340	0.1827	0.0331	0.6655	0.2264	0.2152	0.7955	0.2138
0.5	0.3292	0.8110	0.2247	0.0013	0.6159	0.2723	0.1951	0.7667	0.2620



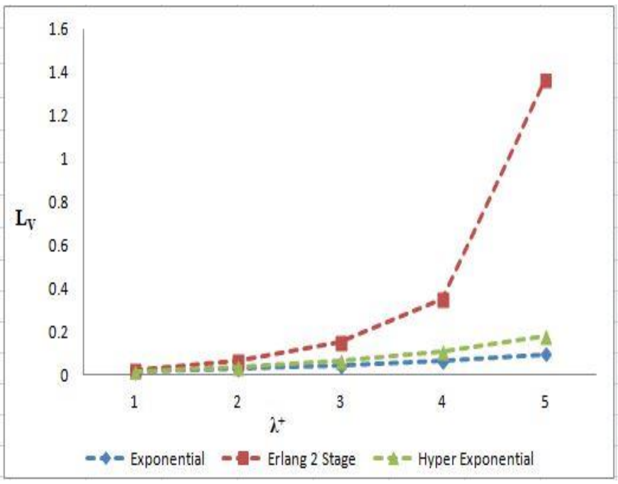
(a) L_I versus λ^+



(b) L_P versus λ^+

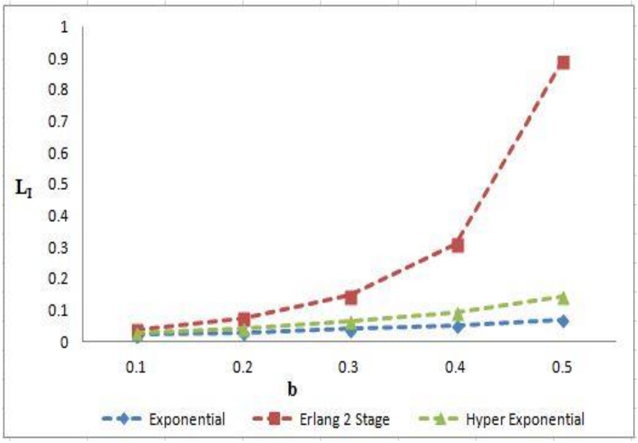


(c) L_R versus λ^+

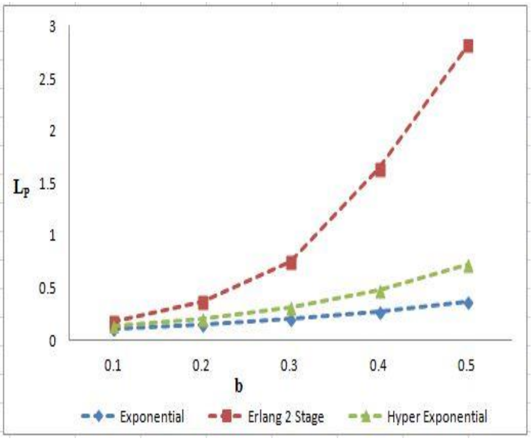


(d) L_V versus λ^+

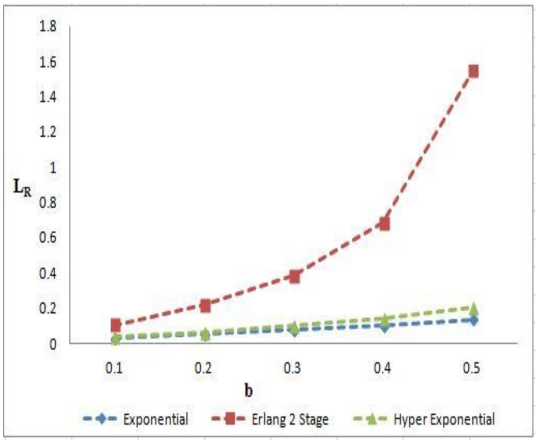
Fig. 6.2 Effect of ' λ^+ ' on Orbit Size L_I , L_P , L_R and L_V



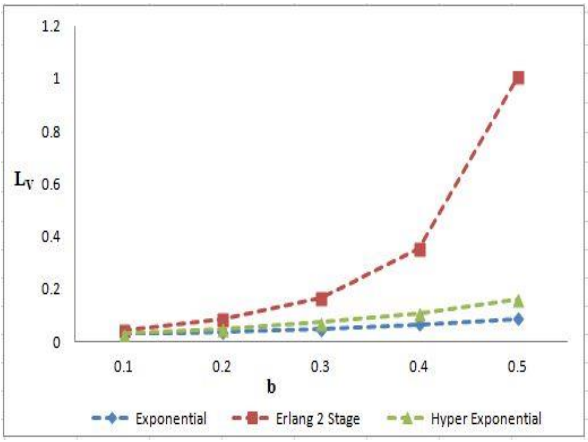
(a) L_I versus b



(b) L_P versus b

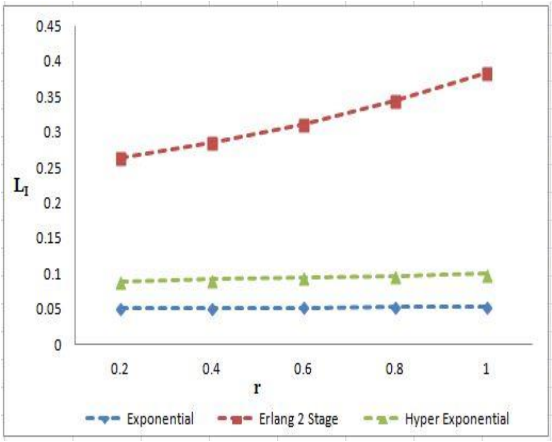


(c) L_R versus b

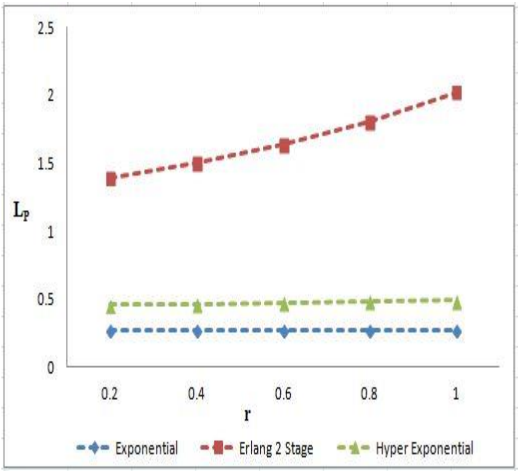


(d) L_V versus b

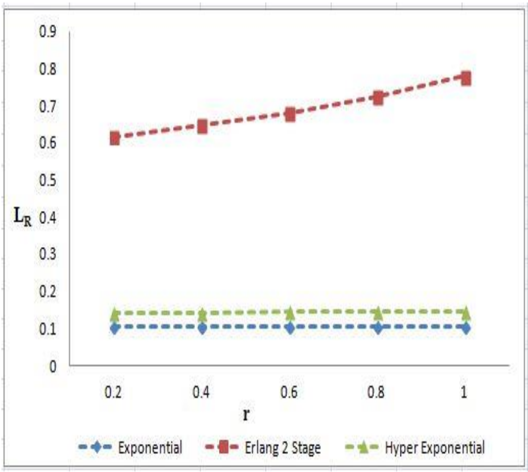
Fig. 6.3 Effect of ' b' on Orbit Size L_I , L_P , L_R and L_V



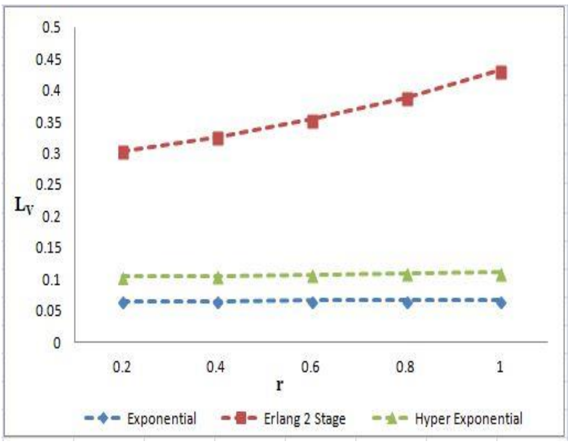
(a) L_I versus r



(b) L_P versus r



(c) L_R versus r



(d) L_V versus r

Fig. 6.4 Effect of ' r ' on Orbit Size L_I , L_P , L_R and L_V