

Chapter V



CHAPTER V

LINEAR OPERATORS PRESERVING RANK AND PERIMETER OF RANK-1 MATRICES OVER SEMIRINGS

Definition : 5.1

The rank of $A \in \mathbf{M}_{m,n}(F_+)$ is 1 if and only if there exist nonzero vectors $\mathbf{a} \in \mathbf{M}_{m,1}(F_+)$ and $\mathbf{b} \in \mathbf{M}_{n,1}(F_+)$ such that $A = \mathbf{a}\mathbf{b}^t$. We call \mathbf{a} the **left factor**, and \mathbf{b} the **right factor** of A . But these vectors \mathbf{a} and \mathbf{b} are not uniquely determined by A .

For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} = \dots\dots\dots$$

Definition :5.2

Let A be any rank-1 matrix in $\mathbf{M}_{m,n}(F_+)$. We define the **perimeter** of A , $P(A)$, as $|\mathbf{a}|+|\mathbf{b}|$ for arbitrary factorization $A = \mathbf{a}\mathbf{b}^t$.

Definition : 5.3

Let T is a linear operator $T \in \mathbf{M}_{m,n}(S)$. Then

- (i) T **preserve rank 1** if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathbf{M}_{m,n}(S)$.
- (ii) T **preserve perimeter k** of rank-1 matrices if $P(T(A)) = k$ whenever $P(A) = k$ for all $A \in \mathbf{M}_{m,n}(S)$ with $r(A) = 1$.

Theorem : 5.4

If T is a (P,Q,B) -operator on $\mathbf{M}_{m,n}(S)$, then T preserves both rank and perimeter of rank-1 matrices.

Proof :

If T is a (P,Q,B)-operator on $M_{m,n}(S)$, there exist $m \times m$ and $n \times n$ permutation matrices P and Q, respectively such that $T(A) = (A \circ B)Q$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $M_{m,n}(S)$, where $B \in M_{m,n}(S)$ is a matrix with all nonzero entries and $r(B) = 1$. Then we can write $B = cd^t$, where none of entries c or d is zero. Let A be a rank-1 matrix in $M_{m,n}(S)$ with a factorization $A = ab^t$. For the case $T(A) = P(A \circ B)Q$, we have the following :

$$T(A) = P(ab^t \circ cd^t)Q = P(a \circ c)(b \circ d)^t Q = (P(a \circ c))(Q^t(b \circ d))^t \text{ -----} (*)$$

Thus (*) implies that

$$r(T(A)) = r((P(a \circ c)) (Q^t(b \circ d))^t) = 1 \text{ and}$$

$$P(T(A)) = |P(a \circ c)| + |Q^t(b \circ d)| = |a \circ c| + |b \circ d| = |a| + |b| = P(A)$$

For the case $m=n$ and $T(A) = (A^t \circ B)Q$, we can show that $r(T(A)) = 1$ and $P(T(A)) = P(A)$ by the similar method as above.

Lemma : 5.5

Let T be a linear operator on $M_{m,n}(S)$. If T preserves rank and perimeter 2 of every rank-1 matrix, then the following statements hold :

- (i) T maps a cell into a cell with nonzero scalar multiplication ;
- (ii) T maps a line matrix into a line matrix.

Proof :

(i) follows from the property that T preserves perimeter 2. (ii) If not, there exist two distinct cells E and F in same row (or column) such that T(E) and T(F) lie in two different columns. Then we have $r(E+F) = 1$, while $r(T(E+F)) = r(T(E)+T(F)) = 2$. This contradicts to the fact that T preserves rank 1.

The following is an example of a linear operator that preserves rank and perimeter 2 of rank 1 matrices, but it does not preserve perimeter $2n$ ($n \geq 2$) and is not a (P,Q,B)-operator.

Example : 5.6

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbf{S})$ with $n \geq 2$ defined by

$$T(A) = \left(\sum_{i,j=1}^n a_{ij} \right) E_{kk} = \max\{a_{ij} \mid i, j = 1, 2, \dots, n\} E_{kk}$$

for all $A = [a_{ij}] \in \mathbf{M}_{n,n}(\mathbf{S})$, where k is a fixed integer in $\{1, 2, \dots, n\}$. Then it is easy to verify that T is a linear operator and preserves rank and perimeter 2 of each rank-1 matrix. But T does not preserve perimeter $2n$; for, if $J \in \mathbf{M}_{n,n}(\mathbf{S})$ is a matrix whose entries are all 1, then J has rank 1 and perimeter $2n$, but $T(J) = E_{kk}$ has rank 1 and perimeter 2. Hence T is not a (P, Q, B) -operator by Theorem 5.4.

Lemma : 5.7

Let T be a linear operator on $\mathbf{M}_{n,n}(\mathbf{S})$. Suppose that T preserves rank and perimeters 2 and $p (\geq 3)$ of rank-1 matrices. Then

- (i) T maps two distinct cells in a row (column) into two distinct cells in a row or in a column with nonzero scalar multiplication;
- (ii) If T maps a row matrix into a row (or column if $m=n$) matrix, then T maps every row matrix into a row (or column if $m=n$) matrix and if T maps a column matrix into a column (or row if $m=n$) matrix then T maps every column matrix into a column (or row if $m=n$) matrix.

Definition :5.8

For a linear operator T on $\mathbf{M}_{n,n}(\mathbf{S})$ preserving rank and perimeter 2 of rank-1 matrices, we define the corresponding mapping $T' : \Delta_{m,n} \rightarrow \Delta_{m,n}$ by $T'(i, j) = (k, l)$ whenever $T(E_{ij}) = b_{ij} E_{kk}$ for some nonzero scalar $b_{ij} \in \mathbf{S}$.

Lemma :5.9

Let T be a linear operator preserving both ranks and perimeters 2 and k ($k \geq 4, k \neq n+1$) of rank-1 matrices. Then T' is a bijection of $\Delta_{m,n}$.

Proof :

By Lemma 5.5, we have that for any $E_{ij} \in E_{m,n}$ there exist $E_{r_l} \in E_{m_n}$ and nonzero $b_{ij} \in S$ such that $T(E_{ij}) = b_{ij}E_{r_l}$. Without loss of generality, we may assume that T maps the i^{th} row of a matrix into the r^{th} row with nonzero scalar multiplication. Suppose that $T'(i,j) = T'(p,q)$ for some distinct pairs $(i,j), (p,q) \in \Delta_{m,n}$. By the definition of T' , we have $T(E_{ij}) = b_{ij}E_{r_l}$ and $T(E_{pq}) = b_{pq}E_{r_l}$ for some nonzero scalars $b_{ij}, b_{pq} \in S$. Lemma 5.7 implies that $i \neq p$ and $j \neq q$. Furthermore T maps the i^{th} row and the p^{th} row of a matrix into the r^{th} row.

Case 1

$4 \leq k \leq n$: claim : we can choose a $2 \times (k-2)$ submatrix A from i^{th} and p^{th} row, but $T(A)$ is a $1 \times k$ submatrix in the r^{th} row. If the claim is true, then $P(A) = k$, while $P(T(A)) = k+1$, a contradiction.

Proof of the claim :

By Lemma 5.7, T maps distincy cells in each row (or column) to distinct cells with nonzero scalar multiplication. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq} . Since there is a cell $E_{ph_1}(h_1 \neq j, q)$ in the p^{th} row such that $T'(p, h_1) = T'(i, q)$ but $T'(i, h_1) \neq T'(p, j)$, we choose a $2 \times (4-2)$ submatrix $E_{ij} + E_{ih_1} + E_{pj} + E_{ph_1}$ whose image under T is 1×4 submatrix in the r^{th} row. Therefore the claim is satisfied for $k = 4$. Assume that for $k = s$ with $4 \leq s \leq n-1$, the claim is true. Then there is a $2 \times (s-2)$ submatrix.

$$X = E_{ij} + \sum_{t=1}^{s-3} E_{ih_t} + E_{pj} + \sum_{t=1}^{s-3} E_{ph_t}$$

such that $T(X)$ is an $1 \times s$ submatrix in the r^{th} row, where $\{j, q, h_1, \dots, h_{s-3}\}$ is the set of distinct indices. Now, we can choose a cell $E_{ph_{s-2}}(h_{s-2} \neq j, q, h_1, \dots, h_{s-3})$ such that $T'(i, h_{s-2}) \neq T'(p, j), T'(p, q), T'(p, h_1), \dots, T'(p, h_{s-3})$. Then we have a

$2 \times ((s+1)-2)$ submatrix $A = E_{ij} + \sum_{t=1}^{s-2} E_{ih_t} + E_{pj} + \sum_{t=1}^{s-2} E_{ph_t}$ such that $T(A)$ is an

$1 \times (s+1)$ submatrix in the r^{th} row. Thus the claim is satisfied for $k = s+1$. By the mathematical induction, the claim is true.

Case 2

$k = n + \alpha \geq n + 2$. Consider a matrix

$$Y = \sum_{s=1}^n E_{is} + \sum_{t=1}^n E_{pt} + \sum_{h=1}^{\alpha-2} \sum_{g=1}^n E_{hg}$$

with rank 1 and perimeter k . Then T maps the i^{th} and p^{th} row Y into the r^{th} row with nonzero scalar multiplication by Lemma 5.7. Thus the perimeter of $T(Y)$ is less than k , a contradiction.

Hence $T'(i, j) \neq T'(p, q)$ for any two distinct pairs $(i, j), (p, q) \in \Delta_{m,n}$.

Therefore T' is bijection on $\Delta_{m,n}$. Hence the lemma.

The condition of $k \neq n+1$ Lemma 5.9 must be necessary. The following example shows that T is a linear operator preserving both ranks and perimeters 2 and $n+1$ of rank-1 matrices, but T' is not a bijection of $\Delta_{m,n}$.

Example : 5.10

Consider a linear operator T on $\mathbf{M}_{2,3}(\mathbf{S})$ defined by

$$T = \left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right) = \begin{bmatrix} a+e & b+f & c+d \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we can easily shows that T preserves both ranks and perimeters 2 and of rank-1 matrices. But T' is not a bijection of $\Delta_{2,3}$.

Notation : 5.11

Let $\mathcal{R}_i = \{E_{ij} \mid 1 \leq j \leq n\}$, $\mathcal{C}_j = \{E_{ij} \mid 1 \leq i \leq m\}$, $\mathcal{R} = \{\mathcal{R}_i \mid 1 \leq i \leq m\}$ and $\mathcal{C} = \{\mathcal{C}_j \mid 1 \leq j \leq n\}$. For a linear operator T on $\mathbf{M}_{n,n}(\mathbf{S})$, define $T^*(\mathcal{R}_i) = \{T^*(E_{ij}) \mid 1 \leq j \leq n\}$ for each $i = 1, 2, \dots, m$ and $T^*(\mathcal{C}_j) = \{T^*(E_{ij}) \mid 1 \leq i \leq m\}$ for each $j = 1, 2, \dots, n$.

Theorem : 5.12

Let T be a linear operator on $\mathbf{M}_{n,n}(\mathbf{S})$, where \mathbf{S} is a chain semiring. Then the following are equivalent :

- (i) T is a (P,Q,B) -operator;
- (ii) T preserves both rank and perimeter of rank-1 matrices;
- (iii) T preserves both rank and perimeters 2 and k ($k \geq 4, k \neq n+1$) of rank-1 matrices.

Proof :

(i) \Rightarrow (ii): Clear by Theorem 5.4. (ii) \Rightarrow (iii): It is obvious.

(iii) \Rightarrow (i): Assume (iii). Then the corresponding mapping $T : \Delta_{m,n} \rightarrow \Delta_{m,n}$ is a bijection by Lemma 5.9. Furthermore, there are two cases :

- (a) T^* maps \mathcal{R} onto \mathcal{R} and maps \mathcal{C} onto \mathcal{C} , or
- (b) T^* maps \mathcal{R} onto \mathcal{C} and maps \mathcal{C} onto \mathcal{R} .

Case (a)

We note that $T^*(\mathcal{R}_i) = \mathcal{R}_{\sigma(i)}$ and $T^*(\mathcal{C}_j) = \mathcal{C}_{\tau(j)}$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, where σ and τ are permutations of $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ , respectively. Then for any $E_{ij} \in E_{m,n}$, we can write $T(E_{ij}) = b_{ij}E_{\sigma(i)\tau(j)}$ for some nonzero scalar $b_{ij} \in \mathbf{S}$. Now we claim that $B = [b_{ij}] \in \mathbf{M}_{m,n}(\mathbf{S})$ has rank 1. For, consider an $m \times n$ matrix J , all of whose entries are 1. Then we have

$$\begin{aligned} T(J) &= T\left(\sum_{i=1}^m \sum_{j=1}^n E_{ij}\right) = \sum_{i=1}^m \sum_{j=1}^n T(E_{ij}) \\ &= \sum_{i=1}^m \sum_{j=1}^n b_{ij} E_{\sigma(i)\tau(j)} \\ &= PBQ. \end{aligned}$$

Since J has rank 1, it follows that $r(T(J)) = 1$, and hence $r(B) = r(PBQ) = r(T(J)) = 1$.

Therefore for any $A = [a_{ij}] \in \mathbf{M}_{n,n}(\mathbf{S})$, we have

$$\begin{aligned}
T(A) &= T\left(\sum_{i=1}^m \sum_{j=1}^n a_{ij} E_{ij}\right) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} T(E_{ij}) \\
&= \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} E_{\sigma(i)\tau(j)} \\
&= P(A \circ B)Q.
\end{aligned}$$

Thus T is a (P,Q,B)-operator.

Case (b)

We note that $m = n$ and $T^*(\mathcal{R}_i) = \mathcal{C}_{\sigma(i)}$ and $T^*(\mathcal{C}_j) = \mathcal{R}_{\tau(j)}$ for all i, j , where σ and τ are some permutations of $\{1, 2, \dots, m\}$. By an argument similar to case (a), we obtain that $T(A)$ is of the form $T(A) = P(A^t \circ B)Q$, thus T is a (P,Q,B)-operator.