

Some Interesting Results from the Theory of Numbers

By

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A thesis Submitted to the Avinashilingam Institute
for Home Science and Higher Education for Women
(Deemed University), Coimbatore - 641 043
in partial fulfilment of the requirements
for the degree of Master of Science
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Certificate

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IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS
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CERTIFIED AS BONAFIED RESEARCH WORK

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Introduction

INTRODUCTION

A "SERIOUS" theorem is a theorem which contains 'Significant' ideas and I suppose that I ought to try to analyse a little more closely the quantities which make a mathematical idea significant.

- A MATHEMATICIAN'S APOLOGY.

by G.H.HARDY

The theory of numbers is simultaneously one of the most elementary branches of mathematics in that it deals, essentially, with the arithmetic properties of the integers 1,2,3, . . . and one of the most difficult branches in so far as it is laden with difficult problems and difficult techniques.

The aim of this dissertation is to collect some interesting results from the Theory of Numbers. With this aim the following papers are studied.

On Mersenne's Numbers

by M.Perisastri [8]

On a technique for obtaining Numbers with a multiplicity of Generators

by A.Narasinga Rao [6]

On Kaprekar's "Junction Numbers"

by Andrzej Makowski [1]

On the form of odd perfect Numbers

by M.Raghavachari [9]

Perfect totient Number

by T.Venkataraman [10]

The set of Numbers $\{1, 2, 7\}$

by Neelambigai Thamothers Pillai [7]

On the structure of Magic Matrices - 1

by R.Balakrishnan [3]

The first chapter is devoted to the study of Mersenne's numbers, generated numbers, Kaprekar's junction numbers, odd perfect numbers, perfect totient numbers, the set of numbers $\{1, 2, 7\}$.

1. Numbers of the form $2^p - 1$ where p is prime, are called Mersenne's numbers and denoted by M_p .

2. Given a number A the number $g(A)$ generated by A is formed by adding to a (written in the scale of ten) the

A magic square is a double array of elements arranged in the form of a square whose row sums, column sums as well as the two diagonal sums (herein called associated sum) are all equal. A matrix representing such a square is a magic matrix.

Here many interesting results connected with the magic matrices are discussed.

1. The aggregate of magic matrices form a ring M with unity and aggregate of the magic matrices with associated sum zero constitutes a maximal ideal in M .

2. Any magic matrix is a linear combination of the permutation matrix, we mean a matrix which has unity in each row and column only once while the other elements of the matrix are all zero.

The ring M of magic matrices of order n defined over a scalar field R (of zero characteristic) is a unitary matrix algebra over R of order $(n^2 - 2^n + 2)$

In the third chapter the two common types of density of sequence of integers (Asymptotic density and Schnirelman density) are defined and some interesting results connected with

the asymptotic density are discussed.

Asymptotic density $\delta_1(A)$ of a set A is defined by:

$$\delta_1(A) = \lim_{n \rightarrow \infty} \inf A(n)/n$$

In case the sequence $A(n)/n$ has a limit we say that A has a natural density $\delta(A)$. The main results proved here are the following

" If A is an infinite sequence then

$$\delta_1(A) = \lim_{n \rightarrow \infty} \inf n/a_n$$

If $\delta(A)$ exists, then $\delta(A) = \lim_{n \rightarrow \infty} n/a_n$

"The set of square - free integers has natural density

6/ π^2 "

Chapter - 1

CHAPTER - I

SECTION 1 ON MERSENNE'S NUMBERS

Definition 1.1

The numbers of the form $\frac{1}{2} n(n+1)$, ($n = 1, 2, 3, \dots$) are called triangular numbers. The numbers 1, 3, 6, 10, . . . etc, are triangular numbers.

Definition 1.2

Numbers of the form $2^n - 1$ where n is prime, are called Mersenne's Numbers and denoted by M_n .

Theorem 1.3

$M_n = x^y$ has no solutions in positive integers, x, y with $x > 1$, $y > 1$.

Proof :-

Let $x > 1$ and $y = 2k$, $k > 1$ [since y is even]

If $M_n = x^y$.

We know that M_n is of the form $2^n - 1$

(i.e) M_n is always odd [$2^n - 1 = \text{even} - 1 = \text{odd}$].

$$\begin{aligned} M_n &= 2^n - 1 \\ &= x^y = x^{2k} \end{aligned}$$

$$M_n = 2^n - 1 = x^y = x^{2k}.$$

$$\therefore 2^n - 1 = x^{2k}.$$

$$2^{n-2} = x^{2k-1} \equiv 0 \pmod{8}.$$

Since x is odd. So, $2^{n-1} \equiv 1 \pmod{4}$ for every $n > 1$.

Now let $x > 1$, $y = 2^{k+1}$, $k \geq 1$ (y -odd)

$$\text{If } M_n = x^y$$

$$\text{We have } M_n = 2^n - 1 = x^y = x^{2k+1}$$

$$2^n - 1 = x^{2k+1}$$

$$\begin{aligned} 2^n &= x^{2k+1} + 1 \\ &= (x+1) \sum_{r=0}^{2k} (-1)^r x^r \end{aligned}$$

Since x is necessarily odd and $k \geq 1$, the second factor on the right is odd.

Hence the result.

Theorem 1.4

There are an infinity of triangular numbers which are not Mersenne's numbers.

Proof :-

A triangular number is of the form $\frac{1}{2} n(n+1)$

$$\text{If } n(n+1) = a^2 \text{ then } \quad [\text{since } (2n+1)^2 = 1 + 4n^2 + 4n]$$

$$\begin{aligned} (2n+1)^2 &= 1 + 8a^2 & &= 1 + 4n(n+1) \\ & & &= 1 + 4 \left[\frac{n(n+1)}{2} \right] \cdot 2 \\ & & &= 1 + 8a^2 \end{aligned}$$

$(2n+1)^2 = 1 + 8a^2$ which has infinitely many solutions with $x, y > 0$.

Therefore, infinitely many triangular numbers are squares, while M_n is not a square for $n > 1$.

Hence the theorem.

Theorem 1.5

$$\begin{array}{l}
 \text{i.} \quad 2^{20k+13} \equiv 92 \\
 \text{ii.} \quad 2^{20k+17} \equiv 72 \\
 \text{iii.} \quad 2^{20k+21} \equiv 52
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{i.} \\ \text{ii.} \\ \text{iii.} \end{array}} \right\} \pmod{10^2}, k \geq 0.$$

Proof :-For $k = 0$ in (i)

$$2^{20(0)+13} = 2^{13} = 8192 \equiv 92 \pmod{100}$$

Put $k = k+1$

$$\begin{aligned}
 \text{But } 2^{20(k+1)+13} - 92 &= 2^{(20k + 20)+13} - 92 \\
 &= 2^{20k+13} 2^{20} - 92 \times 2^{20} + 92 \times 2^{20} - 92 \\
 &= (2^{20k+13} - 92)2^{20} + 92(2^{20} - 1) \\
 &= (2^{20k+13} - 92)2^{20} + 92(2^{10} + (2^{10} - 1))
 \end{aligned}$$

Since $(2^{20k+13} - 92) \equiv 0 \pmod{100}$, $4/92$ and $25/2^{10}+1$ the above equation shows that (i) is true for $k+1$, if is true for k .

Hence by induction (i) is proved.

$$\text{ii. } 2^{20k+17} \equiv 72 \pmod{100}$$

For $k = 0$

$$2^{20(0)+17} = 2^{17} = 131072 \equiv 72 \pmod{100}$$

put $k = k + 1$.

$$\begin{aligned}
2^{20(k+1)+17} - 72 &= 2^{(20k+20)+17} - 72 \\
&= 2^{20k+17} 2^{20} - 72 \times 2^{20} + 72 \times 2^{20} - 72 \\
&= (2^{20k+17} - 72)2^{20} + 72(2^{20} - 1) \\
&= (2^{20k+17} - 72)2^{20} + 72(2^{10} + (2^{10} - 1))
\end{aligned}$$

Since $2^{20k+17} - 72 \equiv 0 \pmod{100}$, $4/72$ and $25/2^{10}+1$, the equation shows that (ii) is true for $K+1$, if it is true for k .

Hence by induction (ii) is proved.

$$\text{iii. } 2^{20k+21} \equiv 52 \pmod{100}, k \geq 0.$$

For $k = 0$

$$2^{20(0)+21} = 2^{21} = 2097152 \equiv 52 \pmod{100}$$

Put $k = k+1$.

$$\begin{aligned}
2^{20(k+1)+21} - 52 &= 2^{(20k+20)+21} - 52 \\
&= 2^{20k+21} 2^{20} - 52 \times 2^{20} + 52 \times 2^{20} - 52 \\
&= (2^{20k+21} - 52)2^{20} + 52(2^{20} - 1) \\
&= (2^{20k+21} - 52)2^{20} + 52(2^{10} + (2^{10} - 1))
\end{aligned}$$

Since $2^{20k+21} - 52 \equiv 0 \pmod{100}$, $4/52$ and $25/2^{10} + 1$ the above equation shows that (iii) is true for $k + 1$, if it is true for k .

Hence by induction (iii) is proved.

Hence the theorem.

Theorem 1.6

M_{2n} is not triangular for $n = 10k + 5, 10k + 7, 10k + 9$.

Proof :-

First let $n = 10k + 5$, then if M_{2n} is triangular.

We know $1 + 8M_{2n} = a^2$

Hence $2^{2n+3} - 7 = a^2$

(i.e) $2^{20k+13} - 7 = a^2$

By theorem 1.5 (i) we get $a^2 \equiv 85 \pmod{100}$. Thus a must be of the form $10b + 5$ and we would get

$$(10b + 5)^2 \equiv 85 \pmod{10^2}$$

$$25 \equiv 85 \pmod{10^2}$$

Hence M_{2n} is not triangular. Similarly, we can prove that the theorem when $n = 10k + 7$ and $n = 10k + 9$.

SECTION 2 GENERATED NUMBERS AND KAPREKAR'S "JUNCTION NUMBERS".

Consider any number A written in the scale of ten. Add to it the sum of all its digits. The number thus obtained will be called the number generated by A and will be denoted by $g(A)$.

From $g(A)$ by applying the generating process we can get another number which we shall denote by $g^2(A)$. It is obvious that $g^n(A)$ exists for all positive integral values of n and forms an ordered set of chain which we shall call a digital chain.

Sometimes we can go backward and find a number $g^{-1}(A)$. Which generates A . Going backwards like this we shall come to a stop at a certain number which cannot be obtained from another smaller number by the above digit addition process. Thus starting from $A = 34$, we get forward and get successively $34 + 4 + 3 = 41$, then 46, 56 etc and going backward $g^{-1}(34) = 26$, $g^{-1}(26) = g^{-2}(34) = 20$ and $g^{-3}(34) = 20$. But 2 has no generator. We call such numbers "Self numbers". Every chain may be supposed to start from some self number. The first few self numbers are 1, 3, 7, 20, 31, 42, 53, 64, 75, 86, 97, 108, 121, 132, etc. Sometime a number has more generators than one we call it a junction number.

The number 101 can be generated both from 91 and from 100 so, it is a Junction number being the smallest of all the Junction Numbers.

The next Junction numbers are 103, 105, 107, 111, 113, 115, 117 but the junction number after that is 202, generated by 191 and 200.

In 1965, Andrzej Makowski (Warsaw) proved that, they are e.g., the numbers,

$$m_k = 10^{n_k + k} + 1 \quad (K = 1, 2, 3, \dots)$$

where $n_k = \frac{1}{9} (10^k - 1)$.

Proof :-

For such m_k we can take $x = (10^{n_k} - 1)10^k + 1$ and $y = 10^{n_k + k}$. We have $t_x = 9n_k + 1 = 10^k$.

$$\text{Hence } x + t_x = 10^{n_k + k} + 1 ; t_y = 1,$$

$$y + t_y = 10^{n_k + k} + 1,$$

which completes the proof.

Sometimes, we have 3 chains meeting at a number with three generators. The least number with three generators according to Kaprekar is $10^{13} + 1$ which has the following generators 10^{13} and $9_{10} 901$ and $9_{10} 892$. Here 9_{10} means the digit 9 repeated ten times. So that the second generator written above is 9 written ten times followed by the digits 901.

The least number 4 generators is believed to be the 25

digit number $10_{21}102$. This last result has been found by Prof. Gunjekar and Sri Kaprekar by starting with some large numbers with 4 generators and property. The least number with 5 generators is still not known but is be a very huge number .

KAPREKAR'S JUNCTION THEOREM

Kaprekar found a very interesting theorem in 1962.

"Let A and B be two numbers and let A have m generators A_1, A_2, \dots, A_m and let B have n generators B_1, B_2, \dots, B_n all the generators A_r and B_s having the same number d of digits. If $A - B = 2$, then we can get a new number with $m + n$ generators as follows. Prefix the digits 9_k to each of the numbers $A_1, A_2, A_3, \dots, A_m$, and prefix the digits 10_k to each of the numbers B_1, B_2, \dots, B_n . Then all the $m + n$ numbers so obtained will be generators of a single number provided $k = 1_d$ that is $1111\dots$ (d times)".

Take $A = 521$ (which has generators 499 and 508) and the number $B = 519$ (which has the generators 498 and 507). Here $A - B = 2$. The generators are all 3 digit numbers so that $k=1_3=111$. According to the theorem $9_{111}499, 9_{111}508, 10_{111}498$, and $10_{111}507$ are all cogenerators of a single number namely $10_{111}520$ which is a number of 115 digits.

This result is easily verified as also the junction theorem.

The merit of Kaprekar's junction theorem is that it gives a simple method for combining 2 numbers with m and n generators so as to get a junction number with $m + n$ generators.

GENERALISATION OF THE JUNCTION THEOREM

"Let A and B be two junction numbers with m and n generators, all of d digits each and let $A - B = 2 + 9p$ where p is a positive integer. Then we get a new number with $m + n$ generators by prefixing 9_k to each of the generators of A , and 10_k to each of the generators of B , where $k = 1_d - p$."

A NUMBER WITH THREE GENERATORS

To get a number with 3 generators, we shall combine a number with two generators with a number with one generator. Since the smallest number with 2 generators has generators with three digits, we cannot make d smaller than 3. The smallest number with one generator is 0 whose generator is also 0. We make it into a three digit number by writing it as 000 and take this as our B . For A we search for the biggest number with two generators each of 3 digits and have a choice among 925, 923, 921, 919, etc. and choose the biggest such that $A - B = A$ (since $B = 0$) is of the form $2 + 9p$. This is 911 which has the two generators 892 and 901. The difference $A - B = 911 = 2 + 9p$ where $p = 101$. Since $d = 3$, we have to take $k = 1_d - p = 111 - 101 = 10$. So we have to insert $9_k = 9_{10}$ before 892 and 901 and 10_{10} before 000

giving $9_{10}901$, $9_{10}892$ and 10_{13} as the 3 generators of the number with 3 generators namely $10_{12}1$.

A NUMBER WITH 4 GENERATORS

Here we take B to be the smallest number with 2 generators, namely 101 and A has the biggest number with 2 generators each of 3 digits consistent with $A - B = 2 + 9p$. It will be found that 913 is the biggest number which fulfils this requirement. Its generators are 893 and 902. The difference $A - B = 913 - 101 = 2 + 9p$ where $p = 90$. Since $d = 3$ in this case we take $k = 111 - 90 = 21$. Inserting 9_{21} before A_1 , A_2 and 10_{21} before B_1 , B_2 we get $9_{21}893$, $9_{21}902$, $10_{21}091$ and $10_{21}100$ as the four generators of $10_{21}102$.

I believe that is the smallest number with 4 generators.

A NUMBER WITH 5 GENERATORS

We combine a number with 3 generators with one with 2 generators. This will give a much smaller number than combining one with 4 generators with one with one generator. It is found that we shall get a smaller junction number if we make A the bigger number have 3 generators and B the smaller number have 2 generators. For the smaller number B we take the smallest number with 2 generators namely 101 and convert its generators into 14 digit numbers by adding zeros in front. Its two generators are thus $B_1 = 0_{11}091$ and $B_2 = 0_{11}100$. For the bigger number A with 3 generators

we start with a number with 3 generators such as $10_{10}102$ which has generators $10_{10}100$, $10_{10}091$, and $09_{10}992$ and increase its till further by adding 8 to the first digit of each of the generators. We now make a suitable addition to the last digits of each of three generators so as to make the difference $A - B = 2 + 9p$ with p as an integer.

It will be found that adding 1 to the last digit does the trick. Thus we get $90_{10}101$, $90_{10}092$ and $89_{10}993$ as the three generators of the number A which is $90_{10}112$. The difference between A and B turns out to be $2 + 9p$ where $p = 10_{12}1$. Since all the generators are of 14 digits, $d = 14$. Subtracting p from $1_d = 1_{14}$. We get $k = 1_{12}0$. Hence the generators of the 5 generator number sought for are

$$\begin{array}{l}
 10_{k+11}091 \\
 10_{k+11}100 \\
 9_{k+1}0_{10}101 \\
 9_{k+1}0_{10}092 \\
 9_k89_{10}993
 \end{array}
 \quad
 \begin{array}{l}
 \text{leading to } N = 10_{k+11}102 \\
 = 10_{k+14} + 102 \\
 \text{where } k = 111111111110.
 \end{array}$$

The number of digits in the number N with 5 generators is $k + 15 = 1_{11}25$ and magnitude of the number is slightly $> 10^{12}$.

NUMBERS WITH SIX, SEVEN AND EIGHT GENERATORS

To find a number with 6 generators, we seek two junction numbers. A and B each having 3 generators, such that $A - B = 2 + 9p$ and in order to make $A - B$ as large as possible. For any number with 3 generators we can get another much bigger number with 3 generators by adding the same number < 10 to the first digit of all the generators of that number, and finding the number generator by all of them. Infact we can add any particular number < 10 to the units, or tens or hundreds, . . . digits of all the cogenerators of a junction number without affecting the cogenerator property, provided that by this addition we do not make any digit > 9 . In this way we find a big number A and a small number B with the same number of digits such that $A - B = 2 + 9p$.

For a number with SIX generators found by this process.

$9_k 89_{10} 993$ all of which are generators of $10_k 10_{12}^2$

$9_k 90_{10} 092$ where $K = 2, 222, 222, 222, 210 = 2_{11} 10$

$9_k 90_{10} 101$ a number of $k + 15$ digits. The magnitude

$10_k 09_{10} 901$ of the number is roughly $10^{k+14} = 10^{2 \times 10^{12}}$

$10_k 09_{10} 8992$ approx.

$10_k 10_{10} 000$

For a number with SEVEN generators

$9_k^{89} 21^{894}$
 $9_k^{89} 21^{903}$ all of which are generators of
 $9_k^{90} 21^{092}$ $M = 10_{k+11} 10_{12}^2$ where $K = 111, 111,$
 $9_k^{90} 21^{101}$ $111, 112, 222, 222, 222, 210 = 1_{11} 2_{11} 10.$
 $10_{k+11} 10_{10}^{000}$ M is number of $1_{11} 2_{11} 3 6$ digits and
 $10_{k+11} 09_{10}^{901}$ it is of order $10^{k+25} = 10^{10^{23}}$ approx.
 $10_{k+11} 09_{10}^{892}$

M is a number of $1_{11} 2_{11} 36$ digits and it is of order $10^{k+25} = 10^{10^{23}}$ approx.

Similarly for a junction number with 8 generators.

$9_k^{90} 21^{093}$
 $9_k^{90} 21^{102}$ all of which are generators of
 $9_k^{89} 21^{904}$ $P = 10_k 10_{21}^{103}$ where $K = 2_{23}^1 = 2_{24}^{-1}$
 $9_k^{89} 21^{895}$ Numbers of digits of p are $k + 26$
 $10_k^{09} 21^{893}$ The order of P is $10^{k+25} = 10^{2 \times 10^{23}}$
 $10_k^{09} 21^{902}$ $= 100^{100^{11.5}}$
 $10_k^{10} 21^{091}$
 $10_k^{18} 100_{21}$

A GENERAL THEOREM ON DOUBLING THE NUMBER OF GENERATORS OF A JUNCTION NUMBER.

" Given a number B of digits whose first digit is 1 and having d digits, and given also that the different between the biggest and the smallest units digit in the several generators does not exceed 7, it is possible to find another number which has $2m$ generators. The value of this number is $10^{k+d} + B + 1$ where $k = 2^{d-1} - 1$ ".

This we do by adding 8 to the highest digit of each generator of B to get the generator of A_2 and adjust the unit digits so that $A - B - 2$ is divisible by 9.

SECTION 3 ON THE FORM OF ODD PERFECT NUMBERS.

Definition 1.7

A number is called perfect number if it equals to the sum of all its proper divisor (less than that Number) If that number is odd then it is called Odd Perfect Numbers.

Example 1.8

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

6 and 28 are examples of even perfect numbers.

Theorem 1.9

If odd perfect numbers exist, they are of the forms $12m + 1$ or $36m + 9$.

Proof :-

Let n be an odd perfect number with representation

$$n = p^\alpha q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$$

p and α are of the form $4k + 1$ and $\beta_1, \beta_2, \dots, \beta_t$ are all even.

since $n \equiv 1 \pmod{4}$. This is because $p \equiv 1 \pmod{4}$ and

$q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$ being an odd perfect square, is $\equiv 1 \pmod{4}$

also for the same reason, $q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t} \equiv 1 \pmod{3}$,

$$q_i \neq 3 ; i = 1, 2, \dots, t$$

so that $q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t} \equiv 1 \pmod{12}$, $q_i \neq 3 ; i = 1, 2, \dots, t$

Now we distinguish two cases.

Case [1] $3/n$

$3/n \Rightarrow n$ is either $12m + 3$ or $12m + 9$.

Since $12m + 3 = 1 \pmod{4}$,

n cannot be of the form $12m + 3$.

Hence n is of the form $12m + 9$.

Since $p \neq 3$, $3^2/(12m + 9)$

which implies $3/3 \Rightarrow n$ is of the form $36m + 9$.

Case [2]

3 is not a divisor of n .

$\Rightarrow n$ is one of the form $12m + 1$, $12m + 5$, $12m + 7$ and $12m + 11$.

Since $12m + 7$ and $12m + 1$ are not congruent to 1 (mod 4) we can rule out these two forms.

Next we observe that n cannot be of the form $12m + 5$.

Therefore $q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t} \equiv 1 \pmod{12}$, $q_i \neq 3$; $i = 1, 2, \dots, t$
and $\alpha \equiv 1 \pmod{4}$

which implies that p is of the form $12m + 5$.

Since α is odd, this means $3/(1 - p + p^2 + \dots + p^\alpha)$

$3/n$, which is a contradiction

Thus n can only be of the form $12m + 1$.

Theorem 1.10

Let n be an odd perfect with the representation

$n = p^\alpha q_1^{\beta_1} \dots q_t^{\beta_t}$. Assume 3 is not a divisor of n . Then $p \equiv 1 \pmod{12}$ and $\alpha \equiv 1 \pmod{12}$ or $\alpha \equiv 9 \pmod{12}$

Proof :-

From case 2 of theorem (1.9) we have $n \equiv 1 \pmod{12}$ and
 $q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t} \equiv 1 \pmod{12}$

$$p^\alpha \equiv 1 \pmod{12}$$

$$p \equiv 1 \pmod{12}, \text{ since } \alpha \equiv 1 \pmod{4}$$

New α can be of the form $12s + 1$, $12s + 5$ and $12s + 9$.

Let if possible, $\alpha \equiv 5 \pmod{12}$

We have now $1 + p + p^2 + \dots + p^{12s+5} = (1+p)$

$(1 + p^2 + p^4 + \dots + p^{12s+4})$ since $p \equiv 1 \pmod{12}$,

$$1 + p^2 + \dots + p^{12s+4} \equiv 0 \pmod{3}$$

$\Rightarrow 3/n$ which is a contradiction.

Thus α is one of the forms $12s - 1$ or $12s + 9$.

SECTION 4 PERFECT TOTIENT NUMBER

Definition 1.11

If $n \geq 1$ the Euler totient $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n ; thus

$$\phi(n) = \sum_{\substack{k=1 \\ k \text{ relatively prime to } n}}^n 1,$$

where the ' indicates that the sum is extended over those k relatively prime to n .

Here is a short table of values of $\phi(n)$.

n	1	2	3	4	5	6	7	8	9	10
$\phi(n)$	1	1	2	2	4	2	6	4	6	4

Definition 1.12

Let $\phi(n)$ be Euler's function. If $\phi(n) = n_1$, $\phi(n_1) = n_2$, $\phi(n_2) = n_3$, so on, then define the totient sum function as

$$\phi_p(n) = \phi(n) + \phi(n_1) + \dots + 1$$

when $\phi_p(n) = n$, n is called a perfect totient number.

Theorem 1.13

If $n = 3(4 \cdot 3^{m-1} + 1)$ and $4 \cdot 3^{m-1}$ is a prime then n is a perfect number ($m \geq 1$).

Proof :-

$$\text{We know } \phi(n) = n \prod_{p/n} (1 - 1/p)$$

$$\phi_p(n) = \phi_p(3(4 \cdot 3^{m-1} + 1))$$

$$\phi_p(3(4 \cdot 3^{m-1})) = 3(4 \cdot 3^{m-1} + 1) (1 - 1/3) \left(1 - \frac{1}{4 \cdot 3^{m-1} + 1}\right)$$

$$= 3(4 \cdot 3^{m-1} + 1) \left(\frac{2}{3}\right) \left(\frac{4 \cdot 3^{m-1} + 1 - 1}{4 \cdot 3^{m-1} + 1}\right)$$

$$= 8 \cdot 3^{m-1} = n_1$$

$$\phi_p(n_1) = 8 \cdot 3^{m-1} \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right)$$

$$= 8 \cdot 3^{m-1} \left(\frac{2}{3}\right) \left(\frac{2}{3}\right)$$

$$= 8 \cdot 3^{m-2} = n_2$$

$$\phi_p(n_2) = 8 \cdot 3^{m-2} \cdot \frac{1}{2} \cdot \frac{2}{3} = 8 \cdot 3^{m-3} = n_3$$

proceeding like this, we get

$$\phi_p(n) = 8(3^{m-1} + 3^{m-2} + \dots + 3 + 1) + 4 + 2 + 1$$

$$= 3(4 \cdot 3^{m-1} + 1) = n.$$

Theorem 1.14

$n/3 \neq k^t$, for any $k > 1$, $t > 1$.

Proof ;-

If t is even $k^{t-1} \equiv 0 \pmod{8}$ as k is odd

But $4 \cdot 3^{m-1} \not\equiv 0 \pmod{8}$ }
 So, t must be odd } - - - - - (1)

If t is odd $(k - 1) (k^{t-1} + k^{t-2} + \dots + k + 1) = 4 \cdot 3^{m-1}$

So, $k - 1 = 4$ or $4 \cdot 3^t$ and

$1 + k + k^2 + \dots + k^{t-1} = 3^{m-1}$ or 3^h , $l + h = (m - 1)$

By definition (1.12) $t = 2$ in contradiction to (1)

Hence the theorem.

Note :

If we apply quadratic character of 3, we say that $4 \cdot 3^{m-1} + 1$ can have factors of the form $12s + 5$, $12s + 1$ or $12s + 7$. We are not able to prove for all conditions that $\phi_p(n) < n$. Under certain conditions $\phi_p(n) < n$.

Example 1.15

$n = 3 \cdot 5 (125 + 1)$ or $3 \cdot 5 (125 + 7)$. It is seen that

$$\phi_p(n) < \frac{8}{15} n + \frac{8}{45} (1 + \frac{1}{2} + \frac{1}{2^2} + \dots) n = \frac{8}{9} n < n.$$

SECTION 5 THE SET OF NUMBERS $\{1,2,7\}$

In this section we present an interesting result proved by 'Neelambigai Thamotharam Pillai'.

The set of numbers $\{1,2,7\}$ has the property that the product of any two distinct number of the set increased by 2 is a square. But this property doesnot hold for the set $\{1,2,7,C\}$ where C is any positive integer. To prove this fact we shall prove the following theorem.

Theorem 1.16

There exists no positive C such that the product of every pair of numbers of the set $\{1,2,7,C\}$ increased by 2 is a square.

Proof :-

To prove this it is sufficient to prove that there exist no positive integers x,y and z satisfying the Diophantine equations.

$$C + 2 = x^2 \quad \text{-----} \quad (1)$$

$$2C + 2 = y^2 \quad \text{-----} \quad (2)$$

$$2C + 2 = z^2 \quad \text{-----} \quad (3)$$

From (2) we get $2(C + 1) = y^2$

$$C + 1 = y^2/2$$

$$2 \text{ divides } y^2$$

$$2 \text{ divides } y$$

Let $y = 2Y$ where Y is any integer.

Therefore equation (2) becomes

$$2C + 2 = 4Y^2$$

$$C + 1 = 2Y^2 \quad \text{-----} \quad (4)$$

From the equation (1) (3) and (4) we get

$$x^2 - 2Y^2 = 1 \quad \text{-----} \quad (5)$$

$$z^2 - 7x^2 = 12 \quad \text{-----} \quad (6)$$

Now we will prove that the Diophantine equations (5) and (6) do not hold simultaneously.

The general solution of the equation (5) is given by x_n
 $x_n + \sqrt{2} Y_n = (1 + \sqrt{2})^{2n}$ where n is an integer.

Hence we get

n	0	1	2	3	4
x_n	1	3	17	99	577
y_n	0	2	12	70	408

$$\begin{aligned} X_{m+n} + \sqrt{2} Y_{m+n} &= (1 + \sqrt{2})^{2(m+n)} \\ &= (1 + \sqrt{2})^{2m} (1 + \sqrt{2})^{2n} \\ &= (x_m + \sqrt{2} Y_m) (x_n + \sqrt{2} Y_n) \end{aligned}$$

$$(i.e) \quad X_{m+n} + \sqrt{2} Y_{m+n} = (X_n X_m + 2 Y_n Y_m) + \sqrt{2} (x_n Y_m + x_m Y_n)$$

Hence we obtain

$$X_{m+n} = X_n X_m + 2 Y_n Y_m \quad \text{-----} \quad (7)$$

$$Y_{m+n} = X_n Y_m + X_m Y_n \quad \text{-----} \quad (8)$$

Putting $m=n$ in the above equation (7) and (8) we get

$$x_{2n} = x_n^2 + 2y_n^2 = 2x_n^2 - 1 = 4y_n^2 - 1 \quad \text{---} \quad (9)$$

$$\text{and} \quad y_{2n} = 2x_n y_n \quad \text{-----} \quad (10)$$

Putting $m = 2r$ in the equation (7) and (8) we get

$$x_n + 2r = x_n x_{2r} + 2y_n y_{2r}$$

$$\text{(i.e.) } x_n + 2r = x_n(2x_r^2 - 1) + 2y_n(2x_r y_r)$$

(Since from(9) and (10))

$$\text{(i.e.) } x_{n+2r} = 2x_n x_r^2 - x_n + 4x_r y_r y_n$$

$$x_{n+2r} \equiv -x_n \pmod{x_r} \quad \text{-----} \quad (11)$$

We also have

$$x_{n+2r} = x_n x_{2r} + 2y_n y_{2r}$$

$$\text{(i.e.) } x_{n+2r} = x_n(4y_r^2 + 1) + 2y_n(2x_r y_r)$$

$$\text{(i.e.) } x_{n+2r} = 4x_n y_r^2 + 4x_r y_r y_n + x_n$$

$$x_{n+2r} \equiv x_n \pmod{y_r} \quad \text{-----} \quad (12)$$

From equation (6) we get

$$z_n^2 = 7x_n^2 - 12 \quad \text{-----} \quad (13)$$

The proof is now accomplished in three stages

a. When $n \equiv 0 \pmod{4}$ by equation (11) we have,

$$x_n \equiv \pm x_0 \pmod{x_2}$$

$$\equiv \pm 1 \pmod{17}$$

Therefore by (13)

$$Z_n^2 \equiv -5 \pmod{17}$$

which is impossible, since -5 is not a quadratic residue modulo 17.

Hence (13) is impossible when $n \equiv 0 \pmod{4}$

b. When $n \equiv 1, 3 \pmod{4}$

By equation (12) we have

$$x_n \equiv x_1, x_3 \pmod{y_2}$$

In both cases we have $x_n \equiv 0 \pmod{3}$

$$\therefore x_n^2 \equiv 0 \pmod{9}$$

$$\therefore z_n^2 \equiv -12 \pmod{9}$$

which is impossible.

Hence (13) is impossible when $n \equiv 1, 3 \pmod{4}$

c. when $n \equiv 2 \pmod{4}$

by equation (11) we have

$$x_n \equiv \pm x_2 \pmod{x_2}$$

$$\equiv 0 \pmod{17}$$

So, $Z_n^2 \equiv -12 \pmod{17}$

Which is impossible. Since -12 is not a quadratic residue mod 17.

Hence (13) is impossible when $n \equiv 2 \pmod{4}$.

Thus we have shown that the Diophantine equations (5) and (6) cannot hold simultaneously.

Hence the theorem.

We would like to collect some interesting triplets of numbers which have the similar property mentioned in the above theorem.

Sets of numbers	Property
$\{1, 2, 7\}$	Product of any two distinct numbers increased by 2 is a square.
$\{1, 3, 8\}$	Product of any two distinct numbers increased by 1 is a square.
$\{1, 4, 9\}$	Product of any two distinct numbers itself is a square.
$\{1, 5, 10\}$	Product of any two distinct number decreased by 1 is a square.
$\{1, 6, 11\}$	Product of any two distinct number decreased by 2 is a square.
$\{1, 7, 12\}$	Product of any two distinct number decreased by 3 is a square.
$\{1, 8, 13\}$	Product of any two distinct number decreased by 4 is a square.
$\{1, 9, 14\}$	Product of any two distinct number decreased by 5 is a square.
$\{1, 10, 15\}$	Product of any two distinct number decreased by 6 is a square.
$\{1, 11, 16\}$	Product of any two distinct number decreased by 7 is a square.
$\{1, 12, 17\}$	Product of any two distinct number decreased by 8 is a square.
$\{1, 13, 18\}$	Product of any two distinct number decreased by 9 is a square.

Chapter - II

CHAPTER - II

ON THE STRUCTURE OF MAGIC MATRICES

Definition 2.1

A Magic Square, is a double array of elements arranged in the form of a square whose row sums, column sums as well as diagonal sums are all equal.

We shall relax the restriction on the diagonals and consider here all 'squares' whose row and column sums are equal.

Lemma 2.2

A matrix A of order n is a magic matrix with associated sum α if, and only if, the following two conditions are satisfied.

$$AE = \alpha \cdot E$$

$$\text{and } E'A = \alpha \cdot E'$$

Proof :-

$$AE = \alpha \cdot E$$

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \quad n \times n$$

Let E be the column vector all of whose elements are unity.

$$(i.e) \quad E = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}$$

$$A E = \begin{pmatrix} a_{11} + a_{12} + \dots + a_{1n} \\ a_{21} + a_{22} + \dots + a_{2n} \\ \dots \dots \dots \\ a_{n1} + a_{n2} + \dots + a_{nn} \end{pmatrix}_{n \times 1}$$

$$\alpha E = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \vdots \\ \vdots \\ \alpha \end{pmatrix} = \alpha E$$

$$\therefore AE = \alpha E$$

ii. $E'A = \alpha E'$

$$E' = (1 \ 1 \ 1 \ \dots \ 1)_{1 \times n}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}_{n \times n}$$

$$E'A = (a_{11} + a_{21} + \dots + a_{n1} \quad a_{12} + a_{22} + \dots + a_{n2} \quad \dots \quad a_{n1} + a_{n2} + \dots + a_{nn})$$

(Since Matrix A of order 'n' is magic matrix with associative sum α).

$$\therefore E'A = (\alpha \ \alpha \ \dots \ \alpha)$$

$$\alpha E' = (1 \ 1 \ \dots \ 1) = (\alpha \ \alpha \ \dots \ \alpha) = E'A$$

$$\therefore E'A = \alpha E'$$

Lemma 2.3

If A and B are magic matrices with associated sums α and β , then their sum $A + B$ and product AB are magic matrices with associated sums $\alpha + \beta$ and $\alpha\beta$ respectively.

(i.e) To prove $(A+B)E = (\alpha + \beta)E$,

$$E'(A+B) = (\alpha + \beta)E'$$

and $(AB)E = (\alpha\beta)E$,

$$E'(AB) = (\alpha\beta)E'$$

Proof :-

$$(A+B)E = AE + BE$$

$$= \alpha E + \beta E \text{ (since by lemma 2.2)}$$

$$= (\alpha + \beta)E$$

$$E'(A+B) = E'A + E'B$$

$$= \alpha E' + \beta E' \text{ (since by lemma 2.2)}$$

$$= (\alpha + \beta)E'$$

Similarly $(AB)E = A(BE)$

$$= A(\beta E)$$

$$= A(E\beta)$$

$$\begin{aligned}
 &= (AE) \beta \\
 &= (\alpha E) \beta \\
 &= (\alpha \beta) E
 \end{aligned}$$

$$\therefore (AB)E = (\alpha \beta) E$$

$$\begin{aligned}
 E'(AB) &= (E'A)B \\
 &= (\alpha E')B \\
 &= \alpha (E'B) \\
 &= \alpha (\beta E') \\
 &= (\alpha \beta) E'
 \end{aligned}$$

$$\therefore E'(AB) = (\alpha \beta) E'$$

Result 2.4

By taking $AB =$ unit matrix I , the inverse of A (when it exists) is also a magic matrix with associated sum α^{-1} .

Proof :

$$\text{Take } (AB)E = (\alpha \beta) E$$

$$IE = (\alpha \beta) E$$

$$E = (\alpha \beta) E$$

$$\therefore \alpha \beta = 1 \implies \beta = \alpha^{-1}$$

\therefore The inverse of A is also a magic matrix with associated sum α^{-1}

To prove the theorem connected with magic matrices we need the following definitions and results in Algebra.

Definition 2.5

A ring is a set A with two binary operations defined by $+$ and \cdot satisfying the following axioms.

R_1 : $(A, +)$ is abelian group

R_2 : The operation \cdot is associative

R_3 : $a, b, c \in A$ such that

$$a \cdot (b+c) = a \cdot b + a \cdot c \text{ (left distributive law)}$$

$$(b+c) \cdot a = b \cdot a + c \cdot a \text{ (Right distributive law)}$$

Definition 2.6

A homomorphism from a ring A to a ring B is a map of $f : A \rightarrow B$ such that $a, b \in A$.

1. $f(a+b) = f(a) + f(b)$ and
2. $f(ab) = f(a) f(b)$ for every $a, b \in A$.

Definition 2.7

A nonempty subset U of R is said to be a (two - sided) ideal of R if

1. U is a subgroup of R under addition
2. For every $u \in U$ and $r \in R$, both Ur and rU in U .

Definition 2.8

An associative ring A is called an algebra over F if A is a vector space over F such that for all $a, b \in A$ and $\alpha \in F$,

$$\alpha(ab) = (\alpha a)b = a(\alpha b).$$

Definition 2.9

A field F is said to be of Characteristic 0 if $ma \neq 0$ for $a \neq 0$ in F and $m > 0$, an integer.

Results 2.10

1. Let R, R' be rings and ϕ a homomorphism of R onto R' with Kernel U . Then R' is isomorphic to R/U .

2. If R is a commutative ring with element and M is an ideal of R , then M is a maximal ideal of R iff R/M is a field.

Remark 2.11

The magic matrices forms a ring M with unity, the zero and unity elements of ring being the null and unit matrices.

Let A be a magic matrix with associated sum α .

Consider $\phi : A \rightarrow \alpha, \alpha \in R$

To prove this mapping is a homomorphism such that

$$\begin{aligned}\phi(A) &= \alpha, \quad \phi(B) = \beta \\ \phi(A+B) &= \alpha + \beta \\ &= \phi(A) + \phi(B). \\ \phi(AB) &= \alpha\beta \\ &= \phi(A) \cdot \phi(B)\end{aligned}$$

Now define a mapping $f : K \rightarrow M$

$$\text{Let } K = \{A / \phi(A) = 0\}$$

To prove kernel ' K ' is an ideal.

(i.e) To prove that $f: K \rightarrow M$ such that forevery $A \in k$ and $I \in M$ both AI and IA in K .

By known results 2.10 from these we get a homomorphism between rings. This implies that M/K is isomorphic to R . But since R is a field, this gives that the residue class ring M/K is a field and hence K is a maximal ideal in M .

Theorem 2.12

The aggregate of magic matrices with associated sum zero constitutes a maximal ideal in M .

Proof :-

To prove $f : K \rightarrow M$ such that for every $A \in K$ and $B \in M$ both AB and BA in K .

Consider $B \in M$ then the associated sum of B is α and the associated sum of A is zero.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}_{n \times n}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}_{n \times n}$$

$$\begin{array}{l}
 \text{AB} = \left(\begin{array}{cccc}
 a_{11} b_{11} + a_{12} b_{21} + \dots + a_{1n} b_{n1} & a_{11} b_{12} + a_{12} b_{22} + \dots + a_{1n} b_{n2} & \dots & \\
 \dots a_{11} b_{1n} + a_{12} b_{2n} + \dots + a_{1n} b_{nn} & & & \\
 a_{21} b_{11} + a_{22} b_{21} + \dots + a_{2n} b_{n1} & a_{21} b_{12} + a_{22} b_{22} + \dots + a_{2n} b_{n2} & \dots & \\
 \dots a_{21} b_{1n} + a_{22} b_{2n} + \dots + a_{2n} b_{nn} & & & \\
 \dots & \dots & \dots & \\
 \dots & \dots & \dots & \\
 a_{n1} b_{11} + a_{n2} b_{21} + \dots + a_{nn} b_{n1} & a_{n1} b_{12} + a_{n2} b_{22} + \dots + a_{nn} b_{n2} & \dots & \\
 \dots a_{n1} b_{1n} + a_{n2} b_{2n} + \dots + a_{nn} b_{nn} & & &
 \end{array} \right)
 \end{array}$$

Consider the first row of the above matrix.

$$\begin{aligned}
 &= a_{11} (b_{11} + b_{12} + \dots + b_{1n}) + a_{12} (b_{21} + b_{22} + \dots + b_{2n}) + \dots \\
 &\quad \dots + a_{nn} (b_{n1} + b_{n2} + \dots + b_{nn}) \\
 &= a_{11} (\alpha) + a_{12} (\alpha) + \dots + a_{nn} (\alpha) \\
 &= \alpha (a_{11} + a_{12} + \dots + a_{nn}) \\
 &= \alpha \cdot 0 = 0.
 \end{aligned}$$

Consider the last row

$$\begin{aligned}
 &= a_{n1} (b_{11} + b_{12} + \dots + b_{1n}) + a_{n2} (b_{21} + b_{22} + \dots + b_{2n}) + \dots \\
 &\quad \dots + a_{nn} (b_{n1} + b_{n2} + \dots + b_{nn}) \\
 &= a_{n1} (\alpha) + a_{n2} (\alpha) + \dots + a_{nn} (\alpha) \\
 &= \alpha (a_{n1} + a_{n2} + \dots + a_{nn}) \\
 &= \alpha \cdot (0) = 0.
 \end{aligned}$$

$$(i.e) \phi(AB) = 0$$

(i.e) the associated sum of AB is zero

$$\therefore AB \in K.$$

Similarly to prove that $BA = 0$

$$\therefore AB \text{ and } BA \in K$$

Theorem 2.13

The matrix complex $N = \{\lambda U\}$, $\lambda \in R$ forms a two sided ideal in M .

Proof :-

Consider the matrix complex N .

$$N : \{\lambda U\}, \lambda \in R$$

If $\beta, \alpha \in R$, $\beta U - \alpha U = (\beta - \alpha) U \in N$

Also $A \in M$ with associated sum α .

To prove that it is an ideal

Define $f : N \rightarrow M$ such that $\lambda U \in N$ and $A \in M$ and P.T $\lambda UA \in N$.

$$A(\lambda U) = \lambda(AU) = \lambda(AU)$$

$$\begin{aligned} (AU) &= \lambda \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & 1 & 1 & 1 & \dots & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & | & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & | & 1 & 1 & 1 & \dots & 1 \end{pmatrix} \\ &= \lambda \begin{pmatrix} \alpha & \alpha & \alpha & \dots & \alpha \\ \alpha & \alpha & \alpha & \dots & \alpha \\ \dots & \dots & \dots & \dots & \dots \\ \alpha & \alpha & \alpha & \dots & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \lambda\alpha & \lambda\alpha & \dots & \lambda\alpha \\ \lambda\alpha & \lambda\alpha & \dots & \lambda\alpha \\ \dots & \dots & \dots & \dots \\ \lambda\alpha & \lambda\alpha & \dots & \lambda\alpha \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= (\lambda\alpha E, \lambda\alpha E, \dots, \lambda\alpha E) \\
&= \lambda\alpha(E, E, \dots, E) \\
&= \lambda\alpha U \in N, \lambda\alpha \in R
\end{aligned}$$

$\therefore N = \{\lambda U\}, \lambda \in R$ forms a two sided ideal in M .

It is infact a two sided ideal $AU = UA = \alpha U$

\therefore The matrix complex $N = \{\lambda N\}, \lambda \in R$ forms a two sided ideal in M .

Definition : 2.14

By a permutation matrix, we mean a matrix which has unity in each row and column only once while the other elements of the matrix are all zero.

Note

It is obvious that there are $n!$ permutation matrices of a given order n . Moreover the product of any two permutation matrices is a permutation matrix and likewise the inverse of a permutation matrix is again a permutation matrix.

Theorem 2.15

Any magic matrix as a linear combination of the permutation matrices.

Proof :-

Consider the $n!$ permutation matrices of the given order

n . Among these $n!$ permutation matrices, there will be $(n-1)!$ permutations which have 1 in the i -th row or j -th column and among these $(n-2)!$ permutations will have 1 in any place other than that in the i -th row or j -th column. Let us sum these $(n-1)!$ matrices. This matrix sum will then have $(n-1)!$ in the (i,j) -th place and zero in the remaining positions of the i -th row and j -th column while $(n-2)!$ in all the rest of the places.

Removing the factor $(n-2)!$ we call the resulting matrix B_{ij} . Thus B_{ij} is a matrix which has $(n-1)$ in its (i,j) -th place, zero in the remaining positions of the i -th row and j -th column and unity elsewhere and therefore it is a magic matrix with associated sum $(n-1)$. We have corresponding to each of n^2 positions n^2 matrices, B_{ij} , permutation matrices.

Let R , as before, stand for the scalar field of zero characteristic and let the elements of the matrices of the ring M be drawn from this field. By this assumption, it follows readily that the scalar multiplication of the matrices of M by the elements of R is well defined. We shall now examine the general structure of the matrix ring M .

We write the n^2 matrices B_{ij} in the form of a double array thus

$$\Omega : \begin{matrix} B_{11} & B_{12} & \dots & B_{1s} & \dots & B_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ B_{r1} & B_{r2} & \dots & B_{rs} & \dots & B_{rn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{ns} & \dots & B_{nn} \end{matrix}$$

Now select a paper set of base matrices and prove that every magic matrix is expressible as a linear combination of these matrices. Suppose $A = (a_{ij})$ is a magic matrix with associated sum α . Let us denote by D_{ij} , the sum of all the elements of the matrix - minor of a_{ij} in A .

Then

$$D_{ij} = n\alpha - (2\alpha - a_{ij}) = (n-2)\alpha + a_{ij} \quad \text{----- (1)}$$

Consider the sum $\sum a_{ij} B_{ij}$ consisting of n^2 terms corresponding to the n^2 elements of A and Ω . Then, the (i,j) -th element in this matrix sum is

$$(n-1)a_{ij} + D_{ij} = na_{ij} + (n-2)\alpha \quad \text{----- (2) (since from (1))}$$

$$\text{Consequently } \sum_{i,j} a_{ij} B_{ij} = nA + (n-2)\alpha U \quad \text{----- (3)}$$

We observe that the matrix sum of each row and column in Ω is $(n-1)U$. Delete the matrices in the r -th row and s -th column of Ω and consider the remaining $(n-1)^2$ matrices. But the sum of these $(n-1)^2$ matrices is not a multiple of U . Consequently we take in addition to these $(n-1)^2$ matrices of Ω , the matrix U also and (3) now ensures that every magic matrix such as A can

be expressed as a linear combination of these $(n-1)^2+1 = n^2-2n+2$ matrices composed of U and the $(n-1)^2$ matrices obtained by suppressing the matrices in the r -th row and s -th column of Ω .

Moreover, as the matrices in Ω are all linear combination of the permutation matrices, we have, in effect expressed any magic matrix as a linear combination of the permutation matrices.

In fact, it can be verified that if

$$\begin{aligned} w_{ij} &= a_{ij} + a_{rs} - a_{is} - a_{rj} \quad i \neq r, j \neq s \\ &= n\alpha - n(n-1) a_{rs} \end{aligned} \quad (4)$$

then
$$\sum_{\substack{i \neq r \\ j \neq s}} w_{ij} B_{ij} + \lambda U = nA \quad (5)$$

and therefore, the compounding coefficients in the linear sum for A are explicitly expressed by (4).

Theorem 2.16

The ring M of magic matrices of order n defined over a scalar field R (of zero characteristic) is a unitary matrix algebra over R of order $(n^2 - 2n + 2)$.

Proof :-

Now we prove that the (n^2-2n+2) base elements that have chosen a linearly independent set. Consider the sum consisting of (n^2-2n+2) terms

$$\sum_{\substack{i \neq r \\ j \neq s}} \lambda_{ij} B_{ij} + \mu U \quad (1)$$

In order to introduce symmetry, we now take all n^2 λ 's instead of $(n-1)^2$ λ 's only and we then have the double array.

$$\begin{matrix}
 & \lambda_{11} & \lambda_{12} & \dots & \lambda_{15} & \dots & \lambda_{1n} \\
 & \dots & \dots & \dots & \dots & \dots & \dots \\
 A = & \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{r5} & \dots & \lambda_{rn} \\
 & \dots & \dots & \dots & \dots & \dots & \dots \\
 & \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{n5} & \dots & \lambda_{nn}
 \end{matrix}$$

Now we take all the r -th row and j -th column of A to be zero. The introduction of these λ 's will, in no way, affect the sum in (1). Consequently, the sum in (1) can be rewritten without any restriction, as

$$\sum \lambda_{ij} B_{ij} + \mu U \text{ ----- (2)}$$

Now, in order to prove that the base elements are linearly independent, we must show that if we equate the matrix sum in (2) to zero, then all λ_{ij} and μ should be zero. Firstly we note that the sum (2), by virtue of lemma 2.3, is a magic matrix with associated sum $(n-1) \sum \lambda_{ij} + n\mu$ and this must be zero. This gives

$$\sum \lambda_{ij} = -n\mu / (n-1) \text{ ----- (3)}$$

Further, the (i,j) -th element of the matrix sum in (2) is $(n-1) \lambda_{ij} + D_{ij} + \mu$ ----- (4)

We know D_{ij} is the sum of all the elements in the minors of λ_{ij} in Λ . Now, if R_i and C_j denote the sum of the elements in the i -th row and j -th column of A respectively,

Then we have

$$\begin{aligned}
 (n-1) \lambda_{ij} + D_{ij} + \mu &= (n-1) \lambda_{ij} + \{ \sum \lambda_{ij} - (R_i + C_j - \lambda_{ij}) + \mu \} \\
 &= n \lambda_{ij} - (R_i + C_j) + \lambda_{ij} + \mu \\
 &= n \lambda_{ij} - (R_i + C_j) - \mu / (n-1) \text{ ----(5)}
 \end{aligned}$$

from (4)

In the matrix sum in (2) is the null matrix, then the expression in (5) is zero for all i and j . In particular choosing $i=r$ and $j=s$, we get $\mu = 0$. Similarly putting $j=s$ and $i=r$ separately we find that $R_i=0$ and $C_j=0$ for all i and j .

Hence $\lambda_{ij} = 0$ for all i and j . Thus it is established that any magic matrix A is a linear combination of the $(n^2 - 2n - 2)$ linearly independent magic matrices comprising U and $(n-1)^2$ matrices chosen out the n^2 matrices in Ω by suppressing the matrices in any one row and any one column.

From these considerations, and from our mode of definition of scalar multiplication in M , we conclude that M , regarded as a module forms a vector spaces of dimension $(n^2 - 2n + 2)$ over R . Moreover, since M is a ring and the associative law holds good for general matrix products, it follows that M is a unitary matrix algebra over R of order $(n^2 - 2n + 2)$.

∴ The ring M of magic matrices of order n defined over a scalar field R is a unitary matrix algebra over R of order $(n^2 - 2n + 2)$.

Chapter - III

CHAPTER - III

THE DENSITY OF SEQUENCES OF INTEGERS

Two common types of density are Asymptotic density and Schnirelman density. Density will be define on a set A of the distinct positive integers. The elements of A will be arranged in a sequence according to size $a_1 < a_2 < a_3 < \dots$, and we will also denote A by $\{a_i\}$. We will use both the terms set and sequence to discribe A . The set A may be infinite or finite. If A is empty then it will be denoted by 0 .

The number of positive integers in a set A that are less than or equal to x is denotedd by $A(x)$. For example, if A consists of the even integers $2, 4, 6, \dots$, then $A(1) = 0$, $A(2) = 1$, $A(6) = 3$, $A(7) = 3$, $A(15/2) = 3$. Infact $A(x) = (X/2)$ if $x \geq 0$. On the otherhand, for any set $A = \{a_i\}$ we have $A(a_j) = j$.

Definition 3.1.

The asymptotic density of a set A is

$$\delta_1(A) = \liminf_{n \rightarrow \infty} \frac{A(n)}{n}$$

In case the sequence $A(n)/n$ has a limit, we say that A has a natural density, $\delta(A)$. Thus $\delta(A) = \delta_1(A) = \lim_{n \rightarrow \infty} \frac{A(n)}{n}$

if A has a natural density. If A is a finite sequence, it is clear that $\delta(A) = 0$.

Example 3.2

Set	Natural Density Value
a. the set of even positive integers	1/2
b. the set of odd positive integers	1/2
c. the positive multiple of 3	1/3
d. the positive integers of the form $4K+2$	1/4
e. all the positive integers a satisfying $a \equiv b \pmod{m}$ where b and $m > 1$ are fixed	1/m
f. the set of primes	0
g. the set $\{ar^n\}$ with $n=1,2,3 \dots$ and fixed $a \geq 1$, fixed $r > 1$.	0
h. the set of all perfect square	0
i. the set of all positive cubes	0
j. the set of all positive power, that is, all numbers of the form a^n with $a \geq 1$, $n \geq 2$.	0

Definition 3.3

The Schnirelman Density $d(A)$ of a set A of non-negative integers is $d(A) = \inf_{n \geq 1} \frac{A(n)}{n}$, where $A(n)$ is the number of positive integers $\leq n$ in the set A . Here in this chapter we discuss asymptotic density. It is interesting to note that the following sets has a natural density.

Theorem 3.4

If A is an infinite sequence, then

$$\delta_1(A) = \liminf_{n \rightarrow \infty} \frac{n}{a_n}$$

If $\delta(A)$ exists, then $\delta(A) = \lim_{n \rightarrow \infty} n/a_n$

Proof :-

The sequence k/a_k is a subsequence of $A(n)/n$ and hence

$$\liminf_{n \rightarrow \infty} \frac{A(n)}{n} \leq \liminf_{k \rightarrow \infty} \frac{k}{a_k}.$$

If n is any integer $\geq a_1$ and a_k is the smallest integer in A that exceeds n , then $a_{k-1} \leq n \leq a_k$ and

$$\text{and } \frac{k}{a_k} - \frac{A(n)}{n} = \frac{k}{a_k} - \frac{k-1}{n} < \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$$

It follows that $\frac{k}{a_k} < \frac{A(n)}{n} + \frac{1}{n}$

$$\liminf_{k \rightarrow \infty} \frac{k}{a_k} \leq \liminf_{n \rightarrow \infty} \frac{A(n)}{n} + \frac{1}{n}$$

and so the theorem is proved.

Definition 3.5

An integer is square-free if it is divisible by no perfect square $a^2 > 1$. We will prove that the set of square-free integers has natural density $6/\pi^2$.

Lemma 3.6

The function $\tau(n)$, representing the number of positive divisors of n , satisfies the inequality $\tau(n) \leq 2\sqrt{n}$ for $n \geq 1$.

Proof :-

Consider the positive divisors d , of n . Corresponding to each $d \geq \sqrt{n}$ is the distinct divisor $d' = n/d$ and $1 \leq d' \leq \sqrt{n}$. Therefore $\tau(n)$ cannot exceed twice the number of divisor d such that $1 \leq d \leq \sqrt{n}$. Clearly the number of these d cannot exceed \sqrt{n} , and we have $\tau(n) \leq 2\sqrt{n}$

Theorem 3.7

$$\text{We have } \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = 1$$

Proof :-

We can write

$$P_m = \sum_{n=1}^m \frac{\mu(n)}{n^2} \sum_{n=1}^m \frac{1}{n^2} = R_m S_m$$

We consider first a fixed integer $j \leq m$. If d is any divisor of j , say $j = dq$, then $\mu(d)/d^2$ is a term of R_m and $1/q^2$ is a term of S_m and $\mu(d)/(dq)^2 = \mu(d)/j^2$ occurs in P_m .

Then $1/j^2$ occurs in P_m with coefficient

$$\sum_{d|j} \mu(d) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

In case $j > m$ the product $\mu(d)/(dq)^2$ may appear in p_m for some divisor d of j . Therefore we can write

$$P_m = \sum_{j=1}^m \left(\sum_{d|j} \mu(d) \right) \frac{1}{j^2} + \sum_{j=m+1}^{m^2} \left(\sum_{d|j} \mu(d) \right) \frac{1}{j^2}$$

Where \sum' denotes a sum over the appropriate divisor d of j .

Thus we have

$$P_m^{-1} = \sum_{j=m+1}^{m^2} \sum'_{d|j} \frac{\mu(d)}{j^2} = \sum_{j=m+1}^{m^2} \frac{c_j}{j^2},$$

$$c_j = \sum'_{d|j} \mu(d)$$

and using lemma 3.6, we observe that

$$|c_j| \leq \sum'_{d|j} |\mu(d)| \leq \sum'_{d|j} 1 \leq \sum_{d|j} 1 = \tau(j) \leq 2 \leq \sqrt{j}$$

Now we have

$$|P_m^{-1}| \leq \sum_{j=m+1}^m \frac{|c_j|}{j^2} \leq \sum_{j=m+1}^{m^2} \frac{2\sqrt{j}}{j^2}$$

$$= \sum_{j=m+1}^{m^2} \frac{2}{j^{3/2}}$$

Applying Cauchy's condition to the convergent series $\sum 2/j^{3/2}$ we see that $|P_m^{-1}|$ tends to zero as m tends to infinity.

This establishes the theorem.

Corollary 3.8

We have
$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2}$$

Proof :-

It is well known from elementary results in the theory of fourier series that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

For instance, it follows by setting $x = 0$ in the result.

$$\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

which is valid for x in the range $0 \leq x \leq 2\pi$.

Theorem 3.9

The set of square-free integers has natural density $6/\pi^2$.

Proof :-

Let S denote the sequence $1, 2, 3, 5, 6, 7, 10 \dots$ of square-free integers. For any positive integer n let p_1, p_2, \dots, p_r denote all the primes such that $p_j^2 \leq n$

We first wish to prove

$$S(n) = \sum (-1)^{\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_r} \left[\frac{n}{(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})^2} \right] \text{----- (1)}$$

Where the sum ranges over the 2^r terms obtained by setting each $\alpha_j = 0$ or 1 .

Now (n/t^2) is the number of integers $\leq n$ that are divisible by t^2 , and we can interpret each term on the right side of (1) as a count of those integers $m \leq n$ that are divisible by $(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})^2$

If m is square-free, $1 \leq m \leq n$, then m is counted by the term $[n]$ and by no other terms. If $1 \leq m \leq n$ and m is divisible by p_i^2 but by no other p_j^2 then m is counted by the terms $[n]$ and $-[n/p_i^2]$, once positively and once negatively, a net count of $1 - 1 = 0$. To take the general case, consider an integer m , $1 \leq m \leq n$, that is divisible by $p_{j_1}^2, p_{j_2}^2, \dots, p_{j_s}^2$, $s \geq 1$, but not by any of the other p_j^2 . Then m is counted by the terms

$$(-1)^{\alpha_{j_1} + \alpha_{j_2} + \dots + \alpha_{j_s}} \left[\frac{n}{(p_{j_1}^{\alpha_{j_1}} p_{j_2}^{\alpha_{j_2}} \dots p_{j_s}^{\alpha_{j_s}})^2} \right] \text{----- (2)}$$

The net count for this m is thus

$$\sum (-1)^{\alpha_j} + \alpha_{j^2} + \dots + \alpha_{j^s}$$

$$= \sum (-1)^{\alpha_j} \sum (-1)^{\alpha_{j^2}} \dots \sum (-1)^{\alpha_{j^s}} = 0$$

since $\sum (-1)^{\alpha_j} = 1 + (-1) = 0$

This establishes (2).

Next we note that (2) can be written as

$$S(n) = \sum_{d | p_1 p_2 \dots p_r} \mu(d) \left[\frac{n}{d^2} \right] \text{ ----- (3)}$$

In this sum any term which $d^2 > n$ has the factor $[n/d^2] = 0$, and we can restrict d in (3) to be such that $d^2 \leq n$.

In fact, we have

$$(n) = \sum_{d^2 \leq n} \mu(d) \left[\frac{n}{d^2} \right] \text{ ----- (4)}$$

where d ranges over all positive integers such that $d^2 \leq n$. Since any term in (4) that is not in (3) will belong to a value of d that is not square-free.

In this case the term has the factor $\mu(d) = 0$.

Using corollary 3.8 and (4) we find

$$S(n) - \frac{6}{\pi^2} = \sum_{d^2 \leq n} \mu(d) \left(\left[\frac{n}{d^2} \right] - \frac{n}{d^2} \right) - \sum_{d^2 > n} \mu(d) \frac{n}{d^2}$$

and hence

$$\left| \frac{S(n)}{n} - \frac{6}{\pi^2} \right| \leq \frac{1}{n} \sum_{d^2 \leq n} \left| \left[\frac{n}{d^2} \right] - \frac{n}{d^2} \right| + \sum_{d^2 > n} \frac{1}{d^2} \quad \text{-----(5)}$$

But $\frac{1}{n} \sum_{d^2 \leq n} \left| \left[\frac{n}{d^2} \right] - \frac{n}{d^2} \right| \leq \frac{1}{n} \sum_{d^2 \leq n} 1 \leq \frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$

and $\sum_{d^2 > n} \frac{1}{d^2} \rightarrow 0$ as $n \rightarrow \infty$

Since $\sum \frac{1}{d^2}$ converges. Therefore the right side of (5) tends to zero, we have $S(n)/n \rightarrow \frac{6}{\pi^2}$ as $n \rightarrow \infty$.

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