5

$\psi^*\alpha$-Homeomorphisms in Topological Spaces

5.1 Introduction

Several weaker and stronger forms of closed maps in topological spaces were introduced and investigated by many topologists. Malghan (1982) introduced the concept of generalized closed maps in topological spaces. Ramya and Parvathi (2013) introduced the concept of ψg-closed maps in topological spaces. The notion of homeomorphism plays an important role in topology. A homeomorphism between two topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. Maki et al. (1991a) introduced g-homeomorphisms and gc-homeomorphisms.

In this chapter a new class of closed maps called $\psi^*\alpha$-closed maps is introduced and dependency and independency relations with various closed maps are obtained. Further quasi $\psi^*\alpha$-open maps, quasi $\psi^*\alpha$-closed maps and $\psi^*\alpha$-quotient maps are introduced and studied their properties and characterizations. Using $\psi^*\alpha$-open maps a new class of maps called $\psi^*\alpha$-homeomorphisms is introduced and their properties are discussed. $\psi^*\alpha\mathcal{C}$-homeomorphisms using $\psi^*\alpha$-irresolute maps is defined and proved that $\psi^*\alpha\mathcal{C}$-homeomorphisms form a group under the operation composition of maps. Also $\psi^*\alpha$-compact spaces and $\psi^*\alpha$-connected spaces are defined and their properties are discussed.

5.2 $\psi^*\alpha$-closed maps

In this section, $\psi^*\alpha$-closed maps and $\psi^*\alpha$-open maps are introduced and some of their properties are discussed. It is shown that the composition of two $\psi^*\alpha$-closed ($\psi^*\alpha$-open) maps need not be a $\psi^*\alpha$-closed ($\psi^*\alpha$-open) map.

Definition 5.2.1 A map $f : (X, \tau) \to (Y, \sigma)$ is called **$\psi^*\alpha$-closed** if f(A) is $\psi^*\alpha$-closed in (Y, σ) for each closed set A in (X, τ).

Example 5.2.2 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = { ϕ, {a}, {b}, {a, b}, {a, c}, Y}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a $\psi^*\alpha$-closed map.

Proposition 5.2.3

(i) Every closed map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map.

(ii) Every α-closed map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map.

(iii) Every regular closed map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map.

Proof: Since every closed set, α-closed set and regular closed set is a $\psi^*\alpha$-closed, the result follows.

The converse of the statements in the above proposition need not be true as seen from the following example.

Example 5.2.4 Let X = Y = {a, b, c}, τ = {ϕ, {c}, X} and σ = {ϕ, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is a $\psi^*\alpha$-closed map but not a closed map, not an α-closed map and not a regular closed map, since for the closed set {a, b} in (X, τ), f({a, b}) = {a, c} is not closed, not α-closed and not regular closed in (Y, σ).

Proposition 5.2.5

(i) Every $\psi^*\alpha$-closed map f : (X, τ) → (Y, σ) is a $\tilde{g}_\alpha$-closed map.

(ii) Every $\psi^*\alpha$-closed map f : (X, τ) → (Y, σ) is a gα-closed map.

(iii) Every $\psi^*\alpha$-closed map f : (X, τ) → (Y, σ) is a αg-closed map.

(iv) Every $\psi^*\alpha$-closed map f : (X, τ) → (Y, σ) is a ψ-closed map.

(v) Every $\psi^*\alpha$-closed map f : (X, τ) → (Y, σ) is a ψg-closed map.

Proof: Since every $\psi^*\alpha$-closed set is $\tilde{g}_\alpha$-closed, gα-closed, αg-closed, ψ-closed and ψg -closed, the result follows.

The converse of the statements in the above proposition need not be true as seen from the following examples.

Example 5.2.6 Let X = Y = {a, b, c, d}, τ = {ϕ, {a}, X} and σ = {ϕ, {d}, {a, b}, {a, b, d}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = c, f(c) = a, f(d)=d. Then f is a $\tilde{g}_\alpha$-closed map but not a $\psi^*\alpha$-closed map, since for the closed set {b, c, d} in (X, τ), f({b, c, d}) = {a, c, d} is not $\psi^*\alpha$-closed in (Y, σ).

Example 5.2.7 Let X = Y = {a, b, c}, τ = {ϕ, {b}, {c}, {b, c}, X} and σ = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is a gα closed map and αg-closed map but not a $\psi^*\alpha$-closed map in (Y, σ), since {a, c} is closed in (X, τ) but f({a, c}) = {a, b} is not $\psi^*\alpha$-closed in (Y, σ).

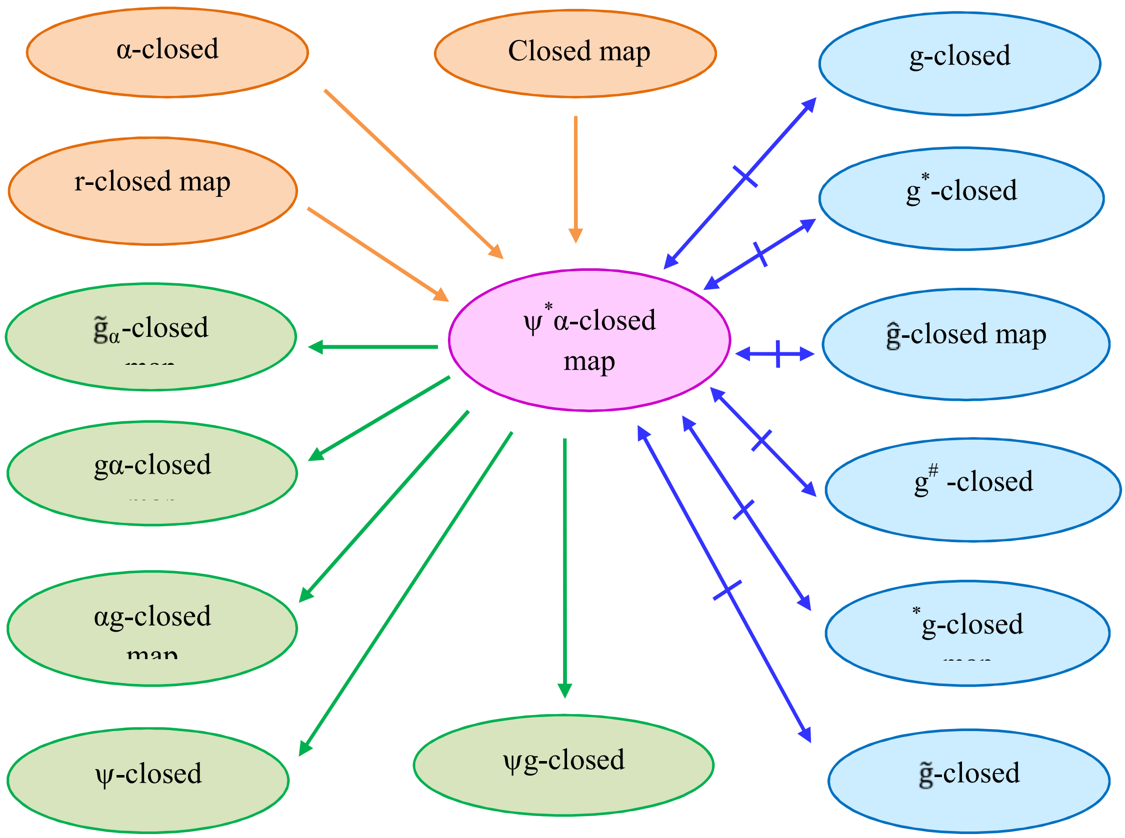
Example 5.2.8 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a}, {b}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is a ψ-closed map and a ψg-closed map but not a $\psi^*\alpha$-closed map in (Y, σ), since {c} is closed in (X, τ) but f({ c}) = {a} is not $\psi^*\alpha$-closed in (Y, σ).

Remark 5.2.9 The following examples show that $\psi^*\alpha$-closed map is independent from a g-closed map, a g^*-closed map, a $\hat{g}$-closed map, a $g^\#$-closed map, a $^*g$-closed map and a $\tilde{g}$-closed map.

Example 5.2.10 Let X = Y = {a, b, c}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b, Then f is a $\psi^*\alpha$-closed map but not a g-closed, g^*-closed, $\hat{g}$-closed, $g^\#$-closed, $^*g$-closed and $\tilde{g}$-closed maps, since for the closed set {c} in (X, τ), f({c}) = {b} is not g-closed, g^*-closed, $\hat{g}$-closed, $g^\#$-closed, $^*g$-closed and $\tilde{g}$-closed in (Y, σ).

Example 5.2.11 Let X = Y = {a, b, c, d}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {d}, {a, b}, {a, b, d} Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = d, f(c) = c, f(d)=b. Then f is a g-closed, g^*-closed, $\hat{g}$-closed, $g^\#$-closed, $^*g$-closed and $\tilde{g}$-closed map but not a $\psi^*\alpha$-closed map, since for the closed set {c, d} in (X, τ), f({c, d}) = {b, c} is g-closed, g^*-closed, $\hat{g}$-closed, $g^\#$-closed, $^*g$-closed and $\tilde{g}$-closed in (X, τ) but not $\psi^*\alpha$-closed in (Y, σ).

Remark 5.2.12 The above observations are depicted by the following diagram.



Properties of $\psi^*\alpha$-closed maps

Proposition 5.2.13 A map f : (X, τ) → (Y, σ) is $\psi^*\alpha$-closed if and only if $\psi^*\alpha cl(f(U)) \subseteq f(cl(U))$ for every subset U of (X, τ).

Proof: Let U be any subset of (X, τ). Since cl(U) is closed in (X, τ) and f is $\psi^*\alpha$-closed, f(cl(U)) is $\psi^*\alpha$-closed in (Y, σ). Since $f(U) \subseteq f(cl(U))$, $\psi^*\alpha cl(f(U)) \subseteq \psi^*\alpha cl(f(cl(U))) = f(cl(U))$. Therefore $\psi^*\alpha cl(f(U)) \subseteq f(cl(U))$.

Conversely, let U be any closed set in (X, τ). Then U = cl(U) and so f(U) = f(cl(U)). By hypothesis, $\psi^*\alpha cl(f(U)) \subseteq f(cl(U)) = f(U)$. Hence $\psi^*\alpha cl(f(U)) = f(U)$. Therefore f(U) is $\psi^*\alpha$-closed set in (Y, σ). Hence f is a $\psi^*\alpha$-closed map.

Theorem 5.2.14 A map f : (X, τ) → (Y, σ) is $\psi^*\alpha$-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there exists a $\psi^*\alpha$-open set V of (Y, σ) such that S □ V and $f^{-1}(V)$ □ U.

Proof: Let S be a subset of (Y, σ) and U be a open set in (X, τ) containing $f^{-1}(S)$. As U is open, X – U is closed. Since f is a $\psi^*\alpha$-closed map, f(X – U) is a $\psi^*\alpha$-closed set in (Y, σ). Hence Y – f(X – U) is a $\psi^*\alpha$-open set. Let V = Y – f(X – U). Since $f^{-1}(S)$ □ U, X – U □ X – $f^{-1}(S)$ = $f^{-1}(Y – S)$ implies f(X – U) □ Y – S ie., S □ Y – (f(X – U)) = V. Since V = Y – f(X – U), f(X – U) □ Y – V implies X – U □ $f^{-1}(Y – V)$ = X – $f^{-1}(V)$. Hence $f^{-1}(V)$ □ U.

Conversely, let S be a closed set of (X, τ). Then f^{-1}[Y – f(S)] □ X – S and X – S is open. By hypothesis, there exists a $\psi^*\alpha$-open set V of (Y, σ) such that Y – f(S) □ V and $f^{-1}(V)$ □ X – S which implies S □ X – $f^{-1}(V)$. Hence Y – V □ f(S) □ f[X – $f^{-1}(V)$] □Y – V. Therefore f(S) = Y –V. Since Y – V is $\psi^*\alpha$-closed, f(S) is $\psi^*\alpha$-closed. Hence f is a $\psi^*\alpha$-closed map.

Theorem 5.2.15 If f : (X, τ) → (Y, σ) is a pre α-closed and ψg-irresolute map and A is a $\psi^*\alpha$-closed subset of (X, τ), then f(A) is a $\psi^*\alpha$-closed set in (Y, σ).

Proof: Let U be any ψg-open set in (Y, σ) such that f(A) □ U. Since f is ψg-irresolute, $f^{-1}(U)$ is ψg-open in (X, τ). Since A is a $\psi^*\alpha$-closed subset of (X, τ), αcl(A) □ $f^{-1}(U)$. Since f is pre α-closed and αcl(A) is α-closed in (X, τ), f(αcl(A)) is a α-closed set contained in the ψg-open set U which gives αcl(f(αcl(A))) □ U and so A □ αcl(A). Therefore f(A)) □ f(αcl(A)), αcl(f(A)) □ αcl[f(αcl(A))] □ U. Hence f(A) is a $\psi^*\alpha$-closed set in (Y, σ).

Theorem 5.2.16 If f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map and A is a closed set in (X, τ) then f_A : (A, τ_A) → (Y, σ) is $\psi^*\alpha$-closed.

Proof: Let A be a closed set in (X, τ) and B be a closed set in (A, τ_A). Then B = A ∩ F, for some closed set F in (X, τ) which implies B is closed in (X, τ). Since f is $\psi^*\alpha$-closed, f(B) = (f_A)(B) is $\psi^*\alpha$-closed in (Y, σ). Hence f_A is $\psi^*\alpha$-closed.

Composition of $\psi^*\alpha$-closed maps

Remark 5.2.17 The composition of two $\psi^*\alpha$-closed maps need not be a $\psi^*\alpha$-closed map as seen from the following example.

Example 5.2.18 Let X = Y = Z = {a, b, c}, τ = {ϕ, {b}, {c}, {b, c}, X}, σ = {ϕ, {a, b}, Y} and η = {ϕ, {a}, {b, c}, Z}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b and g : (Y, σ) → (Z, η) be a map defined by g(a) = b, g(b) = c, g(c) = a. Then f and g are $\psi^*\alpha$-closed maps but their composition g∘f : (X, τ) → (Z,η) is not a $\psi^*\alpha$-closed map, since for the closed set {a, c} in (X, τ), (g∘f)({a, c}) = {a, c} in not $\psi^*\alpha$-closed in (Z, η).

Proposition 5.2.19 If f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) are $\psi^*\alpha$-closed maps and (Y, σ) is a $_{\psi^*\alpha}T_c$-space, then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-closed map.

Proof: Let U be any closed set in (X, τ). Then f(U) is $\psi^*\alpha$-closed in (Y, σ) as f is $\psi^*\alpha$-closed. Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space, f(U) is closed in (Y, σ). Since g is $\psi^*\alpha$-closed, (g∘f)(U) = g(f(U)) is $\psi^*\alpha$-closed in (Z, η). Thus g∘f is a $\psi^*\alpha$-closed map.

Corollary 5.2.20 If f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map and g : (Y, σ) → (Z, η) is a pre α-closed and ψg-irresolute map, then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-closed map.

Proof: Let A be any closed set in (X, τ). Then f(A) is $\psi^*\alpha$-closed in (Y, σ), as f is $\psi^*\alpha$-closed. Since g is both pre α-closed and ψg-irresolute, then by **Theorem 5.2.15** g(f(A)) = (g∘f)(A) is $\psi^*\alpha$-closed in (Z, η). Therefore g∘f is a $\psi^*\alpha$-closed map

Proposition 5.2.21 If f : (X, τ) → (Y, σ) is a closed map and g : (Y, σ) → (Z, η) is a $\psi^*\alpha$-closed map, then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-closed map.

Proof: Let B be a closed set in (X, τ). Then f(B) is closed in (Y, σ) as f is closed and (g∘f)(B) = g(f(B)) is $\psi^*\alpha$-closed in (Z, η) as g is $\psi^*\alpha$-closed. Thus g∘f is a $\psi^*\alpha$-closed map.

Remark 5.2.22 If f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-closed map and g : (Y, σ) → (Z, η) is a closed map then their composition g∘f : (X, τ) → (Z, η) need not be a $\psi^*\alpha$-closed map as seen from the following example.

Example 5.2.23 Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {a, b}, X}, σ = {ϕ, {a, b}, Y} and η = {ϕ, {a}, {b, c}, Z}. Let f : (X, τ) → (Y, σ) be the identity map and g : (Y, σ) → (Z, η) be a map defined by g(a) = c, g(b) = b, g(c) = a. Then f is a $\psi^*\alpha$-closed map and g is a closed map but their composition g∘f : (X, τ) → (Z, η) is not a $\psi^*\alpha$-closed map, since for the closed set {b, c} in (X, τ), (g∘f)({b, c}) = {a, b} in not $\psi^*\alpha$-closed in (Z, η).

Proposition 5.2.24 If f : f : (X, τ) → (Y, σ) is a closed map and g : (Y, σ) → (Z, η) is a closed (resp. an α-closed and a regular closed) map, then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-closed map.

Proof: Let A be a closed set in (X, τ). Then f(A) is a closed set in (Y, σ) as f is a closed map and (g∘f)(A) = g(f(A)) is a closed (resp. α-closed and regular closed) set in (Z, η), as g is a closed (resp. α-closed and regular closed) map. Since every closed (resp. α-closed and regular closed) set is a $\psi^*\alpha$-closed set, g∘f is a $\psi^*\alpha$-closed map

Proposition 5.2.25 Let f : (X, τ) → (Y, σ) be any map and g : (Y, σ) → (Z, η) be an injective $\psi^*\alpha$-irresolute map. If g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-closed map, then f is a $\psi^*\alpha$-closed map

Proof: Let A be any closed set in (X, τ). Since g∘f is $\psi^*\alpha$-closed, (g∘f)(A) is $\psi^*\alpha$-closed in (Z, η). Since g is $\psi^*\alpha$-irresolute and injective, g^{-1}(g∘f)(A) = f(A) is a $\psi^*\alpha$-closed set in (Y, σ). Hence f is a $\psi^*\alpha$-closed map.

Proposition 5.2.26 If f : (X, τ) → (Y, σ) is a closed map and g : (Y, σ) → (Z,η) is a $\psi^*\alpha$-closed map, then g∘f : (X, τ) → (Z, η) is a $\tilde{g}_\alpha$-closed (resp. gα-closed, αg-closed, ψ-closed and ψg-closed) map.

Proof: Let A be a closed set in (X, τ). Then f(A) is a closed set in (Y, σ) as f is closed and (g∘f)(A) = g(f(A)) is $\psi^*\alpha$-closed in (Z, η) as g is $\psi^*\alpha$-closed. Since every $\psi^*\alpha$-closed set is $\tilde{g}_\alpha$-closed (resp. gα-closed, αg-closed, ψ-closed and ψg-closed), g∘f is a $\tilde{g}_\alpha$-closed (resp. gα-closed, αg-closed, ψ-closed and ψg-closed) map.

Theorem 5.2.27 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two maps such that $g \circ f : (X, \tau) \to (Z, \eta)$ is a $\psi^*\alpha$-closed map. Then the following statements are true.

(i) If f is a surjective continuous map, then g is a $\psi^*\alpha$-closed map.

(ii) If f is a surjective completely continuous map then g is a $\psi^*\alpha$-closed map.

(iii) If f is a surjective α-continuous (resp. g-continuous, αg-continuous and g^*-continuous) map and (X, τ) is an α-space (resp. $T_{1/2}$ -space, ${}_{\alpha}T_b$-space and $T^*_{1/2}$-space) then g is a $\psi^*\alpha$-closed map.

(iv) If g is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) and injective map, then f is a closed map.

Proof: (i) Let V be any closed set in (Y, σ). Since f is continuous, $f^{1}(V)$ is closed in (X, τ). Since $g \circ f$ is $\psi^*\alpha$-closed and f is surjective, $(g \circ f)(f^{1}(V)) = g(V)$ is $\psi^*\alpha$-closed in (Z, η). Therefore g is a $\psi^*\alpha$-closed map.

(ii) Let V be any closed set in (Y, σ). Since f is completely continuous, $f^{-1}(V)$ is regular closed in (X, τ). Since every regular closed set is closed, $f^{1}(V)$ is closed in (X, τ). Since $g \circ f$ is $\psi^*\alpha$-closed and f is surjective, $(g \circ f)(f^{1}(V)) = g(V)$ is $\psi^*\alpha$-closed in (Z, η). Therefore g is a $\psi^*\alpha$-closed map.

(iii) Let V be any closed set in (Y, σ). Since f is α-continuous (resp. g-continuous, αg-continuous and g^*-continuous) $f^{1}(V)$ is α-closed (resp. g-closed, αg-closed and g^*-closed) in (X, τ). Since (X, τ) is an α-space (resp. $T_{1/2}$-space, ${}_{\alpha}T_b$ - space and $T^*_{1/2}$-space), $f^{1}(V)$ is closed in (X, τ). Since $g \circ f$ is $\psi^*\alpha$-closed and f is surjective, $(g \circ f)(f^{1}(V)) = g(V)$ is $\psi^*\alpha$-closed in (Z, η). Therefore g is a $\psi^*\alpha$-closed map.

(iv) Let U be any closed set in (X, τ). Since $g \circ f$ is $\psi^*\alpha$-closed in (Z, η), $(g \circ f)(U)$ is $\psi^*\alpha$-closed in (Z, η). Since g is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) and injective, $g^{-1}((g \circ f)(U)) = f(U)$ is closed (resp. clopen) in (Y, σ). Therefore f is a closed map

$\psi^*\alpha$-open maps

Definition 5.2.28 A map $f : (X, \tau) \to (Y, \sigma)$ is called **$\psi^*\alpha$-open** if f(A) is $\psi^*\alpha$-open in (Y, σ) for each open set A in (X, τ).

Example 5.2.29 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, {b}, {a, b}, {a, c}, Y}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a $\psi^*\alpha$-open map.

Theorem 5.2.30 Let f : (X, τ) → (Y, σ) be a bijective map. Then the following statements are equivalent.

(a) f is a $\psi^*\alpha$-open map

(b) f is a $\psi^*\alpha$-closed map

(c) f^{-1} : (Y, σ) → (X, τ) is $\psi^*\alpha$-continuous.

Proof: (a) ⇒ (b) Let A be any closed set in (X, τ). Then X – A is open in (X, τ). By (a), f(X – A) = Y – f(A) is $\psi^*\alpha$-open in (Y, σ). Therefore f(A) is $\psi^*\alpha$-closed in (Y, σ) and hence f is $\psi^*\alpha$-closed.

(b) ⇒ (c) Let A be any closed set in (X, τ). Since f is $\psi^*\alpha$-closed, f(A) = $(f^{-1})^{-1}$(A) is $\psi^*\alpha$-closed in (Y, σ). Hence f^{-1} is $\psi^*\alpha$-continuous.

(c) ⇒ (a) Let A be an open set in (X, τ). By (c), f(A) = $(f^{-1})^{-1}$(A) is $\psi^*\alpha$-open in (Y, σ). Hence f is $\psi^*\alpha$-open.

Theorem 5.2.31 A map f : (X, τ) → (Y, σ) is $\psi^*\alpha$-open if and only if f(int(U)) □ $\psi^*\alpha$int(f(U)) for every subset U of (X, τ).

Proof: Let U be any subset of (X, τ). Since int(U) is open in (X, τ) and f is $\psi^*\alpha$-open, f(int(U)) is $\psi^*\alpha$-open in (Y, σ). Since int(U) $\subseteq$ U, $\psi^*\alpha$int(f(int(U))) $\subseteq$ $\psi^*\alpha$int(f(U)). Since f(int(U)) is $\psi^*\alpha$-open in (Y, σ), $\psi^*\alpha$int(f(int(U))) = f(int(U)). Hence f(int(U)) □ $\psi^*\alpha$int(f(U)).

Conversely, let U be an open set in (X, τ). Then U = int(U), f(U) = f(int(U)) $\subseteq$ $\psi^*\alpha$ int(f(U)). Hence f(U) = $\psi^*\alpha$ int(f(U)). Therefore f(U) is a $\psi^*\alpha$-open set in (Y, σ). Thus f is $\psi^*\alpha$-open.

5.3 Quasi $\psi^*\alpha$-open maps and quasi $\psi^*\alpha$-closed maps

In this section, quasi $\psi^*\alpha$-open maps and quasi $\psi^*\alpha$-closed maps are introduced and some of their properties and characterizations are obtained.

Definition 5.3.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **quasi $\psi^*\alpha$-open** if $f(A)$ is open in (Y, σ) for each $\psi^*\alpha$-open set A in (X, τ).

Example 5.3.2 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is quasi $\psi^*\alpha$-open.

Theorem 5.3.3 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi $\psi^*\alpha$-open if and only if for every subset U of (X, τ), $f(\psi^*\alpha int(U)) \subset int(f(U))$.

Proof: Let f be a quasi $\psi^*\alpha$-open map. Let U be a subset of (X, τ). Since $int(U) \subset U$ and $\psi^*\alpha int(U)$ is a $\psi^*\alpha$-open set and $\psi^*\alpha int(U)) \subset U$, $f(\psi^*\alpha int(U)) \subset f(U)$. As f is quasi $\psi^*\alpha$-open, $f(\psi^*\alpha int(U))$ is open and $f(\psi^*\alpha int(U)) = intf(\psi^*\alpha int(U)) \subset int(f(U))$. Therefore $f(\psi^*\alpha int(U)) \subset int(f(U))$.

Conversely, assume that U is a $\psi^*\alpha$-open set in (X, τ).Then $f(U) = f(\psi^*\alpha int(U)) \subset int(f(U))$ but $int(f(U)) \subset f(U)$. Consequently, $f(U) = int(f(U))$ and hence f is quasi $\psi^*\alpha$-open.

Theorem 5.3.4 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi $\psi^*\alpha$-open then $\psi^*\alpha\ int(f^{-1}(G)) \subset f^{-1}(int(G))$, for every subset G of (Y, σ).

Proof: Let G be any arbitrary subset of (Y, σ). Then $\psi^*\alpha int(f^{-1}(G))$ is a $\psi^*\alpha$-open set in (X, τ). Since f is quasi $\psi^*\alpha$-open, $f(\psi^*\alpha int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Thus $\psi^*\alpha\ int(f^{-1}(G)) \subset f^{-1}(int(G))$.

Definition 5.3.5 A subset A is called a **$\psi^*\alpha$-neighbourhood** of a point x of (X, τ) if there exists a $\psi^*\alpha$-open set U such that $x \in U \subset A$.

Theorem 5.3.6 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map. Then the following statements are equivalent:

(i) f is a quasi $\psi^*\alpha$-open map

(ii) For each subset U of (X, τ), $f(\psi^*\alpha int(U)) \subset int(f(U)$

(iii) For each $x \in X$ and each $\psi^*\alpha$-neighbourhood U of x in (X, τ), there exists a neighbourhood V of f(x) in (Y, σ) such that $V \subset f(U)$.

Proof: (i) $\Rightarrow$ (ii) It follows from **Theorem 5.3.3.**

(ii) $\Rightarrow$ (iii) Let $x \in X$ and U be an arbitrary $\psi^*\alpha$ - neighbourhood of x in (X, τ). Then there exists a $\psi^*\alpha$-open set V in (X, τ) such that $x \in V \subset U$. Then by (ii), $f(V) = f(\psi^*\alpha int(V)) \subset int(f(V))$ and hence $f(V) = int(f(V))$. Therefore f(V) is open in (Y, σ) such that $f(x) \in f(V) \subset f(U)$.

(iii) $\Rightarrow$ (i) Let U be an arbitrary $\psi^*\alpha$-open set in (X, τ). Then for each $y \in f(U)$, by (iii) there exists a neighbourhood V_y of y in (Y, σ) such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exists an open set W_y in (Y, σ) such that $y \in W_y \subset V_y$. Thus $f(U) = \cup \{W_y : y \in f(U)\}$ which is an open set in (Y, σ). This implies that f is a quasi $\psi^*\alpha$-open map.

Theorem 5.3.7 A map $f : (X, \tau) \to (Y, \sigma)$ is quasi $\psi^*\alpha$-open if and only if for any subset B of (Y, σ) and for any $\psi^*\alpha$-closed set F in (X, τ) containing $f^{-1}(B)$, there exists a closed set G of (Y, σ) containing B such that $f^{-1}(G) \subset F$.

Proof: Suppose f is a quasi $\psi^*\alpha$-open map. Let $B \subset Y$ and F be a $\psi^*\alpha$-closed set in (X, τ) containing $f^{-1}(B)$. Let $G = Y - [f(X - F)]$. It is clear that $f^{-1}(B) \subset F$ which implies $B \subset G$. Since f is quasi $\psi^*\alpha$-open, G is a closed set of (Y, σ) and $f^{-1}(G) \subset F$.

Conversely, let U be a $\psi^*\alpha$-open set in (X, τ) and let $B = Y - f(U)$. Then $X - U$ is a $\psi^*\alpha$-closed set in (X, τ) containing $f^{-1}(B)$. By hypothesis, there exists a closed set F in (Y, σ) such that $B \subset F$ and $f^{-1}(F) \subset X - U$. Hence $f(U) \subset Y - F$. Now $B \subset F$ implies $Y - F \subset Y - B = f(U)$. Thus $f(U) = Y - F$ is open and hence f is a quasi $\psi^*\alpha$-open map.

Theorem 5.3.8 A map $f : (X, \tau) \to (Y, \sigma)$ is quasi $\psi^*\alpha$-open if and only if $f^{-1}(cl(B)) \subset \psi^*\alpha cl(f^{-1}(B))$, for every subset B of (Y, σ).

Proof: Suppose f is a quasi $\psi^*\alpha$-open map. For any subset B of (Y, σ), $f^{-1}(B) \subset \psi^*\alpha cl(f^{-1}(B))$. Therefore by **Theorem 5.3.7**, there exists a closed set F in (Y, σ) such that $B \subset F$ and $f^{-1}(F) \subset \psi^*\alpha\, cl(f^{-1}(B))$. Therefore $f^{-1}(cl(B)) \subset f^{-1}(F) \subset \psi^*\alpha cl(f^{-1}(B))$.

Conversely, let $B \subset Y$ and F be $\psi^*\alpha$-closed in (X, τ) containing $f^{-1}(B)$. Let $W = cl(B)$, then $B \subset W$ and W is closed and $f^{-1}(W) \subset \psi^*\alpha cl(f^{-1}(B)) \subset F$. Then by **Theorem 5.3.7**, f is a quasi $\psi^*\alpha$-open map.

Proposition 5.3.9 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z,\eta)$ be two maps and $g \circ f : (X, \tau) \rightarrow (Z,\eta)$ is a quasi $\psi^*\alpha$-open map. If g is a continuous and injective map, then f is a quasi $\psi^*\alpha$-open map.

Proof: Let U be a $\psi^*\alpha$-open set in (X, τ). Then $(g \circ f)(U)$ is open in (Z, η) as $g \circ f$ is quasi $\psi^*\alpha$-open. Since g is an injective continuous map, $g^{-1}((g \circ f)(U)) = f(U)$ is open in (Y, σ). Hence f is a quasi $\psi^*\alpha$-open map.

Remark 5.3.10 The above proposition is also true if g is an injective and completely continuous map, since every regular closed set is closed.

Definition 5.3.11 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **quasi $\psi^*\alpha$-closed** if f(A) is closed in (Y, σ) for each $\psi^*\alpha$-closed set A in (X, τ).

Example 5.3.12 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is a quasi $\psi^*\alpha$-closed map.

Theorem 5.3.13 Every quasi $\psi^*\alpha$-closed map is a closed map as well as a $\psi^*\alpha$-closed map but not conversely.

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 5.3.14 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is clearly a closed map as well as a $\psi^*\alpha$-closed map but not a quasi $\psi^*\alpha$-closed map, since $\{c\}$ is $\psi^*\alpha$-closed in (X, τ) but $f(\{c\}) = \{b\}$ is not closed in (Y, σ).

Theorem 5.3.15 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi $\psi^*\alpha$-closed if and only if for every subset U of (X, τ), $cl(f(U)) \subset f(\psi^*\alpha cl(U))$.

Proof: This proof is similar to the proof of **Theorem 5.3.3.**

Theorem 5.3.16 A map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-closed if and only if for any subset B of (Y, σ) and for any $\psi^*\alpha$-open set G in (X, τ) containing $f^{-1}(B)$, there exists an open set U of (Y, σ) containing B such that $f^{-1}(U) \subseteq G$.

Proof: This proof is similar to the proof of **Theorem 5.3.7.**

Theorem 5.3.17 Let (X, τ) and (Y, σ) be any two topological spaces. Then f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-closed if and only if f(X) is closed in (Y, σ) and f(V) – [f(X – V)] is open in f(X) whenever V is $\psi^*\alpha$-open in (X, τ).

Proof: (Necessity) Suppose f : (X, τ) → (Y, σ) is a quasi $\psi^*\alpha$-closed map. Since X is $\psi^*\alpha$-closed, f(X) is closed in (Y, σ) and f(V) – [f(X – V)] = f(V) ∩ f(X) – [f(X – V)] is open in f(X) when V is $\psi^*\alpha$-open in (X, τ).

(Sufficiency): Suppose f(X) is closed in (Y, σ), f(V) – [f(X – V)] is open in f(X) when V is $\psi^*\alpha$-open in (X, τ) and let C be closed in (X, τ). Then f(C) = f(X) – [f(X – C) –f(C)] is closed in f(X) and hence closed in (Y, σ).

Corollary 5.3.18 Let (X, τ) and (Y, σ) be topological spaces. Then a surjective map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-closed if and only if f(V) – f(X – V) is open in (Y, σ) whenever V is $\psi^*\alpha$-open in (X, τ).

Proof: Obvious

Theorem 5.3.19 Let (X, τ) and (Y, σ) be topological spaces and let f : (X, τ) → (Y, σ) be a $\psi^*\alpha$-continuous and quasi $\psi^*\alpha$-closed surjective map. Then the topology on (Y, σ) is {f(V) – [f(X – V)] : V is $\psi^*\alpha$-open in (X, τ)}.

Proof: Let W be open in (Y, σ). Then $f^{-1}(W)$ is $\psi^*\alpha$-open in (X, τ) and $f[f^{-1}(W)] - [f(X - f^{-1}(W))] = W$. Hence all open sets in (Y, σ) are of the form f(V) – [f(X – V)], V is $\psi^*\alpha$-open in (X, τ). On the other hand, all sets of the form f(V) – [f(X – V)], V is $\psi^*\alpha$-open in (X, τ), are open in (Y, σ) from **Corollary 5.3.18.**

Proposition 5.3.20 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two maps such that $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a quasi $\psi^*\alpha$-closed map. If f is a $\psi^*\alpha$-irresolute surjective map, then g is a closed map.

Proof: Let V be a closed set in (Y, σ). Since every closed set is a $\psi^*\alpha$-closed set and f is a $\psi^*\alpha$-irresolute map, $f^{-1}(V)$ is a $\psi^*\alpha$-closed set in (X, τ). Since $g \circ f$ is a quasi $\psi^*\alpha$-closed map and f is surjective, $(g \circ f)(f^{-1}(V)) = g(V)$ is closed in (Z, η). Hence g is a closed map.

Proposition 5.3.21 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed) maps then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is also a quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed) map.

Proof: Let U be any $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed) set in (X, τ). Then f(U) is open (resp. closed) in (Y, σ) as f is quasi $\psi^*\alpha$-open(resp. quasi $\psi^*\alpha$-closed). Since every open (resp. closed) set is $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed), f(U) is $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed) set in (Y, σ). Since g is quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed), $(g \circ f)(U) = g(f(U))$ is open (resp. closed) in (Z, η). Thus $g \circ f$ is a quasi $\psi^*\alpha$-open (resp. a quasi $\psi^*\alpha$-closed) map.

Definition 5.3.22 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **strongly $\psi^*\alpha$-closed** if f(A) is $\psi^*\alpha$-closed in (Y, σ) for each $\psi^*\alpha$-closed set A in (X, τ).

Example 5.3.23 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a strongly $\psi^*\alpha$-closed map.

Proposition 5.3.24 Every strongly $\psi^*\alpha$-closed map is a $\psi^*\alpha$-closed map but not conversely

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 5.3.25 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is clearly $\psi^*\alpha$-closed but not strongly $\psi^*\alpha$-closed, since $\{b\}$ is $\psi^*\alpha$-closed in (X, τ) but $f(\{b\}) = \{a\}$ is not $\psi^*\alpha$-closed in (Y, σ).

Proposition 5.3.26 Every quasi $\psi^*\alpha$-closed map is strongly $\psi^*\alpha$-closed but not conversely

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 5.3.27 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is strongly $\psi^*\alpha$-closed but not quasi $\psi^*\alpha$-closed, since {b} is $\psi^*\alpha$-closed in (X, τ) but f({b}) = {b} is not closed in (Y, σ).

Proposition 5.3.28 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be any two maps.

(i) If f is a quasi $\psi^*\alpha$-closed map and g is a $\psi^*\alpha$-closed, map then g∘f is a strongly $\psi^*\alpha$-closed map.

(ii) If f is a strongly $\psi^*\alpha$-closed map and g is a quasi $\psi^*\alpha$-closed map, then g∘f is a quasi $\psi^*\alpha$-closed map.

(iii) If f and g are quasi $\psi^*\alpha$-closed maps, then g∘f is a strongly $\psi^*\alpha$-closed map.

(iv) If f and g are strongly $\psi^*\alpha$-closed maps, then g∘f is a strongly $\psi^*\alpha$-closed map.

Proof: (i) Let A be a $\psi^*\alpha$-closed set in (X, τ). Since f is quasi $\psi^*\alpha$-closed, f(A) is closed in (Y, σ).Then g(f(A)) = (g∘f)(A) is $\psi^*\alpha$-closed in (Z, η), as g is $\psi^*\alpha$-closed. Hence g∘f is a strongly $\psi^*\alpha$-closed map.

(ii) Let A be $\psi^*\alpha$-closed in (X, τ). Since f is strongly $\psi^*\alpha$-closed, f(A) is $\psi^*\alpha$-closed in (Y, σ). Then g(f(A)) = (g∘f)(A) is closed in (Z, η), as g is quasi $\psi^*\alpha$-closed. Hence g∘f is a quasi $\psi^*\alpha$-closed map.

(iii) The proof follows from the fact every closed set is $\psi^*\alpha$-closed.

(iv) Let A be $\psi^*\alpha$-closed in (X, τ). Then f(A) is $\psi^*\alpha$-closed in (Y, σ), as f is strongly $\psi^*\alpha$-closed. Since g is strongly $\psi^*\alpha$-closed, (g∘f)(A) = g(f(A)) is $\psi^*\alpha$-closed in (Z, η). Hence g∘f is a strongly $\psi^*\alpha$-closed map.

Proposition 5.3.29 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be any two maps such that g∘f : (X, τ) → (Z, η) is a quasi $\psi^*\alpha$-closed map. If g is $\psi^*\alpha$-continuous injective then f is a strongly $\psi^*\alpha$-closed map.

Proof: Let A be a $\psi^*\alpha$-closed set in (X, τ). Since g∘f is quasi $\psi^*\alpha$-closed, (g∘f)(A) is closed in (Z, η). Since g is $\psi^*\alpha$-continuous injective, g^{-1}(g∘f)(A) = f(A) is $\psi^*\alpha$-closed in (Y, σ). Hence f is a strongly $\psi^*\alpha$-closed map.

Remark 5.3.30 The **Proposition 5.3.29** is true if g is continuous, α-continuous and completely continuous, since every closed set, α-closed set and regular closed set is $\psi^*\alpha$-closed.

5.4 $\psi^*\alpha$-homeomorphisms

In this section, two new classes of homeomorphisms namely, $\psi^*\alpha$-homeomorphisms and $\psi^*\alpha\mathcal{C}$-homeomorphisms are introduced in topological spaces. Properties of these homeomorphisms in topological spaces are analyzed.

Definition 5.4.1 A bijective map f : (X, τ) → (Y, σ) is called **$\psi^*\alpha$-homeomorphism** if f is both $\psi^*\alpha$-continuous and $\psi^*\alpha$-open.

Example 5.4.2 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is $\psi^*\alpha$-homeomorphism.

Proposition 5.4.3 Every homeomorphism f : (X, τ) → (Y, σ) is a $\psi^*\alpha$- homeomorphism but not conversely.

Proof: Let f be a homeomorphism. Then f is both continuous and open. Since every continuous map is $\psi^*\alpha$-continuous and every open map is $\psi^*\alpha$-open, f is both $\psi^*\alpha$-continuous and $\psi^*\alpha$-open. Therefore f is a $\psi^*\alpha$-homeomorphism.

Example 5.4.4 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is $\psi^*\alpha$-homeomorphism but not homeomorphism, since for the closed set {c} in (Y, σ) f^{-1}({c}) = {b} is not closed in (X, τ).

Theorem 5.4.5 Let f : (X, τ) → (Y, σ) be a bijective $\psi^*\alpha$-continuous map. Then the following statements are equivalent:

(i) f is a $\psi^*\alpha$-open map

(ii) f is a $\psi^*\alpha$-homeomorphism

(iii) f is a $\psi^*\alpha$-closed map.

Proof: (i) $\Rightarrow$ (ii) Let f be a $\psi^*\alpha$-open map. By hypothesis f is a bijective $\psi^*\alpha$-continuous map. Hence f is a $\psi^*\alpha$-homeomorphism.

(ii) $\Rightarrow$ (iii) Let f be a $\psi^*\alpha$-homeomorphism. Then f is $\psi^*\alpha$-open map and by **Theorem 5.2.30,** f is a $\psi^*\alpha$-closed map.

(iii) $\Rightarrow$ (i) Follows from **Theorem 5.2.30.**

Remark 5.4.6 The composition of two $\psi^*\alpha$-homeomorphisms need not be a $\psi^*\alpha$-homeomorphism as seen from the following example.

Example 5.4.7 Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, σ = {ϕ, {a, b}, Y} and η = {ϕ, {a}, {a, b}, Z}. Let f : (X, τ) $\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) be identity maps. Then both f and g are $\psi^*\alpha$-homeomorphisms but their composition g∘f : (X, τ) $\rightarrow$ (Z, η) is not a $\psi^*\alpha$-homeomorphism, since for the open set {b} in (X, τ), (g∘f)({b}) = {b}which is not a $\psi^*\alpha$-open set in (Z, η).Therefore g∘f is not a $\psi^*\alpha$-open map and so g∘f is not a $\psi^*\alpha$-homeomorphism.

Definition 5.4.8 A bijective map f : (X, τ) $\rightarrow$ (Y, σ) is called **$\psi^*\alpha\mathcal{C}$-homeomorphism** if f and f^{-1} are $\psi^*\alpha$-irresolute.

The family of all $\psi^*\alpha\mathcal{C}$-homeomorphisms of a topological space (X, τ) onto itself is denoted by $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ)

Theorem 5.4.9 If f : (X, τ) $\rightarrow$ (Y, σ) is $\psi^*\alpha\mathcal{C}$-homeomorphism then $\psi^*\alpha cl(f^{-1}(B)) = f^{-1}(\psi^*\alpha cl(B))$, for all B $\subseteq$ Y.

Proof: Since f is a $\psi^*\alpha\mathcal{C}$-homeomorphism, f is $\psi^*\alpha$-irresolute. Since for B $\subseteq$ Y, $\psi^*\alpha cl(B)$ is a $\psi^*\alpha$-closed set in (Y, σ), $f^{-1}(\psi^*\alpha cl(B))$ is $\psi^*\alpha$-closed in (X, τ). Since B $\subseteq$ $\psi^*\alpha cl(B)$, $f^{-1}(B) \subseteq f^{-1}(\psi^*\alpha cl(B))$. Therefore $\psi^*\alpha cl(f^{-1}(B)) \subseteq f^{-1}(\psi^*\alpha cl(B))$.

Again since f is a $\psi^*\alpha\mathcal{C}$-homeomorphism, f^{-1} is $\psi^*\alpha$-irresolute. Since $\psi^*\alpha cl(f^{-1}(B))$ is a $\psi^*\alpha$-closed set in (X, τ), $(f^{-1})^{-1}(\psi^*\alpha cl(f^{-1}(B))) = f(\psi^*\alpha cl(f^{-1}(B)))$ is $\psi^*\alpha$-closed in (Y, σ). Since $B \subseteq (f^{-1})^{-1}(f^{-1}(B)) \subseteq (f^{-1})^{-1}(\psi^*\alpha cl(f^{-1}(B))) = f(\psi^*\alpha cl(f^{-1}(B)))$. Therefore $\psi^*\alpha cl(B) \subseteq f(\psi^*\alpha cl(f^{-1}(B)))$ and hence $f^{-1}(\psi^*\alpha cl(B)) \subseteq f^{-1}(f(\psi^*\alpha cl(f^{-1}(B))) \subseteq \psi^*\alpha cl(f^{-1}(B))$. Thus the equality holds.

Corollary 5.4.10 If $f : (X, \tau) \to (Y, \sigma)$ is $\psi^*\alpha\mathcal{C}$-homeomorphism then $\psi^*\alpha cl(f(B)) = f(\psi^*\alpha cl(B))$, $B \subseteq X$.

Proof: Since $f : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha\mathcal{C}$-homeomorphism, $f^{-1} : (Y, \sigma) \to (X, \tau)$ is also a $\psi^*\alpha\mathcal{C}$-homeomorphism. Therefore by **Theorem 5.4.9**, $\psi^*\alpha cl((f^{-1})^{-1}(B)) = (f^{-1})^{-1}(\psi^*\alpha cl(B))$, for all $B \subseteq X$. Hence $\psi^*\alpha cl(f(B)) = f(\psi^*\alpha cl(B))$.

Corollary 5.4.11 If $f : (X, \tau) \to (Y, \sigma)$ is $\psi^*\alpha\mathcal{C}$-homeomorphism then $f(\psi^*\alpha int(B)) = \psi^*\alpha\, int(f(B))$, for all $B \subseteq X$.

Proof: For any subset $B \subseteq X$, $\psi^*\alpha int(B) = X - \psi^*\alpha cl(X - B)$. Thus $f(\psi^*\alpha int(B)) = f(X - \psi^*\alpha cl(X - B)) = Y - f(\psi^*\alpha cl(X - B)) = Y - \psi^*\alpha cl(f(X - B)) = \psi^*\alpha int(f(B))$ (since by **Corollary 5.4.10**).

Theorem 5.4.12 If $f : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha\mathcal{C}$-homeomorphism, then $f^{-1}(\psi^*\alpha int(B)) = \psi^*\alpha int(f^{-1}(B))$, for all $B \subseteq Y$.

Proof: Since f is a $\psi^*\alpha\mathcal{C}$-homeomorphism, $f^{-1} : (Y, \sigma) \to (X, \tau)$ is also $\psi^*\alpha\mathcal{C}$-homeomorphism. The proof follows from **Corollary 5.4.11.**

Theorem 5.4.13 Every $\psi^*\alpha$-homeomorphism f from ${}_{\psi^*\alpha}T_c$-space (X, τ) into another ${}_{\psi^*\alpha}T_c$-space (Y, σ) is $\psi^*\alpha\mathcal{C}$-homeomorphism.

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha$-homeomorphism. Then f is bijective, $\psi^*\alpha$-continuous and $\psi^*\alpha$-open. Let V be a $\psi^*\alpha$-closed set in (Y, σ). Then V is closed, as (Y, σ) is a ${}_{\psi^*\alpha}T_c$-space. Since f is $\psi^*\alpha$-continuous, $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence f is a $\psi^*\alpha$-irresolute map. Let U be $\psi^*\alpha$-open in (X, τ). Then U is open in (X, τ). Since f is $\psi^*\alpha$-open, f(U) is $\psi^*\alpha$-open in (Y, σ). ie.,$(f^{-1})^{-1}(U)$ is $\psi^*\alpha$-open in (Y, σ) and hence f^{-1} is a $\psi^*\alpha$-irresolute map. Thus f is $\psi^*\alpha\mathcal{C}$-homeomorphism.

Theorem 5.4.14 If f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) are $\psi^*\alpha\mathcal{C}$-homeomorphisms then g∘f : (X, τ) → (Z, η) is also a $\psi^*\alpha\mathcal{C}$-homeomorphism.

Proof: Let U be $\psi^*\alpha$-closed in (Z, η). Consider $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, V is $\psi^*\alpha$-closed in (Y, σ) and $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is $\psi^*\alpha$-irresolute.

Let G be a $\psi^*\alpha$-closed set in (X, τ). Consider (g∘f)(G) = g(f(G)) = g(W), where W = f(G). By hypothesis, f(G) is $\psi^*\alpha$-closed in (Y, σ) and g(f(G)) is $\psi^*\alpha$-closed in (Z, η). Hence (g∘f)(G) is $\psi^*\alpha$-closed in (Z, η) and therefore $(g\circ f)^{-1}$ is $\psi^*\alpha$-irresolute. Thus g∘f is a $\psi^*\alpha\mathcal{C}$-homeomorphism.

Theorem 5.4.15 The set $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) is a group under the composition of maps.

Proof: Define a binary operation ∗ : $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) × $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) → $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) by (f ∗ g) = (g ∘ f), for all f, g ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ), where ∘ is the usual operation of composition of maps. Then by **Theorem 5.4.14**, g ∘ f ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ). Hence $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) is closed. The composition of maps is always associative. The identity map I : (X, τ) → (X, τ) is a $\psi^*\alpha\mathcal{C}$-homeomorphism and I ∈$\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ). Also I∘f = f∘I = f for every f ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ). For any f ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ), f^{-1} ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) and $f\circ f^{-1} = f^{-1}\circ f = I$. Therefore the set $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) is a group under the operation of composition of maps.

Theorem 5.4.16 Let f : (X, τ) → (Y, σ) be a $\psi^*\alpha\mathcal{C}$-homeomorphism. Then f induces an isomorphism from the group $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) onto the group $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(Y, σ).

Proof: Let θ_f : $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) → $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(Y, σ) be a map defined by $\theta_f(h) = f^{-1}\circ h\circ f$, for every h ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$ (X, τ). For $h_1 \neq h_2$, if $\theta_f(h_1) = \theta_f(h_2)$ then $f^{-1}\circ h_1\circ f = f^{-1}\circ h_2\circ f$. Hence $h_1 = h_2$, which is a contradiction. Therefore θ_f is one - one. For g $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(Y, σ), choose h $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ) such that $\theta_f(h) = f^{-1}\circ h\circ f = g$. That is, $h = f\circ g\circ f^{-1}$ ∈ $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ). Therefore θ_f is onto. Hence θ_f is bijective. Further, for all h_1, h_2 $\psi^*\alpha\mathcal{C}$-$\mathcal{h}$(X, τ), $\theta_f(h_1\circ h_2) = f^{-1}\circ(h_1\circ h_2)\circ f = (f^{-1}\circ h_1\circ f) \circ (f^{-1}\circ h_2\circ f) = \theta_f(h_1) \circ \theta_f(h_2)$. Hence θ_f is a homomorphism. Thus θ_f is an isomorphism induced by f.

Theorem 5.4.17 $\psi^*\alpha\mathcal{C}$-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof: Let Π be the collection of topological spaces. For (X, τ), (Y, σ) ∈ Π define a relation R on Π such that (X, τ) R (Y, σ) if and only if (X, τ) and (Y, σ) are $\psi^*\alpha\mathcal{C}$-homeomorphic to each other.

Reflexive : (X, τ) R (X, τ) is immediate.

Symmetry : (X, τ) R (Y, σ) □ (Y, σ) R (X, τ)

Transitivity : Follows from **Theorem 5.4.14.**

Hence $\psi^*\alpha\mathcal{C}$-homeomorphism is an equivalence relation in the collection of all topological spaces.

5.5 ψ^* α-quotient maps

In this section, $\psi^*\alpha$-quotient maps, strongly $\psi^*\alpha$-quotient maps and completely $\psi^*\alpha$-quotient maps are introduced and studied the relationship between these maps. It is also proved that composition of two completely $\psi^*\alpha$-quotient maps is a completely $\psi^*\alpha$-quotient map.

Definition 5.5.1 A surjective map f : (X, τ) → (Y, σ) is called a **$\psi^*\alpha$-quotient map** if f is $\psi^*\alpha$-continuous and $f^{-1}(V)$ is open in (X, τ) implies V is a $\psi^*\alpha$-open set in (Y, σ)

Example 5.5.2 Let X = {a, b, c}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O$(X, τ) = {ϕ, {a}, {a, b}, {a, c}, X} and $\psi^*\alpha O$(Y, σ) = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is a $\psi^*\alpha$-quotient map.

Proposition 5.5.3 If a map f : (X, τ) → (Y, σ) is surjective, $\psi^*\alpha$-continuous and $\psi^*\alpha$-open, then f is a $\psi^*\alpha$-quotient map.

Proof: It is enough to prove that $f^{-1}(V)$ is open in (X, τ) implies V is a $\psi^*\alpha$-open set in (Y, σ). Let $f^{-1}(V)$ is open in (X, τ). Then $f(f^{-1}(V))$ is $\psi^*\alpha$-open, since f is $\psi^*\alpha$-open. As f is surjective $f(f^{-1}(V)) = V$ and so V is a $\psi^*\alpha$-open set in (Y, σ). Hence f is a $\psi^*\alpha$-quotient map.

Definition 5.5.4 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjective map. Then f is called **strongly $\psi^*\alpha$-quotient map** provided a set U of (Y, σ) is open in (Y, σ) if and only if $f^{-1}(U)$ is a $\psi^*\alpha$-open set in (X, τ).

Example 5.5.5 Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$, $Y = \{l, m, n\}$, $\sigma = \{ \phi, \{l\}, \{m\}, \{l, m\}, Y\}$, $\psi^*\alpha O(X, \tau) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\},\{a, c, d\}, X\}$ and $\psi^*\alpha O(Y, \sigma) = \{ \phi, \{l\}, \{m\}, \{l, m\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = l = f(b)$, $f(c) = m$, $f(d) = n$. Then f is a strongly $\psi^*\alpha$-quotient map.

Proposition 5.5.6 Every strongly $\psi^*\alpha$-quotient map is a $\psi^*\alpha$-open map but not conversely.

Proof: $f : (X, \tau) \rightarrow (Y, \sigma)$ be a strongly $\psi^*\alpha$-quotient map. Let V be an open set in (X, τ).Since every open set is $\psi^*\alpha$-open and hence V is $\psi^*\alpha$-open in (X, τ). That is $f^{-1}(f(V))$ is $\psi^*\alpha$-open in (X, τ). Since f is strongly $\psi^*\alpha$-quotient, f(V) is open and hence f(V) is $\psi^*\alpha$-open in (Y, σ). Therefore f is a $\psi^*\alpha$-open map.

Example 5.5.7 Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, X\}$, $Y = \{l, m, n\}$, $\sigma = \{ \phi, \{l\}, \{m\}, \{l, m\}, \{l, n\}, Y\}$, $\psi^*\alpha O(X, \tau) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ and $\psi^*\alpha O(Y, \sigma) = \{ \phi, \{l\}, \{m\}, \{l, m\}, \{l, n\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a) = l = f(b)$, $f(c) = n$, $f(d) = m$. Then f is $\psi^*\alpha$-open but not strongly $\psi^*\alpha$-quotient, since $f^{-1}(\{m\}) = \{d\}$ is not $\psi^*\alpha$-open in (X, τ)

Proposition 5.5.8 Every strongly $\psi^*\alpha$-quotient map is a strongly $\psi^*\alpha$-open map but not conversely.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a strongly $\psi^*\alpha$-quotient map. Let V be a $\psi^*\alpha$-open set in (X, τ). That is $f^{-1}(f(V))$ is $\psi^*\alpha$-open in (X, τ). Since f is strongly $\psi^*\alpha$-quotient, f(V) is open and hence f(V) is $\psi^*\alpha$-open in (Y, σ).Therefore f is a strongly $\psi^*\alpha$-open map.

Example 5.5.9 Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {b}, {a, b}, X} and $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is strongly $\psi^*\alpha$-open but not strongly $\psi^*\alpha$-quotient, since f^{-1}({l}) = {a} is $\psi^*\alpha$-open in (X, τ) but the set {l} is not open in (Y, σ).

Definition 5.5.10 A surjective map f : (X, τ) → (Y, σ) is called a **completely $\psi^*\alpha$-quotient map** if f is $\psi^*\alpha$-irresolute and f^{-1}(U) is $\psi^*\alpha$-open set in (X, τ) implies U is open in (Y, σ).

Example 5.5.11 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {l, m}, {l, n}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then f is a completely $\psi^*\alpha$-quotient map.

Proposition 5.5.12 Every completely $\psi^*\alpha$-quotient map is a $\psi^*\alpha$-irresolute map but not conversely.

Proof: Follows from the **Definition 5.5.10**

Example 5.5.13 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {l, m}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then f is a $\psi^*\alpha$-irresolute map but not completely $\psi^*\alpha$-quotient map, since f^{-1}({l, n}) = {a, b} is $\psi^*\alpha$-open in (X, τ) but the set {l, n} is not open in (Y, σ).

Proposition 5.5.14 Every completely $\psi^*\alpha$-quotient map is a strongly $\psi^*\alpha$-open map but not conversely.

Proof: Let f : (X, τ) → (Y, σ) be a completely $\psi^*\alpha$-quotient map. Let V be a $\psi^*\alpha$-open set in (X, τ). That is f^{-1}(f(V)) is $\psi^*\alpha$-open in (X, τ). Since f is completely $\psi^*\alpha$-quotient, f(V) is open and hence f(V) is $\psi^*\alpha$-open in (Y, σ). Therefore f is a strongly $\psi^*\alpha$-open map.

Example 5.5.15 Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {b}, {a, b}, X} and $\psi^*\alpha O(Y, \sigma)$ = {ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is strongly $\psi^*\alpha$-open but not completely $\psi^*\alpha$-quotient, since f^{-1}({l}) = {a} is $\psi^*\alpha$-open in (X, τ) but the set {l} is not open in (Y, σ).

Proposition 5.5.16

(i) Every quotient map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-quotient map.

(ii) Every α-quotient map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-quotient map.

Proof: (i) Since every continuous map is $\psi^*\alpha$-continuous map and every open set is $\psi^*\alpha$-open, the proof follows from the definition.

(ii) Since every α-continuous map is $\psi^*\alpha$-continuous map and every α-open set is $\psi^*\alpha$-open, the proof follows from the definition.

The converse of the statements in the above proposition need not be true as seen from the following examples.

Example 5.5.17 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then the map f is $\psi^*\alpha$-quotien but not quotient, since for the $\psi^*\alpha$-open set {l, m} in (Y, σ). (f^{-1}({l, m}) = {a, c, d} is open in (X, τ) but the set {l, m} is not open in (Y, σ).

Example 5.5.18 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d},{a, b, c}, {a, b, d}, {a, c, d}, X}, $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {m}, {l, m}, Y} and $\alpha O(Y, \sigma)$ = { ϕ, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = f(d) = n. Then the map f is $\psi^*\alpha$-quotient but not α-quotient, since (f^{-1}({l}) = {a} is open in (X, τ) but the set {l} is not α-open in (Y, σ).

Proposition 5.5.19 Every strongly $\psi^*\alpha$-quotient map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-quotient map but not conversely.

Proof: Let V be an open set in (Y, σ). Since f is strongly $\psi^*\alpha$-quotient, $f^{-1}(V)$ is a $\psi^*\alpha$-open set in (X, τ). Hence f is $\psi^*\alpha$-continuous, Let $f^{-1}(V)$ be a open set in (X, τ). Then $f^{-1}(V)$ is a $\psi^*\alpha$-open set in (X, τ). Since f is strongly $\psi^*\alpha$-quotient, V is open in (Y, σ). Hence f is $\psi^*\alpha$-quotient.

Example 5.5.20 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O$(X, τ) = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O$(Y, σ) = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then the map f is $\psi^*\alpha$-quotient but not strongly $\psi^*\alpha$-quotient, since for the $\psi^*\alpha$-open set {l, m} in (Y, σ). f^{-1}({l, m}) = {a, c, d} is open in (X, τ) but the set {l, m} is not open in (Y, σ).

Proposition 5.5.21 Every completely $\psi^*\alpha$-quotient map f : (X, τ) → (Y, σ) is a strongly $\psi^*\alpha$-quotient map but not conversely.

Proof: Let V be an open set in (Y, σ). Then it is $\psi^*\alpha$-open in (Y, σ). Since f is completely $\psi^*\alpha$-quotient, $f^{-1}(V)$ is a $\psi^*\alpha$-open set in (X, τ). If $f^{-1}(V)$ is $\psi^*\alpha$-open set in (X, τ). Then V is open in (Y, σ) as f is completely $\psi^*\alpha$-quotient. Hence f is a strongly $\psi^*\alpha$-quotient map.

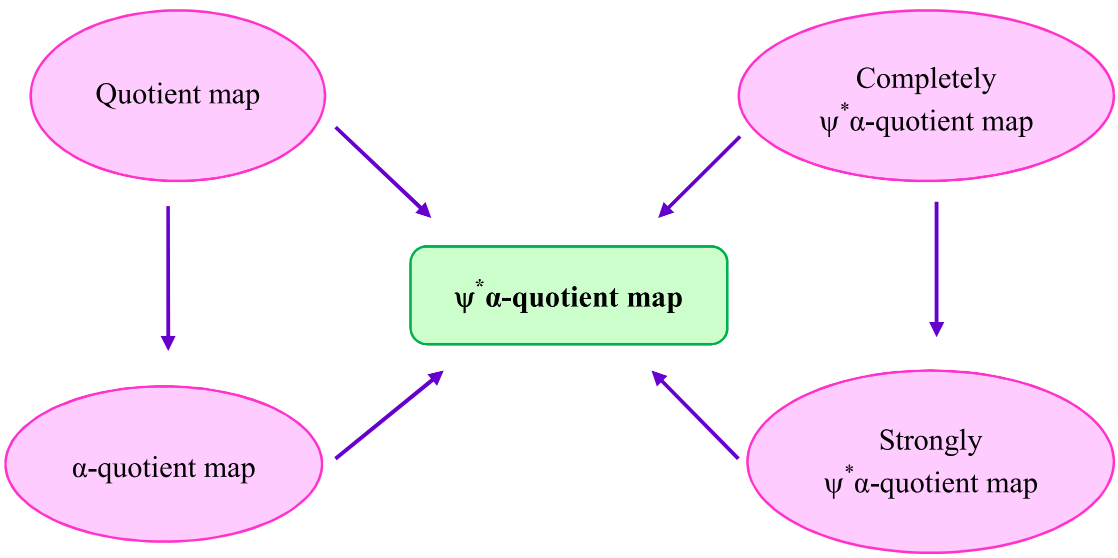
Example 5.5.22 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O$(X, τ) = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X} and $\psi^*\alpha O$(Y, σ) = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n = f(c), f(d) = m. Then the map f is strongly $\psi^*\alpha$-quotient but not completely $\psi^*\alpha$-quotient, since for the $\psi^*\alpha$-open set {l, n} in (Y, σ) f^{-1}({l, n}) = {a, b, c} is $\psi^*\alpha$-open in (X, τ) but the set {l, n} is not open in (Y, σ).

Proposition 5.5.23 Every completely $\psi^*\alpha$-quotient map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-quotient map but not conversely.

Proof: Let f be a completely $\psi^*\alpha$-quotient map. Then f is $\psi^*\alpha$-irresolute and by **Theorem 4.2.7,** f is $\psi^*\alpha$-continuous. Let $f^{-1}(V)$ be an open set in (X, τ). Then $f^{-1}(V)$ is a $\psi^*\alpha$-open set in (X, τ). As f is completely $\psi^*\alpha$-quotient, V is a open set in (Y, σ). It implies that V is a $\psi^*\alpha$-open set in (Y, σ). Hence f is a $\psi^*\alpha$-quotient map.

Example 5.5.24 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = {ϕ, {l}, Y}, $\psi^*\alpha O(X, \tau)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X} and $\psi^*\alpha O(Y, \sigma)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n = f(c), f(d) = m. Then the map f is $\psi^*\alpha$-quotient but not completely $\psi^*\alpha$-quotient, since for the $\psi^*\alpha$-open set {l, m} in (Y, σ) f^{-1}({l, m}) = {a, d} is $\psi^*\alpha$-open in (X, τ) but the set {l, m} is not open in (Y, σ).

Remark 5.5.25 From the above observations we have the following diagram,



Composition of $\psi^*\alpha$-quotient maps

Proposition 5.5.26 If a map f : (X, τ) → (Y, σ) is open, surjective and $\psi^*\alpha$-irresolute and g : (Y, σ) → (Z, η) is a $\psi^*\alpha$-quotient map. Then g ∘ f : (X, τ) → (Z, η) is a $\psi^*\alpha$-quotient map.

Proof: Let V be any open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ), since g is a $\psi^*\alpha$-quotient map. Since f is $\psi^*\alpha$ -irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$-open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ). This shows that g ∘ f is $\psi^*\alpha$-continuous.

Now assume that $(g \circ f)^{-1}(V)$ is open in (X, τ) for a subset $V \subseteq Z$. Since f is open, $f(f^{-1}(g^{-1}(V)))$ is open in (Y, σ). This implies that $g^{-1}(V)$ is open in (Y, σ), as f is surjective. Since g is a $\psi^*\alpha$-quotient map, V is a $\psi^*\alpha$-open set in (Z, η). Hence $g \circ f$ is a $\psi^*\alpha$-quotient map.

Proposition 5.5.27 If $h : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha$-quotient map and $g : (X, \tau) \to (Z, \eta)$ is a continuous map where Z is a space that is constant on each set $h^{-1}(\{y\})$ for each $y \in Y$, then g induces $\psi^*\alpha$-continuous map $f : (Y, \sigma) \to (Z, \eta)$ such that $f \circ h = g$.

Proof: Since g is constant on $h^{-1}(\{y\})$ for each $y \in Y$, the set $g(h^{-1}(\{y\}))$ is an one point set in (Z, η). If we let f(y) to denote this point, then it is clear that f is well defined and for each $x \in X$, $f(h(x)) = g(x)$. We prove that f is $\psi^*\alpha$ -continuous. For if let V be any open set in (Z, η),then $g^{-1}(V)$ is an open set as g is continuous. But $g^{-1}(V) = h^{-1}(f^{-1}(V))$ is open in (X, τ).Since h is a $\psi^*\alpha$-quotient map, $f^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ). Hence f is $\psi^*\alpha$-continuous.

Proposition 5.5.28 Let $f : (X, \tau) \to (Y, \sigma)$ be strongly $\psi^*\alpha$-open, surjective and $\psi^*\alpha$-irresolute map and $g : (Y, \sigma) \to (Z, \eta)$ be a strongly $\psi^*\alpha$-quotient map. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a strongly $\psi^*\alpha$-quotient map.

Proof: Let V be any open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ), since g is strongly $\psi^*\alpha$-quotient. Since f is $\psi^*\alpha$-irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$-open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ).

Now assume that $(g \circ f)^{-1}(V)$ is a $\psi^*\alpha$-open set in (X, τ) for a subset $V \subseteq Z$. Then $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$-open set in (X, τ). Since f is strongly $\psi^*\alpha$-open, $f(f^{-1}(g^{-1}(V)))$ is $\psi^*\alpha$-open in (Y, σ). This implies that $g^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ), as f is surjective. This gives that V is an open set in (Y, σ), since g is a strongly $\psi^*\alpha$-quotient map. Hence $g \circ f$ is a strongly $\psi^*\alpha$-quotient map.

Proposition 5.5.29 Let $f : (X, \tau) \to (Y, \sigma)$ be a $\psi^*\alpha$-quotient map where (X, τ) and (Y, σ) are ${}_{\psi^*\alpha}T_c$-spaces. A map $g : (Y, \sigma) \to (Z, \eta)$ is quasi $\psi^*\alpha$-continuous if and only if the composite map $g \circ f : (X, \tau) \to (Z, \eta)$ is a quasi $\psi^*\alpha$-continuous map.

Proof: Let g be quasi $\psi^*\alpha$-continuous and U be any $\psi^*\alpha$-open set in (Z, η). Then $g^{-1}(U)$ is open in (Y, σ). Since f is $\psi^*\alpha$-quotient, $f^{-1}(g^{-1}(U)) =(g \circ f)^{-1}(U)$ is $\psi^*\alpha$-open in (X, τ). Since (X, τ) is a ${}_{\psi^*\alpha}T_c$-space, $f^{-1}(g^{-1}(U))$ is in open in (X, τ). Thus $(g \circ f)$ is quasi $\psi^*\alpha$-continuous

Conversely, assume that $(g \circ f)$ is quasi $\psi^*\alpha$-continuous. Let U be any $\psi^*\alpha$-open set in (Z, η). Then $f^{-1}(g^{-1}(U))$ is open in (X, τ). Since f is $\psi^*\alpha$-quotient, $g^{-1}(U)$ is $\psi^*\alpha$-open in (Y, σ). Since (Y, σ). is a ${}_{\psi^*\alpha}T_c$-space, $g^{-1}(U)$ is open in (Y, σ). Hence g is quasi $\psi^*\alpha$-continuous.

Proposition 5.5.30 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be strongly $\psi^*\alpha$-open, surjective and $\psi^*\alpha$-irresolute map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a completely $\psi^*\alpha$-quotient map. Then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a completely $\psi^*\alpha$-quotient map.

Proof: Since f and g are $\psi^*\alpha$-irresolute, $g \circ f$ is $\psi^*\alpha$-irresolute by **Proposition 4.2.40.** Suppose that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ) for a subset $V \subseteq Z$, that is $f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-open in(X, τ). Since f is strongly $\psi^*\alpha$-open and surjective, $f(f^{-1}(g^{-1}(V)) = g^{-1}(V)$ is $\psi^*\alpha$-open in (Y, σ). Since g is completely $\psi^*\alpha$-quotient implies V is open in (Y, σ). Hence $g \circ f$ is a completely $\psi^*\alpha$-quotient map.

Proposition 5.5.31 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a strongly $\psi^*\alpha$-quotient and $\psi^*\alpha$-irresolute map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a completely $\psi^*\alpha$-quotient map. Then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a completely $\psi^*\alpha$-quotient map.

Proof: Let V be a $\psi^*\alpha$-open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ), as g is completely $\psi^*\alpha$-quotient. Since f is $\psi^*\alpha$-irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$-open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ). This shows that $g \circ f$ is $\psi^*\alpha$-irresolute. Let $f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-open in (X, τ) for a subset $V \subseteq Z$. Since f is strongly $\psi^*\alpha$-quotient, $g^{-1}(V)$ is open in (Y, σ). This implies that $g^{-1}(V)$ is a $\psi^*\alpha$-open set in (Y, σ). Since g is completely $\psi^*\alpha$-quotient, V is open in (Z, η). Hence $(g \circ f)$ is a completely $\psi^*\alpha$-quotient map.

Proposition 5.5.32 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are completely $\psi^*\alpha$-quotient maps. Then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is also a completely $\psi^*\alpha$-quotient map.

Proof: Let V be any $\psi^*\alpha$-open set in (Z, η). Then $g^{-1}(V)$ is $\psi^*\alpha$-open in (Y, σ), since g is completely $\psi^*\alpha$-quotient. Since f is completely $\psi^*\alpha$-quotient, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a $\psi^*\alpha$-open in (X, τ). This shows that $g \circ f$ is $\psi^*\alpha$-irresolute. Let $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ). Then $f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-open in (X, τ). Since f is completely $\psi^*\alpha$-quotient, $g^{-1}(V)$ is open in (Y, σ). Since g is a completely $\psi^*\alpha$-quotient map, V is open in (Z, η). Hence $(g \circ f)$ is a completely $\psi^*\alpha$-quotient map.

Proposition 5.5.33 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map where (X, τ) and (Y, σ) are $_{\psi^*\alpha}T_c$-spaces. Then the following statements are equivalent.

(i) f is a completely $\psi^*\alpha$-quotient map

(ii) f is strongly $\psi^*\alpha$-quotient map

(iii) f is a $\psi^*\alpha$-quotient map

Proof: (i) $\Rightarrow$ (ii) Follows from **Proposition 5.5.21**

(ii) $\Rightarrow$ (iii) Follows from **Proposition 5.5.19**

(iii) $\Rightarrow$ (i) Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space and f is a $\psi^*\alpha$-quotient map, f is $\psi^*\alpha$-irresolute by **Theorem 4.2.7.** Suppose $f^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ). Since (X, τ) is a $_{\psi^*\alpha}T_c$-space, $f^{-1}(V)$ is open in (X, τ). By (iii), V is $\psi^*\alpha$-open in (Y, σ). Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space, V is $\psi^*\alpha$-open in (Y, σ). Hence f is a completely $\psi^*\alpha$-quotient map.

5.6 $\psi^*\alpha$-Compact spaces and $\psi^*\alpha$-Connected spaces

In this section, the concept of $\psi^*\alpha$-compact spaces and $\psi^*\alpha$-connected spaces are introduced and characterizations of $\psi^*\alpha$-connected spaces are discussed in topological spaces.

Definition 5.6.1 A collection $\mathcal{A}$ of subsets of a space (X, τ) is said to ***cover* X** or to be a **covering of X** if the union of the elements of $\mathcal{A}$ is equal to X. It is called a **$\psi^*\alpha$-open covering of X** if its elements are $\psi^*\alpha$-open subsets of (X, τ).

Definition 5.6.2 A nonempty collection $\{A_i, i \in \Lambda$, an index set$\}$ of $\psi^*\alpha$-open sets in a topological spaces (X, τ) is called a $\psi^*\alpha$-open cover of a subset B of (X, τ) if $B \subseteq \cup \{A_i, i \in \Lambda\}$.

Definition 5.6.3 A topological space (X, τ) is **$\psi^*\alpha$-compact** if every $\psi^*\alpha$-open cover of X has a finite subcover.

Definition 5.6.4 A subset B of a topological space (X, τ) is called **$\psi^*\alpha$-compact relative to X** if for every collection $\{A_i, i \in \Lambda\}$ of $\psi^*\alpha$-open subsets of (X, τ) such that $B \subseteq \cup \{A_i, i \in \Lambda\}$ there exists a finite subset Λ_o of Λ such that $B \subseteq \cup \{A_i, i \in \Lambda_o\}$.

Theorem 5.6.5 A $\psi^*\alpha$-closed subset of a $\psi^*\alpha$-compact space (X, τ) is $\psi^*\alpha$-compact relative to (X, τ).

Proof: Let A be a $\psi^*\alpha$-closed subset of a $\psi^*\alpha$-compact space (X, τ). Then A^c is $\psi^*\alpha$-open in (X, τ). Let $N = \{G_\alpha: \alpha\in\Lambda\}$ be a $\psi^*\alpha$-open cover of A in (X, τ). Then $N^* = N \cup A^c$ is a $\psi^*\alpha$-open cover of (X, τ). Since (X, τ) is $\psi^*\alpha$-compact, N^* is reducible to a finite subcover of (X, τ), say $X = G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots\ldots\ldots. \cup G_{\alpha_m} \cup A^c$, $G_{\alpha_k} \in N$. But A and A^c are disjoint hence $A \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots\ldots\ldots. \cup G_{\alpha_m}$, $G_{\alpha_k} \in N$ which implies that any $\psi^*\alpha$-open cover N of A contains a finite subcover. Therefore A is $\psi^*\alpha$-compact relative to (X, τ).

Theorem 5.6.6 If a surjective map $f : (X, \tau) \to (Y, \sigma)$ is $\psi^*\alpha$-continuous and if (X, τ) is $\psi^*\alpha$-compact then (Y, σ) is compact.

Proof: Let $\{V_i : i \in \Lambda\}$ be an open cover of (Y, σ). Since f is $\psi^*\alpha$-continuous, $\{f^{-1}(V_i) : i \in \Lambda\}$ is a $\psi^*\alpha$-open cover of (X, τ). Since (X, τ) is $\psi^*\alpha$-compact, it has a finite subcover of (X, τ), say $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)\}$. Since f is surjective, $\{V_1, V_2, \ldots, V_n\}$ is a finite open cover of (Y, σ). Hence (Y, σ) is compact.

Theorem 5.6.7 If a surjective map $f : (X, \tau) \to (Y, \sigma)$ is quasi $\psi^*\alpha$-continuous, where (X, τ) is a compact space, then (Y, σ) is $\psi^*\alpha$-compact.

Proof: Let $\{V_i: i \in \Lambda\}$ be a $\psi^*\alpha$-open cover of (Y, σ). Since f is quasi $\psi^*\alpha$-continuous, $\{f^{-1}(V_i) : i \in \Lambda\}$ is an open cover of (X, τ). Since (X, τ) is compact, it has a finite subcover say, $\{f^{-1}(V_1), f^{-1}(V_2),\ldots\ldots, f^{-1}(V_n)\}$. Since f is surjective, $\{V_1, V_2,\ldots\ldots, V_n\}$ is a finite subcover of (Y, σ) and therefore (Y, σ) is $\psi^*\alpha$-compact.

Corollary 5.6.8 If a surjective map f : (X, τ) → (Y, σ) is perfectly $\psi^*\alpha$-continuous, where (X, τ) is a compact space, then (Y, σ) is $\psi^*\alpha$-compact.

Proof: Follows from the fact that every perfectly $\psi^*\alpha$-continuous map is quasi $\psi^*\alpha$-continuous map and by **Theorem 5.6.7**.

Theorem 5.6.9 If a map f : (X, τ) → (Y, σ) is $\psi^*\alpha$-irresolute and a subset B of (X, τ) is $\psi^*\alpha$-compact relative to (X, τ), then the image f(B) is $\psi^*\alpha$-compact relative to (Y, σ).

Proof: Let $\{A_i : i \in \Lambda\}$ be any collection of $\psi^*\alpha$-open subsets of (Y, σ) such that $f(B) \subseteq \cup\{A_i : i \in \Lambda\}$. Then $B \subseteq \cup \{f^{-1}(A_i) : i \in \Lambda\}$. Since, B is $\psi^*\alpha$-compact relative to (X, τ) there exists a finite subset Λ_o of Λ such that $B \subseteq \cup\{f^{-1}(A_i) : i \in \Lambda_o\}$. Therefore $f(B) \subseteq \cup\{A_i : i \in \Lambda_o\}$, which shows that f(B) is $\psi^*\alpha$-compact relative to (Y, σ).

Definition 5.6.10 A topological space (X, τ) is called **$\psi^*\alpha$-connected** if X cannot be expressed as a union of two disjoint nonempty $\psi^*\alpha$-open sets A and B such that $[A \cap \psi^*\alpha cl(B)] \cup [\psi^*\alpha\ cl(A) \cap B] = \phi$.

Suppose X can be so expressed then (X, τ) is called a **$\psi^*\alpha$-disconnected space** and we write X = A|B and call this **$\psi^*\alpha$- separation** of (X, τ).

Example 5.6.11 Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}. Therefore (X, τ) is $\psi^*\alpha$ -connected.

Example 5.6.12 Let X = {a, b}, τ = {ϕ, X}. Therefore (X, τ) is $\psi^*\alpha$-disconnected.

Theorem 5.6.13 Every $\psi^*\alpha$-connected space is connected but not conversely.

Proof: Let (X, τ) be a $\psi^*\alpha$-connected space. If possible let (X, τ) be not connected. Then X can be written as X = A ∪ B where A and B are disjoint nonempty open sets in (X, τ). Since every open set is $\psi^*\alpha$-open, X = A ∪ B where A and B are disjoint nonempty $\psi^*\alpha$-open sets in (X, τ). This contradicts the fact that (X, τ) is $\psi^*\alpha$-connected. Therefore (X, τ) is connected.

Example 5.6.14 Let X = {a, b}, τ = {ϕ, X}. Clearly (X, τ) is connected but not $\psi^*\alpha$-connected.

Theorem 5.6.15 For a topological space (X, τ) the following are equivalent:

(i) (X, τ) is $\psi^*\alpha$-connected.

(ii) X and ϕ are the only subsets of (X, τ) which are both $\psi^*\alpha$-open and $\psi^*\alpha$-closed.

(iii) Each $\psi^*\alpha$-continuous map of (X, τ) into a discrete space (Y, σ) with at least two points is a constant map.

Proof: (i) □ (ii) Let U be a $\psi^*\alpha$-open and $\psi^*\alpha$-closed subset of (X, τ).Then U^c is both $\psi^*\alpha$-open and $\psi^*\alpha$-closed. Since (X, τ) is the disjoint union of the $\psi^*\alpha$-open sets U and U^c, one of these must be empty, that is U = ϕ or U = X.

(ii) □ (i) Assume X = A ∪ B where A and B are two non-empty disjoint $\psi^*\alpha$-open subsets of (X, τ). Since A = X – B, A is $\psi^*\alpha$-closed and by assumption A = ϕ or X, which is a contradiction. Hence (X, τ) is $\psi^*\alpha$-connected.

(ii) □ (iii) Let f : (X, τ) → (Y, σ) be a $\psi^*\alpha$-continuous map, where (Y, σ) is a discrete space with atleast two points. Since (Y, σ) is a discrete space, for each y ∈ Y, {y} is both open and closed. Since f is a $\psi^*\alpha$-continuous map, $f^{-1}(\{y\})$ is $\psi^*\alpha$-closed and $\psi^*\alpha$-open and $X = \cup\{f^{-1}(\{y\}) : y \in Y\}$. By assumption $f^{-1}(\{y\}) = \phi$ or X for each y ∈ Y. If $f^{-1}(\{y\}) = \phi$ for all y □ Y, then f will not be a map. If $f^{-1}(\{y\}) = X$ for a single y ∈ Y, then there cannot exist one more $y_1 \in Y$ such that $f^{-1}(\{y_1\}) = X$. Hence there exist only one y ∈ Y such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \phi$ where $y_1 \in Y$ and $y_1 \neq y$. This shows that f is a constant map.

(iii) □ (ii) Let U be both $\psi^*\alpha$-open and $\psi^*\alpha$-closed in (X, τ). Suppose U ≠ ϕ. Let f : (X, τ) → (Y, σ) be a $\psi^*\alpha$-continuous map defined by f(U) = {y} and $f(U^c)$ = {w} for some distinct points y and w in (Y, σ). By assumption f is constant. Therefore y = w which implies U = X.

Theorem 5.6.16 If A is a $\psi^*\alpha$-connected subset of a topological space (X, τ) which has the $\psi^*\alpha$-separation X = E | F then A □ E or A ⊆ F.

Proof: Suppose that (X, τ) has the $\psi^*\alpha$-separation $X = E \mid F$. Then $X = E \cup F$ where E and F are two nonempty disjoint sets such that $[E \cap \psi^*\alpha cl(F)] \cup [\psi^*\alpha cl(E) \cap F] = \phi$. Since $E \cap F = \phi$, $E = F^c$ and $F = E^c$. Now $[(E \cap A) \cap \psi^*\alpha cl(F \cap A)] \cup [\psi^*\alpha cl(E \cap A) \cap (F \cap A)] \subseteq [E \cap \psi^*\alpha cl(F)] \cup [\psi^*\alpha cl(E) \cap F] = \phi$. Hence $A = (E \cap A) \mid (F \cap A)$ is a $\psi^*\alpha$-separation of A. Since A is $\psi^*\alpha$-connected, we have either $(E \cap A) = \phi$ or $(F \cap A) = \phi$. Consequently $A \subseteq E^c$ or $A \subseteq F^c$. Therefore $A \subseteq E$ or $A \subseteq F$.

Theorem 5.6.17 If A is $\psi^*\alpha$-connected and $A \subseteq B \subseteq \psi^*\alpha cl(A)$, then B is $\psi^*\alpha$-connected.

Proof: Suppose that B is not $\psi^*\alpha$-connected. Then $B = E \cup F$ where E and F are two nonempty disjoint sets such that $[E \cap \psi^*\alpha cl(F)] \cup [\psi^*\alpha cl(E) \cap F] = \phi$. Since A is $\psi^*\alpha$-connected by **Theorem 5.6.16**, $A \subseteq E$ or $A \subseteq F$. Suppose $A \subseteq E$. Then $F = F \cap B \subseteq F \cap \psi^*\alpha cl(A) \subseteq F \cap \psi^*\alpha cl(E) = \phi$. Consequently $F = \phi$. Similarly $E = \phi$ if $A \subseteq F$. This is a contradiction to the fact that E and F are nonempty. Hence B is $\psi^*\alpha$-connected.

Theorem 5.6.18 The union of any family of $\psi^*\alpha$-connected sets having a nonempty intersection is $\psi^*\alpha$-connected.

Proof: Let I be an index set and i □ I. Let $A = \cup A_i$, where each A_i is $\psi^*\alpha$-connected with $\cap A_i \neq \phi$. Suppose that A is not $\psi^*\alpha$-connected. Then $A = E \cup F$, where E and F are two nonempty disjoint sets such that $[E \cap \psi^*\alpha cl(F)] \cup [\psi^*\alpha cl(E) \cap F] = \phi$ .Since A_i is $\psi^*\alpha$-connected and $A_i \subseteq A$, by **Theorem 5.6.16**, $A_i \subseteq E$ or $A_i \subseteq F$. Since $\cap A_i \neq \phi$, there exists an element $x \in \cap A_i$. Therefore x □ A_i for all i. Suppose $A_i \subseteq E$ for some i. Then $x \in E$ since $E \cap F = \phi$, $x \notin F$. Hence $A_i \subseteq E$, for every i. Therefore $\cup A_i \subseteq E$. Therefore $A \subseteq E$. Hence $F = \phi$. Which is a contradiction. Therefore A is $\psi^*\alpha$-connected.

Theorem 5.6.19 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\psi^*\alpha$-continuous surjection and (X, τ) is $\psi^*\alpha$-connected then (Y, σ) is connected.

Proof: Suppose that (Y, σ) is not connected. Let $Y = A \cup B$ where A and B are disjoint non empty open sets in (Y, σ). Since f is $\psi^*\alpha$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non empty $\psi^*\alpha$-open sets in (X, τ). This contradicts the fact that (X, τ) is $\psi^*\alpha$-connected. Hence (Y, σ) is connected.

Theorem 5.6.20 If f : (X, τ) → (Y, σ) is $\psi^*\alpha$-irresolute surjection and (X, τ) is $\psi^*\alpha$-connected, then (Y, σ) is $\psi^*\alpha$-connected.

Proof: Suppose that (Y, σ) is not $\psi^*\alpha$-connected. Let Y = A ∪ B where A and B are disjoint non empty $\psi^*\alpha$-open sets in (Y, σ). Since f is $\psi^*\alpha$-irresolute and onto, X = $f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non empty $\psi^*\alpha$-open sets in (X, τ). This contradicts the fact that (X, τ) is $\psi^*\alpha$-connected. Hence (Y, σ) is $\psi^*\alpha$-connected.

Theorem 5.6.21 If (X, τ) is both ${}_{\psi^*\alpha}T_c$-space and connected, then (X, τ) is $\psi^*\alpha$-connected.

Proof: Suppose that (X, τ) is connected. Then X cannot be expressed as disjoint union of two nonempty proper open subsets of (X, τ). Suppose (X, τ) is not $\psi^*\alpha$-connected, then X = A ∪ B where A and B are two disjoint nonempty $\psi^*\alpha$-open sets. Since (X, τ) is a ${}_{\psi^*\alpha}T_c$-space, A and B are open in (X, τ). Hence (X, τ) is not connected which is a contradiction. Hence (X, τ) is $\psi^*\alpha$-connected.

Theorem 5.6.22 In a topological space (X, τ) with at least two points, if $\psi^*\alpha O(X, \tau) = \psi^*\alpha\, C(X, \tau)$ then (X, τ) is not $\psi^*\alpha$-connected.

Proof: By hypothesis we have $\psi^*\alpha O(X, \tau) = \psi^*\alpha C(X, \tau)$ Then there exists some nonempty proper subset of (X, τ) which is both $\psi^*\alpha$-open and $\psi^*\alpha$-closed in (X, τ). By **Theorem 5.6.15**, we have (X, τ) is not $\psi^*\alpha$-connected.