Chapter V

CHAPTER V

PREFERENCE ORDERED TOPOLOGICAL SPACES

Definition: 5.1

An asymmetric, negatively transitive binary relation $<$ on a set X is called a **preference**.

Definition: 5.2

A transitive, asymmetric (or irrefflexive) binary relation $<$ is called a **partial order**.

Definition: 5.3

With any partial order $<$ we may associate a binary relation called **indifference relation** , it is denoted by $\sim$ and defined $x \sim y$ if and only if not $x < y$, not $y < x$.

Result: 5.4

If $\sim$ is associated with a preference, then it is an equivalence reelation and conversely.

Definition : 5.5

By $x \leq y$ we mean $x < y$ or $x \sim y$.

Definition: 5.6

The **order topology** associated with a partial order $<$ (denoted by $\tau_{or}(<)$, or simply τ_{or} if no confusion is possible) is the topology which has the family formed by X and all the subsets $\{a \in X : a < x\}$ and $\{a \in X : x < a\}$ with $x \in X$ as a subbase of open sets.

Notation: 5.7

The equivalence classes by an equivalence binary relation R on a set will be denoted by $|x|_R$, or simply $|x|$ if no confusion is possible.

Definition: 5.8

Given a collection $\mathcal{S}$ of subsets of a set X, a binary relation R on X **saturates** $\mathcal{S}$ if x R y, $y \in S$ and $S \in \mathcal{S}$ imply $x \in S$.

Definition: 5.9

A quotient map between topological spaces $f : (X, \tau) \to (X', \tau')$ is a **saturated identification** if, for each $A \in \tau$, $A = f^{-1}f(A)$. We also say that $f : X \to X'$ is a saturated identification with the topologies τ, τ'.

Note: 5.10

Since an open (closed) continuous onto map is a quotient map, a saturated identification can be redefined as an open (closed) continuous onto map with $A = f^{-1}f(A)$ for all $A \in \tau$($C = f^{-1}f(C)$ for all C closed).

Note: 5.11

If f is the quotient map associated with an equivalence relation R on X, the condition that $A = f^{-1}f(A)$ for all $A \in \tau$ is equivalent to R saturates τ. Thus, for any equivalence binary relation on a topological space that saturates the topology, the projection onto the quotient space is open.

Theorem: 5.12

A partial order on a set is α preference if and only if the indifference associated with it saturates its order topology.

Remark: 5.13

If a preference on a set X is denoted by $<$ then $\sim$ will stand (or the indifference that $<$ induces. Furthermore, for this preference $<$ we denoted by $<'$ the linear order on $X / \sim$ defined in a natural way by $|x| <' |y|$ if and only if $a < b$ for all $a \in |x|$ and $b \in |y|$. Conversely, if R is an equivalence binary relation on X, then any linear order $<'$ on X / R induces a preference $<$ on X in a natural way according to the expression $x < y$ if and only if $|x|_R <' |y|_R$.

Theorem: 5.14

If $<$ is a preference on X and $<'$ is the induced linear order on $X / \sim$, then the projection map $p : X \to X / \sim$ is a saturated identification with the order topologies.

Definition: 5.15

A partial order $<$ is compatible with a topology τ if $\tau_{or}(<) \leq \tau$. An alternative term is to say that $<$ is continuous with respect to τ.

Definition: 5.16

Weakly orderable spaces are those which can be endowed with a continuous linear order. Orderable spaces are usually called **LOTS** (linearly ordered topological space), their subspaces are called **GO-spaces** (which stands for generalized ordered space). Thus, a LOTS is a topological space whose topology is induced by a linear order on it.

Definition: 5.17

A **GPO-space (Generalized Preference – Ordered Space)** is a triple $(X, \tau, <)$ where $<$ is a preference on X and τ is a topology on X such that

$\tau_{or} \leq \tau$ and τ has a base formed by convex sets ($C \subseteq X$ is convex if $p \leq z \leq q$ and $p, q \in C$ implies $z \in C$). A **POTS (Preference – Ordered Topological Space)** is a triple $(X, \tau_{or}, <)$ where $<$ is a preference on X.

Definition: 5.18

A **nest** is a collection $\mathcal{N}$ of sets with the property that for any two members N_1 and N_2 of $\mathcal{N}$ it is true that either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Definition: 5.19

A collection $\mathcal{S}$ of sets is **interlocking** provided that every set S_0 of $\mathcal{S}$ which is an intersection of strictly larger members of $\mathcal{S}$ has a representation as a union of strictly smaller members of $\mathcal{S}$, that is,

$$S_0 = \cap\{S : S_0 \subseteq S, S \in \mathcal{S} \setminus \{S_0\}\}$$

$$\Rightarrow \quad S_0 = \cup\{S : S \subseteq S_0, S \in \mathcal{S} \setminus \{S_0\}\}.$$

Theorem: 5.20

A T_1 topological space is a GO-space (resp. a LOTS) if and only if it has an open subbase consisting of two nests (resp. of two interlocking nests). A topological space admits a continuous linear order on it if and only if there exists two nests of open sets that together generate a T_1 topology.

Definition: 5.21

For any $<$ continuous preference on X, the collection of all the subsets $x^u : \{y \in X : x < y\}$ (resp. $x^l : \{y \in X : y < x\}$) with x ranging over X is an interlocking nest. These sets are called **upper (resp. lower) contour sets** associated with x.

Theorem: 5.22

Let (X, τ) be a topological space and let R be an equivalence binary relation on X. The following statements are equivalent:

(1) There exists a preference $<$ on X whose indifference coincides with R and such that $(X, \tau, <)$ is a GPO-space.

(2) R saturates τ, the equivalence classes by R are closed and τ has a subbase constituted by the union of two nests.

Proof

$(1) \Rightarrow (2)$. It is well known that, in any space ordered by a preference, the equivalence classes induced by its indifference are closed in its order topology. Besides, the indifference R saturates τ by the convexity of the open sets, and the nests of open subsets $\{x^u : x \in X\}$ and $\{x^l : x \in X\}$ fulfill the remaining requirement.

$(2) \Rightarrow (1)$. Suppose that τ has a subbase which is consitituted by the union of two nests $\mathcal{U}$ and $\mathcal{L}$. The projection p onto the quotient space by R is open because R saturates τ. We show that $p(\mathcal{U})$ and $p(\mathcal{L})$ are two nests of open sets whose union generates τ_{quot}, which is T_1.

It is clear that $p(\mathcal{U})$ and $p(\mathcal{L})$ are nests of open sets. Now , let $A \in \tau_{quot}$ and $x' \in A$. Take an element $x \in p^{-1}(x')$; there must exist

$\{A_i : i = 1, 2, \ldots. n\} \subseteq \mathcal{U}$ and $\{B_j : j = 1, 2, \ldots. m\} \subseteq \mathcal{L}$ such that

$x \in \cap\{A_i : i = 1, 2, \ldots\ldots n\} \cap \{B_j : j = 1, \ldots\ldots m\} \subseteq p^{-1}(A)$.

Since R saturates τ, it follows that

$p(\cap \{ A_i : i = 1, \ldots\ldots n\} \cap \{B_j : j = 1 \ldots\ldots m\})$

$= \cap \{p(A_i) : i = 1 \ldots\ldots\ldots n\} \cap \{p(B_j) : j = 1 \ldots\ldots\ldots m\}$.

Thus $x' \in \cap \{p(A_i) : i = 1,2,3,\ldots\ldots\ldots n\} \cap \{p(B_j) : j = 1,2,3,\ldots\ldots\ldots m\}$

$= p(\cap \{A_i : i = 1, \ldots\ldots\ldots n\} \cap \{B_j : j = 1 \ldots\ldots\ldots m\}) \subset pp^{-1}(A) = A$.

This shows that the union of $p(\mathcal{U})$ and $p(\mathcal{L})$ is a subbase for τ_{quot}.

From Theorem 5.20, there exists a linear order <' on X / R verifying that $\tau_{or}(<') \leq \tau_{quot}$ and such that the open sets of a base $\mathcal{B}'$ of τ_{quot} are convex respect to <'. Let < be the preference on X that induces <' on X / R.

By Theorem 5.14 the projection p is a saturated identification with $T_{or}(<)$ and $\tau_{or}(<')$, and thus $A \in \tau_{or}(<)$ implies $p(A) \in \tau_{or}(<')$, therefore $p(A) \in \tau_{quot}$. Since R saturates $\tau_{or}(<)$ we deduce from $A \in \tau_{or}(<)$ that $A = p^{-1}p(A) \in \tau$. Therefore $\tau_{or}(<) \leq \tau$.

On the other hand, the collection $\mathcal{B} = \{p^{-1}(B'); B' \in \mathcal{B}'\}$ is a base of τ because p is a saturated identification with τ and τ_{quot}. Thus collection is formed by convex sets with respect to the preference <, which permits to conclude that (X, τ, <) is a GPO-space.

Theorem: 5.23

Let (X, τ) be a topological space and let R be an equivalence binary relation on X. The following statements are equivalent:

(1) There exists a preference < on X whose indifference coincides with R and such that (X, τ, $\leq$) is a POTS.

(2) R saturates τ, the equivalence classes by R are closed and τ has a subbase constituted by the union of two interlocking nests.

Proof

Again, the implication (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (1). Suppose that τ has a subbase that is constituted by the union of two interlocking nests, namely $\mathcal{U}$ and $\mathcal{L}$. The projection p onto the quotient space by R is open because R saturates τ. As in Theorem 5.22, $p(\mathcal{U})$ and $P(\mathcal{L})$ are two nests of open subsets that generate τ_{quot}, which is T_1. We show that $p(\mathcal{S})$ is interlocking for each $\mathcal{S}$ interlocking collection of subsets of X such that R saturates $\mathcal{S}$, which implies that $p(\mathcal{U})$ and $P(\mathcal{L})$ are interlocking.

Assume that S_0 is a set of $\mathcal{S}$ such that

$$p(S_0) = \cap\{p(S) : p(S_0) \subseteq p(S), p(S) \in p(\mathcal{S}) \setminus \{p(S_0)\}\}.$$

Since R saturates $\mathcal{S}$.

$$p^{-1}p(S_0) = \cap\{S : p(S_0) \subseteq p(S),\ p(S) \in p(\mathcal{S}) \setminus \{p(S_0)\}\}$$

Now $\{S : p(S_0) \subseteq p(S),\ p(S) \in p(\mathcal{S}) \setminus \{ p(S_0)\}\} = \{S : S_0 \subseteq S,\ S \in \mathcal{S} \setminus \{S_0\}\}$, which will permit to express $S_0 = p^{-1}p\,(S_0) = \cap\{S : S_0 \subseteq S,\ S \in \mathcal{S} \setminus \{S_0\}\}$. Because $\mathcal{S}$ is interlocking we can express $S_0 = \cup\{S : S \subseteq S_0,\ S \in \mathcal{S} \setminus \{S_0\}\}$.

From this it follows that $p(S_0) = \cup\{p(S) : S \subseteq S_0, S \in \mathcal{S}^{\mathcal{S}} \setminus \{S_0\}\}$ We may now conclude the argument because

$$\{p(S) : p(S) \subseteq p(S_0),\ p(S) \in p(\mathcal{S}) \setminus \{p(S_0)\}\}, = \{p(S) : S \subseteq S_0,\ S \in \mathcal{S} \setminus \{S_0\}\},$$

and therefore $p(S_0) = \cup\{p(S) : p(S) \subseteq p(S_0),\ p(S) \in p\,(\mathcal{S}\,) \setminus \{p(S_0)\}\}$.

From Theorem 5.20, there exists a linear order <' on X / R such that $\tau_{quot} = \tau_{or}\,(<')$. Let < be the preference on X that induces <' on X / R , we show that $\tau = \tau_{or}\,(<)$.

Indeed, form Theorem 5.14 the projection p is a saturated identification with $\tau_{or}\,(<)$ and $\tau_{or}\,(<')$; therefore, from $A \in \tau_{or}(<)$ we deduced $p(A) \in \tau_{or}(<')$, or equilently, ,$p(A) \in \tau_{quot}$. Besides R saturates $\tau_{or}(<)$ because it is the indifference associated with < , thus $A \in \tau_{or}(<) \Rightarrow p^{-1}p(A) = A$.

It follows that $\tau_{or}(<) \leq \tau$ because

$A \in \tau_{or}(<) \Rightarrow p(A) \in \tau_{quot} \Rightarrow p^{-1}p(A) \in \tau \Rightarrow A \in \tau$.

On the other hand, if $A \in \tau$ then $p(A) \in \tau_{quot}$ and now $\tau_{or}(<') = \tau_{quot}$ yields $p^{-1}p(A) \in \tau_{or}(<)$ because $p : (X , \tau_{or} (<)) - (X / R, \tau_{or} (<'))$ is continuous (Theorem 5.14). From $A \in \tau$ we obtain that $A = p^{-1}p(A)$ because R saturates τ, and thus $A \in \tau$ yields $p^{-1}p(A) \in \tau_{or}(<)$ and so $A \in \tau_{or}(<)$. Therefore $\tau \leq \tau_{or}(<)$.

Theorem: 5.24

Let (X, τ) be a topological space and let R be an equivalence binary relation on X. The following statements are equivalent:

(1) There exists a preference on X which is continuous with respect to τ and whose indifference coincides with R.

(2) There exist two nests of open sets of τ whose union generates τ' such that R saturates τ' and the equivalence classes by R are closed in τ'.

Proof

The implication (1) $\Rightarrow$ (2) immediate.

(2) $\Rightarrow$ (1). As $p : (X, \tau') \rightarrow (X / R, (\tau')_{quot})$ is open, then the elements of $p(\mathcal{U})$ and $p(\mathcal{L})$ are open in τ_{quot}. As in Theorem 5.22 $p(\mathcal{U})$ and $p(\mathcal{L})$ are two nests whose union generates $(\tau')_{quot}$, which is T_1. From Theorem 5.20 there exists a linear order <' on X / R that is compatible with T_{quot}.

Let < be the preference on X whose indifference coincides with R and naturally induces <' on X/R. Then the fact that the indifference R associated with < saturates the order topology $\tau_{or}(<)$ and Theorem 5.14 yield that < is the desired preference.