CHAPTER 5

Aggregation Operators for Neutrosophic Variants

(5.1) Arithmetic and Geometric Operators

(5.2) Efficient CODAS Technique for MCDM using Neutrosophic Variants

This chapter introduces novel operators such as arithmetic mean and geometric mean, score function and accuracy functions for neutrosophic variants. Furthermore, it introduces the CODAS (Combinative Distance-based Assessment) technique tailored for addressing multiple-critria decision making problems utilizing the newly defined operators.

5.1 Arithmetic and Geometric Operators

The aggregation operators for NSS and FNS are defined and claimed that its play crucial role in decision-making processes, especially in contexts where multiple criteria need to be considered in complex uncertain environments. These operators help in combining multiple inputs into a single value, facilitating better decision-making.

This section discusses the aggregation of NSS and FNS through arithmetic operators, geometric operators, and de-neutrosophication processes.

Neutrosophic spherical aggregation operators

Definition 5.1.1

Let Z be a non empty set (universe of discourse). Let A_{S1} and A_{S2} be two NSSs in Z denoted by $A_{S1} = \{< s, (T_{A_{S1}}(s), I_{A_{S1}}(s), F_{A_{S1}}(s)) > |s \in Z\}$, $A_{S2} = \left\{< s, \left(T_{A_{S2}}(s), I_{A_{S2}}(s), F_{A_{S2}}(s)\right) > \middle| s \in Z\right\}$. Then the basic operational laws of NSS are defined as below:

$$A_{S1} \oplus A_{S2} = \begin{Bmatrix} \left({T_{A_{S1}}}^2 + {T_{A_{S2}}}^2 - {T_{A_{S1}}}^2 {T_{A_{S2}}}^2\right)^{\frac{1}{2}}, \\ \left({I_{A_{S1}}}^2 + {I_{A_{S2}}}^2 - {I_{A_{S1}}}^2 {I_{A_{S2}}}^2\right)^{\frac{1}{2}}, \\ \left({F_{A_{S1}}}^2 + {F_{A_{S2}}}^2 - {F_{A_{S1}}}^2 {F_{A_{S2}}}^2\right)^{\frac{1}{2}} \end{Bmatrix} \tag{5.1.1}$$

$$A_{S1} \otimes A_{S2} = \left\{\left(T_{A_{S1}} T_{A_{S2}}\right), \left(I_{A_{S1}} I_{A_{S2}}\right), \left(F_{A_{S1}} F_{A_{S2}}\right)\right\} \tag{5.1.2}$$

$$\lambda \bullet A_{S1} = \begin{Bmatrix} \left(1 - \left(1 - {T_{A_{S1}}}^2\right)^{\lambda}\right)^{\frac{1}{2}}, \\ \left(1 - \left(1 - {I_{A_{S1}}}^2\right)^{\lambda}\right)^{\frac{1}{2}}, \\ \left(1 - \left(1 - {F_{A_{S1}}}^2\right)^{\lambda}\right)^{\frac{1}{2}} \end{Bmatrix} \tag{5.1.3}$$

$${A_{S1}}^{\lambda} = \left\{{T_{A_{S1}}}^{\lambda}, {I_{A_{S1}}}^{\lambda}, {F_{A_{S1}}}^{\lambda}\right\}, \lambda > 0 \tag{5.1.4}$$

Proposition 5.1.2

Let $A_{S1} = \left(T_{A_{S1}}, I_{A_{S1}}, F_{A_{S1}}\right)$ and $A_{S2} = \left(T_{A_{S2}}, I_{A_{S2}}, F_{A_{S2}}\right)$ be two NSSs. Let $\lambda, \lambda_1, \lambda_2 > 0$ be positive scalars. The operational laws are as follows:

1. $A_{S1} \oplus A_{S2} = A_{S2} \oplus A_{S1}$ (5.1.5)
2. $A_{S1} \otimes A_{S2} = A_{S1} \otimes A_{S2}$ (5.1.6)
3. $\lambda_1(A_{S1} \oplus A_{S2}) = \lambda_1 A_{S1} \oplus \lambda_1 A_{S2}$ (5.1.7)
4. $\lambda_1 A_{S1} \oplus \lambda_2 A_{S1} = (\lambda_1 + \lambda_2) A_{S1}$ (5.1.8)

$$5.\ (A_{S1}\otimes A_{S2})^{\lambda} = {A_{S1}}^{\lambda}\otimes {A_{S2}}^{\lambda} \tag{5.1.9}$$

$$6.\ {A_{S1}}^{\lambda_1}\otimes {A_{S1}}^{\lambda_2} = {A_{S1}}^{\lambda_1+\lambda_2} \tag{5.1.10}$$

Proof:

1. To prove: $A_{S1}\oplus A_{S2} = A_{S2}\oplus A_{S1}$

$$A_{S1}\oplus A_{S2} = \begin{Bmatrix} \left({T_{A_{S1}}}^{2}+{T_{A_{S2}}}^{2}-{T_{A_{S1}}}^{2}{T_{A_{S2}}}^{2}\right)^{\frac{1}{2}}, \\ \left({I_{A_{S1}}}^{2}+{I_{A_{S2}}}^{2}-{I_{A_{S1}}}^{2}{I_{A_{S2}}}^{2}\right)^{\frac{1}{2}}, \\ \left({F_{A_{S1}}}^{2}+{F_{A_{S2}}}^{2}-{F_{A_{S1}}}^{2}{F_{A_{S2}}}^{2}\right)^{\frac{1}{2}} \end{Bmatrix} \tag{A1}$$

$$A_{S2}\oplus A_{S1} = \begin{Bmatrix} \left({T_{A_{S2}}}^{2}+{T_{A_{S1}}}^{2}-{T_{A_{S2}}}^{2}{T_{A_{S1}}}^{2}\right)^{\frac{1}{2}}, \\ \left({I_{A_{S2}}}^{2}+{I_{A_{S1}}}^{2}-{I_{A_{S2}}}^{2}{I_{A_{S1}}}^{2}\right)^{\frac{1}{2}}, \\ \left({F_{A_{S2}}}^{2}+{F_{A_{S1}}}^{2}-{F_{A_{S2}}}^{2}{F_{A_{S1}}}^{2}\right)^{\frac{1}{2}} \end{Bmatrix} \tag{A2}$$

Equation $(A1)$ = Equation $(A2)$. Hence 1 is proved.

2. To prove: $A_{S1}\otimes A_{S2} = A_{S1}\otimes A_{S2}$

$$A_{S1}\otimes A_{S2} = \left\{\left(T_{A_{S1}}T_{A_{S2}}\right),\left(I_{A_{S1}}I_{A_{S2}}\right),\left(F_{A_{S1}}F_{A_{S2}}\right)\right\} \tag{B1}$$

$$A_{S2}\otimes A_{S1} = \left\{\left(T_{A_{S2}}T_{A_{S1}}\right),\left(I_{A_{S2}}I_{A_{S1}}\right),\left(F_{A_{S2}}F_{A_{S1}}\right)\right\} \tag{B2}$$

Equation $(B1)$ = Equation $(B2)$. Hence 2 is proved.

3. To prove: $\lambda_1(A_{S1}\oplus A_{S2}) = \lambda_1 A_{S1}\oplus \lambda_1 A_{S2}$

$$\lambda_1(A_{S1}\oplus A_{S2}) = \lambda_1\begin{Bmatrix} \left({T_{A_{S1}}}^{2}+{T_{A_{S2}}}^{2}-{T_{A_{S1}}}^{2}{T_{A_{S2}}}^{2}\right)^{\frac{1}{2}}, \\ \left({I_{A_{S1}}}^{2}+{I_{A_{S2}}}^{2}-{I_{A_{S1}}}^{2}{I_{A_{S2}}}^{2}\right)^{\frac{1}{2}}, \\ \left({F_{A_{S1}}}^{2}+{F_{A_{S2}}}^{2}-{F_{A_{S1}}}^{2}{F_{A_{S2}}}^{2}\right)^{\frac{1}{2}} \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(1-\left(1-\left({T_{A_{S1}}}^{2}+{T_{A_{S2}}}^{2}-{T_{A_{S1}}}^{2}{T_{A_{S2}}}^{2}\right)\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-\left({I_{A_{S1}}}^{2}+{I_{A_{S2}}}^{2}-{I_{A_{S1}}}^{2}{I_{A_{S2}}}^{2}\right)\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-\left({F_{A_{S1}}}^{2}+{F_{A_{S2}}}^{2}-{F_{A_{S1}}}^{2}{F_{A_{S2}}}^{2}\right)\right)^{\lambda_1}\right)^{\frac{1}{2}} \end{Bmatrix} \tag{C1}$$

$$\lambda_1 A_{S1} \oplus \lambda_1 A_{S2} = \left\{\begin{matrix} \left(1-\left(1-T_{A_{S1}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-I_{A_{S1}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-F_{A_{S1}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}} \end{matrix}\right\} \oplus \left\{\begin{matrix} \left(1-\left(1-T_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-I_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-F_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \left(1-\left(1-T_{A_{S1}}^2\right)^{\lambda_1}+1-\left(1-T_{A_{S2}}^2\right)^{\lambda_1}-\left(1-\left(1-T_{A_{S1}}^2\right)^{\lambda_1}\right)\left(1-\left(1-T_{A_{S2}}^2\right)^{\lambda_1}\right)\right)^{\frac{1}{2}}, \\ \left(1-\left(1-I_{A_{S1}}^2\right)^{\lambda_1}+1-\left(1-I_{A_{S2}}^2\right)^{\lambda_1}-\left(1-\left(1-I_{A_{S1}}^2\right)^{\lambda_1}\right)\left(1-\left(1-I_{A_{S2}}^2\right)^{\lambda_1}\right)\right)^{\frac{1}{2}}, \\ \left(1-\left(1-F_{A_{S1}}^2\right)^{\lambda_1}+1-\left(1-F_{A_{S2}}^2\right)^{\lambda_1}-\left(1-\left(1-F_{A_{S1}}^2\right)^{\lambda_1}\right)\left(1-\left(1-F_{A_{S2}}^2\right)^{\lambda_1}\right)\right)^{\frac{1}{2}} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \left(1-\left(1-T_{A_{S1}}^2\right)^{\lambda_1}\left(1-T_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-I_{A_{S1}}^2\right)^{\lambda_1}\left(1-I_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-F_{A_{S1}}^2\right)^{\lambda_1}\left(1-F_{A_{S2}}^2\right)^{\lambda_1}\right)^{\frac{1}{2}} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \left(1-\left(1-\left(T_{A_{S1}}^2+T_{A_{S2}}^2-T_{A_{S1}}^2T_{A_{S2}}^2\right)\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-\left(I_{A_{S1}}^2+I_{A_{S2}}^2-I_{A_{S1}}^2I_{A_{S2}}^2\right)\right)^{\lambda_1}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-\left(F_{A_{S1}}^2+F_{A_{S2}}^2-F_{A_{S1}}^2F_{A_{S2}}^2\right)\right)^{\lambda_1}\right)^{\frac{1}{2}} \end{matrix}\right\} \qquad (C2)$$

Equation $(C1)$ = Equation $(C2)$. Hence 3 is proved.

4. To prove: $\lambda_1 A_{S1} \oplus \lambda_2 A_{S1} = (\lambda_1+\lambda_2)A_{S1}$

   Proof is similar to (5.1.7).

5. To prove: $(A_{S1} \otimes A_{S2})^{\lambda} = {A_{S1}}^{\lambda} \otimes {A_{S2}}^{\lambda}$

$$(A_{S1} \otimes A_{S2})^{\lambda} = \left\{\left(T_{A_{S1}}T_{A_{S2}}, I_{A_{S1}}I_{A_{S2}}, F_{A_{S1}}F_{A_{S2}}\right)^{\lambda}\right\}$$

$$= \left\{{T_{A_{S1}}}^{\lambda}{T_{A_{S2}}}^{\lambda}, {I_{A_{S1}}}^{\lambda}I_{A_{S2}}, {F_{A_{S1}}}^{\lambda}{F_{A_{S2}}}^{\lambda}\right\} \qquad (D1)$$

$${A_{S1}}^{\lambda} \otimes {A_{S2}}^{\lambda} = \left\{{T_{A_{S1}}}^{\lambda}, {I_{A_{S1}}}^{\lambda}, {F_{A_{S1}}}^{\lambda}\right\} \otimes \left\{{T_{A_{S2}}}^{\lambda}, {I_{A_{S2}}}^{\lambda}, {F_{A_{S2}}}^{\lambda}\right\}$$

$$= \left\{{T_{A_{S1}}}^{\lambda}{T_{A_{S2}}}^{\lambda}, {I_{A_{S1}}}^{\lambda}{I_{A_{S2}}}^{\lambda}, {F_{A_{S1}}}^{\lambda}{F_{A_{S2}}}^{\lambda}\right\} \qquad \text{(D2)}$$

Equation $(D1) =$ Equation $(D2)$. Hence 5 is proved.

6. To prove: ${A_{S1}}^{\lambda_1} \otimes {A_{S1}}^{\lambda_2} = {A_{S1}}^{\lambda_1+\lambda_2}$
   Proof is similar to (5.1.9).

Definition 5.1.3

Let $A_{S_j} = \langle T_{S_j}, I_{S_j}, F_{S_j}\rangle$ be a collection neutrosophic spherical numbers and $j = 1,2,3,\dots,n$ and n is the number of decision makers and $z = (z_1, z_2, z_3, \dots. z_n)$ be their accociated weight vector with $z_j \in [0,1]$; $\sum_{j=1}^{n} z_j \le \sqrt{3}$. Then the spherical weighted arithmetic mean (SWAM) operator is defined as:

$$SWAM_z\left(\tilde{A}_{S1}, \tilde{A}_{S2}, \dots \tilde{A}_{Sn}\right) = z_1\tilde{A}_{S1} + z_2\tilde{A}_{S2} + z_3\tilde{A}_{S3} + \dots. + z_n\tilde{A}_{Sn}$$

$$\left\{\left[1 - \prod_{j=1}^{n}\left(1 - {T_{A_S}}^2\right)^{z_j}\right]^{\frac{1}{2}}, \left[1 - \prod_{j=1}^{n}\left(1 - {I_{A_S}}^2\right)^{z_j}\right]^{\frac{1}{2}}, \left[1 - \prod_{j=1}^{n}\left(1 - {F_{A_S}}^2\right)^{z_j}\right]^{\frac{1}{2}}\right\} \quad (5.1.11)$$

Definition 5.1.4

Let $A_{S_j} = \langle T_{S_j}, I_{S_j}, F_{S_j}\rangle$ be a collection neutrosophic spherical numbers and $j = 1,2,3,\dots,n$ and n is the number of decision makers and $z = (z_1, z_2, z_3, \dots. z_n)$ be their accociated weight vector with $z_j \in [0,1]$; $\sum_{j=1}^{n} z_j \le \sqrt{3}$. Then the spherical weighted geometric mean (SWGM) operator is defined as:

$$SWGM_z(\tilde{A}_{S1},\tilde{A}_{S2},\ldots\tilde{A}_{Sn}) = \tilde{A}_{S1}{}^{z_1} + \tilde{A}_{S2}{}^{z_2} + \tilde{A}_{S3}{}^{z_3} + \ldots + \tilde{A}_{Sn}{}^{z_n}$$

$$\left\{\prod_{j=1}^{n} T_{A_S}{}^{z_j}, \prod_{j=1}^{n} I_{A_S}{}^{z_j}, \prod_{j=1}^{n} F_{A_S}{}^{z_j}\right\} \tag{5.1.12}$$

Definition 5.1.5

Let $A_S = \{\langle x, T_{A_S}(x), I_{A_S}(x), F_{A_S}(x)\rangle | x \in X\}$ on X be any NSS, then a score function (SF) and accuracy function (AF) for NSS classification are defined as:

$$Score(A_S) = (T_{A_S} - F_{A_S})^2 - (I_{A_S} - F_{A_S})^2 \tag{5.1.13}$$

$$Accuracy(A_S) = T_{A_S}{}^2 + I_{A_S}{}^2 + F_{A_S}{}^2 \tag{5.1.14}$$

Definition 5.1.6

Let $A_S = \{\langle x, T_{A_S}(x), I_{A_S}(x), F_{A_S}(x)\rangle | x \in X\}$,

$B_S = \{\langle x, T_{B_S}(x), I_{B_S}(x), F_{B_S}(x)\rangle | x \in X\}$

be any two NSSs.

$$\left.\begin{array}{l}\text{If } Score(A_S) < Score(B_S) \text{ then } A_S < B_S \\ \text{If } Score(A_S) > Score(B_S) \text{ then } A_S > B_S\end{array}\right\} \tag{5.1.15}$$

If $Score(A_S) = Score(B_S)$ then $Accuracy(A_S)$ is verified.

- If $Accuracy(A_S) > Accuracy(B_S)$ then $A_S > B_S$
- If $Accuracy(A_S) < Accuracy(B_S)$ then $A_S < B_S$

Fermatean neutrosophic aggregation operators

Definition 5.1.7

Consider a non empty set Z (universe of discourse). Let A_F and B_F be two FNSs in Z structured by $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)\rangle | x \in Z\}$,

$B_F = \{\langle x, T_{B_F}(x), I_{B_F}(x), F_{B_F}(x)\rangle | x \in Z\}$. Then the basic operational laws of FNSs are defined as,

$$A_F \oplus B_F = \begin{Bmatrix} \left({T_{A_F}}^3 + {T_{B_F}}^3 - {T_{A_F}}^3 {T_{B_F}}^3\right)^{\frac{1}{3}}, \\ \left({I_{A_F}}^3 + {I_{B_F}}^3 - {I_{A_F}}^3 {I_{B_F}}^3\right)^{\frac{1}{3}}, \\ \left({F_{A_F}}^3 + {F_{B_F}}^3 - {F_{A_F}}^3 {F_{B_F}}^3\right)^{\frac{1}{3}} \end{Bmatrix} \tag{5.1.16}$$

$$A_F \otimes B_F = \{(T_{A_F} T_{B_F}), (I_{A_F} I_{B_F}), (F_{A_F} F_{B_F})\} \tag{5.1.17}$$

$$\lambda \bullet A_F = \{\lambda(T_{A_F}), \lambda(I_{A_F}), \lambda(F_{A_F})\} \tag{5.1.18}$$

$${A_F}^{\lambda} = \{{T_{A_F}}^{\lambda}, {I_{A_F}}^{\lambda}, {F_{A_F}}^{\lambda}\}, \lambda \geq 0 \tag{5.1.19}$$

Proposition 5.1.8

Let $A_F = (T_{A_F}, I_{A_F}, F_{A_F})$ and $B_F = (T_{B_F}, I_{B_F}, F_{B_F})$ be two FNSs. Let $\lambda, \lambda_1, \lambda_2 > 0$ be positive scalars. The operational laws are defined as follows:

i) $A_F \oplus B_F = B_F \oplus A_F$ (5.1.20)

ii) $A_F \otimes B_F = B_F \otimes A_F$ (5.1.21)

iii) $\lambda(A_F \oplus B_F) = \lambda A_F \oplus \lambda B_F$ (5.1.23)

iv) $\lambda_1 A_F \oplus \lambda_2 A_F = (\lambda_1 + \lambda_2) A_F$ (5.1.23)

v) $(A_F \otimes B_F)^{\lambda} = {A_F}^{\lambda} \otimes {B_F}^{\lambda}$ (5.1.24)

vi) ${A_F}^{\lambda_1} \otimes {A_F}^{\lambda_2} = {A_F}^{\lambda_1 + \lambda_2}$ (5.1.25)

Proof : This proof is similar to Proposition 5.1.2.

Definition 5.1.9

Let $A_{F_j} = \langle T_{F_j}, I_{F_j}, F_{F_j} \rangle$ be a collection Fermatean neutrosophic numbers and $j = 1,2,3,\dots,n$ and n is the number of decision makers and $z = (z_1, z_2, z_3, \dots. z_n)$ be their accociated weight vector with $z_j \in [0,1]$; $\sum_{j=1}^{n} z_j \leq 2$. Then the Fermatean weighted arithmetic mean (FWAM) operator is defined as:

$$FWAM_z(A_{F1}, A_{F2}, \dots A_{Fn}) = z_1 A_{F1} + z_2 A_{F2} + z_3 A_{F3} + \dots. + z_n A_{Fn}$$

$$\left\{ \begin{array}{c} \left[1 - \prod_{j=1}^{n} \left(1 - {T_{A_F}}^{3}\right)^{z_j} \right]^{\frac{1}{3}}, \left[\prod_{j=1}^{n} \left(I_{A_F}\right)^{z_j} \right]^{\frac{1}{3}}, \\ \left[\prod_{j=1}^{n} \left(1 - {T_{A_F}}^{3}\right)^{z_j} - \prod_{j=1}^{n} \left(1 - {T_{A_F}}^{3} - {F_{A_F}}^{3}\right)^{z_j} \right]^{\frac{1}{3}} \end{array} \right\} \tag{5.1.26}$$

Definition 5.1.10

Let $A_{F_j} = \langle T_{F_j}, I_{F_j}, F_{F_j} \rangle$ be a collection Fermatean neutrosophic numbers and $j = 1,2,3,\dots,n$ and n is the number of decision makers and $z = (z_1, z_2, z_3, \dots. z_n)$ be their accociated weight vector with $z_j \in [0,1]$; $\sum_{j=1}^{n} z_j \leq 2$. Then the Fermatean weighted geometric mean (FWGM) operator is defined as:

$$FWGM_z\left(A_{F_1}, A_{F_2}, \dots A_{F_n}\right) = {A_{F_1}}^{z_1} + {A_{F_2}}^{z_2} + {A_{F_3}}^{z_3} + \dots. + {A_{F_n}}^{z_n}$$

$$\left\{ \begin{array}{c} \prod_{j=1}^{n} {T_{A_F}}^{z_j}, \left[1 - \prod_{j=1}^{n} \left(1 - {I_{A_F}}^{3}\right)^{z_j} \right]^{\frac{1}{3}}, \\ \prod_{j=1}^{n} \left(1 - {I_{A_F}}^{3}\right)^{z_j} - \prod_{j=1}^{n} \left(1 - {I_{A_F}}^{3} - {F_{A_F}}^{3}\right)^{z_j} \end{array} \right\} \tag{5.1.27}$$

Definition 5.1.11

Let $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)\rangle | x \in X\}$ on X be any FNS, then a score function (SF) and accuracy function (AF) for FNS classification are defined as:

$$Score(A_F) = \left(T_{A_F} - F_{A_F}\right)^3 - \left(I_{A_F} - F_{A_F}\right)^3 \tag{5.1.28}$$

$$Accuracy(A_F) = {T_{A_F}}^3 + {I_{A_F}}^3 + {F_{A_F}}^3 \tag{5.1.29}$$

Definition 5.1.12

Let $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)\rangle | x \in X\}$,

$B_F = \{\langle x, T_{B_F}(x), I_{B_F}(x), F_{B_F}(x)\rangle | x \in X\}$ be any two FNSs.

- If $Score(A_F) < Score(B_F)$ then $A_F < B_F$
- If $Score(A_F) > Score(B_F)$ then $A_F > B_F$

If $Score(A_F) = Score(B_F)$ then $Accuracy(A_F)$ is verified.

- If $Accuracy(A_F) > Accuracy(B_F)$ then $A_F > B_F$
- If $Accuracy(A_F) < Accuracy(B_F)$ then $A_F < B_F$
- If $Accuracy(A_F) = Accuracy(B_F)$ then $A_F = B_F$

5.2 Efficient CODAS Technique for MCDM Method Using Neutrosophic Variants

This section developes a novel type of NS CODAS and FN CODAS that evaluates and ranks alternatives based on multiple criteria to captures fine variations more precisely. This method involves normalizing the data of each criteria, assigning weights, and calculating the Euclidean distances of each alternative from positive ideal solution (represents the best possible outcome for the same set of criteria) and negative-ideal solution (represents the worst possible outcome for the same set of criteria). These distances are combined to form a proximity ratio, with higher scores indicating better alternatives. This developed CODAS methodes are used to select the best alternative when all the alternatives are equally important.

5.2.1 Neutrosophic Spherical / Fermatean Neutrosophic CODAS

Algorithm for CODAS technique for NSS and FNS

Suppose that $S = \{s_1, s_2, s_3, \dots . s_m\}$, $(m \geq 2)$ represents distinct collection of m possible alternatives and $p = 1,2,3, \dots, k$ and p is the number of decision makers and $z = (z_1, z_2, z_3, \dots . z_k)$ be their accociated weight vector derived from every requirement that meet $0 \leq z_p \leq 1$ and $\sum_{p=1}^{k} z_p \leq \sqrt{3}$ in neutrosophic spherical set and $0 \leq z_p \leq 1$ and $\sum_{p=1}^{k} z_p \leq 2$ in Fermatean neutrosophic set.

Step 1. Let Decision Makers (ĐMs) use the linguistic terms (LT) to complete the assessment matrices for decisions and criteria according to the neutrosophic variants.

Step 2. Aggregate the outcomes of ĐMs using neutrosophic spherical / Fermatean neutrosophic linguistic judgements of each criteria. Assemble neutrosophic decision matrix based on ĐMs' views and normalize the decision matrix.

Let $S_i(i = 1,2,\ldots m)$ be the set of alternative, and $K_j(j = 1,2\ldots n)$ be the set of criteria. The NS D-Mx / FN D-Mx is of the form $K_j(S_i) = (T_{ij}, I_{ij}, F_{ij})$, which represents the evaluation information.

At the very start, the evaluation information is normalized because there exists benefit criteria and cost criteria. Such criteria react oppositely, i.e., the bigger value reveals better performance of a benefit criterion but reflects the worse performance of a cost criterion. Therefore, in order to ensure all criteria are compatible, continue to shift the cost criterion into benefit criterion by the following formula.

$$K_j(\tilde{S}_i) = (\tilde{T}_{ij}, \tilde{I}_{ij}, \tilde{F}_{ij}) = \begin{cases} (T_{ij}, I_{ij}, F_{ij}), & K_j \text{ is benefit criterion} \\ (F_{ij}, I_{ij}, T_{ij}), & K_j \text{ is cost criterion} \end{cases}$$

$D = \left(K_j(\tilde{S}_i)\right)_{m\times n}$ is a normalized NS D-Mx / FN D-Mx for MCDM problem using neutrosophic variants, as shown in equation (5.2.1.1).

$$\left(K_j(\tilde{S}_i)\right)_{m\times n} = \begin{pmatrix} (\tilde{T}_{11}, \tilde{I}_{11}, \tilde{F}_{11}) & (\tilde{T}_{12}, \tilde{I}_{12}, \tilde{F}_{12}) & \cdots & (\tilde{T}_{1n}, \tilde{I}_{1n}, \tilde{F}_{1n}) \\ (\tilde{T}_{21}, \tilde{I}_{21}, \tilde{F}_{21}) & (\tilde{T}_{22}, \tilde{I}_{22}, \tilde{F}_{22}) & \cdots & (\tilde{T}_{2n}, \tilde{I}_{2n}, \tilde{F}_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ (\tilde{T}_{m1}, \tilde{I}_{m1}, \tilde{F}_{m1}) & (\tilde{T}_{m2}, \tilde{I}_{m2}, \tilde{F}_{m2}) & \cdots & (\tilde{T}_{mn}, \tilde{I}_{mn}, \tilde{F}_{mn}) \end{pmatrix} \quad (5.2.1.1)$$

Step 3. Calculate the aggregated weighted normalized NS D-Mx / FN D-Mx $D_Z = \left(K_j(\tilde{S}_{iz})\right)_{m\times n}$ using the aggregation operation given in Equations (5.1.1 & 5.1.2) and (5.1.16 & 5.1.17) for neutrosophic spherical and Fermatean neutrosophic sets respectively. The aggregated weighted normalized NS D-Mx / FN D-Mx can be defined as follows:

$$D_Z = \left(K_j(\tilde{S}_{iz})\right)_{m\times n} = \begin{pmatrix} (T_{11z}, I_{11z}, F_{11z}) & (T_{12z}, I_{12z}, F_{12z}) & \cdots & (T_{1nz}, I_{1nz}, F_{1nz}) \\ (T_{21z}, I_{21z}, F_{21z}) & (T_{22z}, I_{22z}, F_{22z}) & \cdots & (T_{2nz}, I_{2nz}, F_{2nz}) \\ \vdots & \vdots & \cdots & \vdots \\ (T_{m1z}, I_{m1z}, F_{m1z}) & (T_{m2z}, I_{m2z}, F_{m2z}) & \cdots & (T_{mnz}, I_{mnz}, F_{mnz}) \end{pmatrix} \quad (5.2.1.2)$$

Step 4. Utilising equation (5.1.13/5.1.28), de-neutrosophy the aggregated weighted normalized D-Mx.

Step 5. Find the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) according to the score function (SF) acquired in Step 4.

Regarding the PIS

$$S^* = \left\{K_j, \max_i \left\langle Score\left(K_j(S_{iz})\right)\right\rangle | j = 1,2, \dots n\right\} \quad (5.2.1.3)$$

$$S^* = \{\langle K_1, (T_1{}^*, I_1{}^*, F_1^*)\rangle, \langle K_2, (T_2{}^*, I_2{}^*, F_2^*)\rangle, \dots \langle K_n, (T_n{}^*, I_n{}^*, F_n^*)\rangle\}$$

Regarding the NIS

$$S^- = \left\{K_j, \min_i \left\langle Score\left(K_j(S_{iz})\right)\right\rangle | j = 1,2, \dots n\right\} \quad (5.2.1.4)$$

$$S^- = \{\langle K_1, (T_1{}^-, I_1{}^-, F_1^-)\rangle, \langle K_2, (T_2{}^-, I_2{}^-, F_2^-)\rangle, \dots \langle K_n, (T_n{}^-, I_n{}^-, F_n^-)\rangle\}$$

Step 6. Calculate the distances between alternative S_i and both NIS and PIS accordingly.

$$D(S_i, S^-) = \sqrt{\tfrac{1}{3}\textstyle\sum_{i=1}^{n}\left(\left(T_{S_i} - T_{S^-}\right)^2 + \left(I_{S_i} - I_{S^-}\right)^2 + \left(F_{S_i} - F_{S^-}\right)^2\right)} \quad (5.2.1.5)$$

$$D(S_i, S^*) = \sqrt{\frac{1}{3}\sum_{i=1}^{n}\left(\left(T_{S_i} - T_{S^*}\right)^2 + \left(I_{S_i} - I_{S^*}\right)^2 + \left(F_{S_i} - F_{S^*}\right)^2\right)} \quad (5.2.1.6)$$

Step 7 Calculate the minimum and maximum distances to the NIS and the PIS, respectively

$$D(S_i, S^-)_{max} = \max_{i \leq i \leq m} D(S_i, S^-) \tag{5.2.1.7}$$

$$D(S_i, S^*)_{min} = \min_{i \leq i \leq m} D(S_i, S^*) \tag{5.2.1.8}$$

Step 8 Compute the revised proximity ratio in Equation (5.2.1.9).

$$\xi(S_i) = \frac{D(S_i, S^-)}{D(S_i, S^-)_{max} - \dfrac{D(S_i, S^*)}{D(S_i, S^*)_{min}}} \tag{5.2.1.9}$$

Equation (5.2.1.9) because the subtraction's second element is at least equal to its first element, the result is zero or negative. To address this, Equation (5.2.1.10) was modified, allowing the outcome to be either zero or a positive value.

$$\xi(S_i) = \frac{D(S_i, S^*)}{D(S_i, S^*)_{min} - \dfrac{D(S_i, S^-)}{D(S_i, S^-)_{max}}} \tag{5.2.1.10}$$

Step 9. Rank the alternatives according to the decreasing values of proximity ratio value. The alternative with the highest assessment score is the best alternative.

5.2.2. A Framework for Neutrosophic Spherical CODAS: Modeling and Application

To depict the neutrosophic spherical CODAS technique, an example of supplier selection model is considered. Let $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}$ be ten vendors and to test their efficacy, the Criteria are given below,

	Arithmetic Mean Criteria	**Geometric Mean Criteria**
(K_1)	Quality	Management Capability
(K_2)	Service	Financial Position
(K_3)	On-Time Delivery	Production Capability
(K_4)	Discounts	Reputation

Let ĐM1, ĐM2, and ĐM3 are three decision-makers take part in the procedure for evaluation. The weights representing their various levels of experience are assumed to be, 0.4, 0.5 and 0.3, respectively. In this example, each criterion is considered equally important; therefore, the CODAS technique is applied to select the best alternative. For the same set of criteria, two different approaches are adopted: one using the arithmetic operator and the other using the geometric operator.

In the considered criteria, indeterminacy can manifest in various forms. That may include ambiguous definitions of criteria, challenges with trade-offs and waiting, uncertain future performance, and uncertain preferences.

Based on this assumption of ambiguity, the judgements made by the decision-makers are compiled using the linguistics variables listed in Table 5.2.2.1.

Table 5.2.2.1. Terms used in linguistics and their associated Spherical Neutrosophic Number

LT		(T, I, F)
Extremely High	EH	0.95, 0.90, 0.30
Very High	VH	0.90, 0.80, 0.40
High	H	0.80, 0.90, 0.50

Above Average	AA	0.70, 0.85, 0.65
Average	A	0.65, 0.80, 0.80
Below Average	BA	0.60, 0.90, 0.83
Low	L	0.50, 0.70, 0.85
Very Low	VL	0.45, 0.50, 0.87
Extremely Low	EL	0.40, 0.45, 0.90

The following tables 5.2.2.2, 5.2.2.3 and 5.2.2.4 elucidates the decisions with respect to the decision makers.

Table 5.2.2.2. Decisions of ĐM1

ĐM1	(K_1)	(K_2)	(K_3)	(K_4)
S_1	VH	H	A	AA
S_2	EH	VH	H	A
S_3	L	BA	VH	VL
S_4	A	VL	L	EL
S_5	BA	VL	H	H
S_6	L	M	VH	H
S_7	H	AA	AA	VL
S_8	BA	L	H	A
S_9	L	BA	H	EH
S_{10}	EH	AA	H	BA

Table 5.2.2.3. Decisions of ĐM2

ĐM2	(K_1)	(K_2)	(K_3)	(K_4)
S_1	EH	H	L	EH
S_2	BA	L	H	A
S_3	H	AA	AA	VL
S_4	VL	A	L	VH
S_5	VH	H	A	AA
S_6	EH	VH	H	A
S_7	BA	VL	H	H
S_8	L	A	VH	H

S_9	L	BA	H	VL
S_{10}	BA	VH	VL	EL

Table 5.2.2.4. Decisions of ĐM3

ĐM3	(K_1)	(K_2)	(K_3)	(K_4)
S_1	H	VH	EH	AA
S_2	BA	EH	VH	BA
S_3	BA	VL	H	H
S_4	L	A	VH	H
S_5	BA	L	H	A
S_6	L	BA	H	EH
S_7	H	AA	AA	VL
S_8	VL	M	L	VH
S_9	VH	H	M	AA
S_{10}	EH	VH	H	A

The normalized NS D-Mxs shown in Tables 5.2.2.5 and 5.2.2.6 are obtained using Equation 5.1.11 and Equation 5.1.12.

Table 5.2.2.5. Normalized NS D-Mx by using SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.41	0.87	0.74	0.81
	0.41	0.91	0.84	0.92
	0.09	0.52	0.82	0.59
S_2	0.32	0.85	0.87	0.64
	0.38	0.84	0.91	0.84
	0.28	0.74	0.52	0.84
S_3	0.26	0.67	0.85	0.65
	0.37	0.85	0.91	0.72
	0.31	0.81	0.59	0.85
S_4	0.19	0.61	0.72	0.83
	0.33	0.77	0.77	0.82
	0.36	0.86	0.85	0.77
S_5	0.34	0.71	0.78	0.68
	0.35	0.82	0.90	0.91

	0.21	0.82	0.58	0.61
S_6	0.37 0.36 0.27	0.82 0.84 0.61	0.87 0.91 0.50	0.85 0.90 0.56
S_7	0.30 0.40 0.20	0.67 0.85 0.82	0.79 0.92 0.61	0.71 0.80 0.85
S_8	0.18 0.32 0.31	0.61 0.81 0.73	0.85 0.86 0.63	0.83 0.88 0.55
S_9	0.26 0.29 0.31	0.71 0.87 0.70	0.82 0.91 0.57	0.82 0.85 0.80
S_{10}	0.42 0.40 0.18	0.89 0.87 0.52	0.75 0.85 0.80	0.57 0.75 0.89

Table 5.2.2.6. Normalized NS D-Mx by using SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.85 0.84 0.31	0.79 0.85 0.41	0.55 0.74 0.59	0.59 0.88 0.41
S_2	0.63 0.80 0.51	0.65 0.74 0.46	0.79 0.85 0.41	0.54 0.76 0.76
S_3	0.58 0.76 0.62	0.55 0.70 0.70	0.75 0.84 0.47	0.50 0.51 0.71
S_4	0.51 0.73 0.78	0.50 0.63 0.79	0.51 0.67 0.68	0.61 0.65 0.51
S_5	0.66 0.76 0.49	0.55 0.64 0.63	0.66 0.83 0.47	0.49 0.85 0.50
S_6	0.60	0.66	0.80	0.69

	0.73 0.54	0.76 0.46	0.84 0.39	0.83 0.44
S_7	0.66 0.83 0.51	0.55 0.65 0.66	0.69 0.88 0.49	0.55 0.58 0.65
S_8	0.49 0.71 0.69	0.50 0.72 0.60	0.70 0.77 0.44	0.71 0.81 0.43
S_9	0.51 0.67 0.62	0.59 0.79 0.58	0.70 0.85 0.46	0.52 0.65 0.56
S_{10}	0.74 0.83 0.44	0.79 0.80 0.39	0.61 0.65 0.58	0.44 0.60 0.72

The decision-makers evaluated the selected criteria and weighted them by using linguistic variables. Which is given in the Table 5.2.2.7

Table 5.2.2.7. The weights assigned to each criterion

Criteria	ĐM1	ĐM2	ĐM3
(K_1)	L	BA	H
(K_2)	AA	A	AA
(K_3)	EH	AA	H
(K_4)	H	H	BA

Based on the evaluations of DMs and Equation (5.1.11), the aggregated spherical weights are calculated which is given in the Table 5.2.2.8

Table 5.2.2.8. Aggregation of Criteria weights using SWAM operator

Criteria	Weights of each criterion
(K_1)	(0.68, 0.75, 0.72)
(K_2)	(0.71, 0.83, 0.71)
(K_3)	(0.85, 0.88, 0.45)
(K_4)	(0.81, 0.85, 0.59)

The aggregated weighted normalized neutrosophic spherical matrices are formulated using Equations (5.1.1) and (5.1.2) following the assignment of criteria weights. The corresponding evaluations are presented in Tables 5.2.2.9 and 5.2.2.10.

Table 5.2.2.9. Weighted normalized NS D-Mx for arithmetic criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.72	0.93	0.93	0.94
	0.79	0.97	0.96	0.98
	0.72	0.79	0.86	0.75
S_2	0.72	0.92	0.96	0.89
	0.79	0.95	0.98	0.95
	0.75	0.88	0.65	0.89
S_3	0.72	0.85	0.96	0.89
	0.79	0.95	0.98	0.93
	0.75	0.91	0.70	0.91
S_4	0.70	0.82	0.93	0.94
	0.78	0.93	0.95	0.95
	0.76	0.93	0.88	0.86
S_5	0.73	0.86	0.94	0.90
	0.78	0.94	0.97	0.97
	0.73	0.91	0.69	0.77
S_6	0.74	0.91	0.96	0.95
	0.78	0.95	0.98	0.97
	0.74	0.82	0.64	0.74
S_7	0.72	0.85	0.94	0.91
	0.79	0.95	0.98	0.94
	0.73	0.91	0.71	0.91
S_8	0.70	0.82	0.96	0.94
	0.78	0.94	0.97	0.97
	0.75	0.87	0.72	0.74
S_9	0.71	0.86	0.95	0.94
	0.77	0.96	0.98	0.96
	0.75	0.86	0.68	0.87
S_{10}	0.75	0.94	0.94	0.87

	0.79 0.73	0.96 0.79	0.96 0.84	0.93 0.92

Table 5.2.2.10. Weighted normalized NS D-Mx for geometric criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.58 0.63 0.23	0.56 0.71 0.29	0.47 0.65 0.27	0.47 0.74 0.23
S_2	0.43 0.60 0.37	0.46 0.61 0.32	0.67 0.74 0.19	0.43 0.65 0.45
S_3	0.40 0.57 0.45	0.39 0.58 0.49	0.64 0.74 0.21	0.40 0.44 0.42
S_4	0.35 0.55 0.56	0.35 0.52 0.55	0.44 0.59 0.31	0.49 0.55 0.30
S_5	0.45 0.57 0.35	0.38 0.53 0.44	0.56 0.73 0.21	0.40 0.72 0.29
S_6	0.41 0.55 0.39	0.46 0.63 0.32	0.68 0.74 0.18	0.56 0.71 0.26
S_7	0.45 0.62 0.37	0.38 0.54 0.46	0.59 0.77 0.22	0.44 0.49 0.38
S_8	0.34 0.53 0.50	0.35 0.60 0.42	0.60 0.67 0.21	0.57 0.69 0.25
S_9	0.35 0.51 0.45	0.41 0.65 0.41	0.59 0.74 0.21	0.42 0.55 0.33
S_{10}	0.51 0.62 0.31	0.56 0.66 0.27	0.51 0.57 0.26	0.35 0.51 0.42

SF is calculated using Equation (5.1.13) and given in Tables 5.2.2.11 and 5.2.2.12, which are based on Tables 5.2.2.9 and 5.2.2.10.

Table 5.2.2.11. SF for arithmetic criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	-0.005	-0.0115	-0.0051	-0.0163
S_2	-0.001	-0.0029	-0.0094	-0.0034
S_3	0.0004	0.0016	-0.0093	-0.0003
S_4	0.0038	0.0109	-0.0022	-0.0013
S_5	-0.002	0.0007	-0.0177	-0.0252
S_6	-0.001	-0.0078	-0.0075	-0.0090
S_7	-0.003	0.0022	-0.0190	-0.0017
S_8	0.002	-0.0028	-0.0038	-0.0106
S_9	0.002	-0.0092	-0.0168	-0.0030
S_{10}	-0.003	-0.0051	-0.0061	0.00284

Table 5.2.2.12. SF geometric criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	-0.0342	-0.0988	-0.1035	-0.2040
S_2	-0.0478	-0.0640	-0.0758	-0.0393
S_3	-0.0131	0.0039	-0.0952	-0.0001
S_4	0.0455	0.0397	-0.0639	-0.0276
S_5	-0.0374	-0.0044	-0.1431	-0.1709
S_6	-0.0262	-0.0748	-0.0592	-0.1068
S_7	-0.0560	0.0003	-0.1666	-0.0086
S_8	0.02577	-0.0246	-0.0670	-0.0893
S_9	0.0049	-0.0586	-0.1393	-0.0425
S_{10}	-0.0546	-0.0693	-0.0348	-0.0023

The PIS and NIS based on the equations (5.2.1.3 and 5.2.1.4) are found according to the score function and are shown in Tables 5.2.2.13 and 5.2.2.14

Table 5.2.2.13. PIS and NIS according to SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Best)	0.72,0.79,0.72	0.93,0.97,0.79	0.94,0.98,0.71	0.94,0.98,0.76
S^- (Worst)	0.70,0.78,0.76	0.83,0.93,0.93	0.93,0.95,0.88	0.87,0.94,0.93

Table 5.2.2.14. PIS and NIS according to SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Great)	0.35,0.55,0.56	0.35,0.52,0.55	0.44,0.59,0.31	0.49,0.55,0.31
S^- (Poor)	0.58,0.63,0.23	0.561,0.70,0.29	0.47,0.65,0.27	0.48,0.75,0.23

The distances between each alternative S_i, and PIS, are calculated using Equation (5.2.1.5) and listed in the Table 5.2.2.15. The distances between each alternative S_i, and NIS, are calculated using Equation (5.2.1.6) are listed in the Table 5.2.2.16.

Table 5.2.2.15. Distance to PIS and NIS for arithmetic criteria

Alternatives	$D(S_i, S^*)$	$D(S_i, S^-)$
S_1	0.0118	0.0041
S_2	0.0119	0.0056
S_3	0.0063	0.0083
S_4	0.0016	0.0134
S_5	0.0114	0.0035
S_6	0.0203	0.0012
S_7	0.0063	0.0076

S_8	0.0119	0.0038
S_9	0.0095	0.0044
S_{10}	0.0065	0.0092

Table 5.2.2.16. Distance to PIS and NIS for geomertric criteria

Alternatives	$D(S_i, S^*)$	$D(S_i, S^-)$
S_1	0.39041	0.0113
S_2	0.2744	0.0140
S_3	0.1514	0.0388
S_4	0.1201	0.0501
S_5	0.2531	0.0161
S_6	0.3133	0.0070
S_7	0.1964	0.0287
S_8	0.2334	0.0219
S_9	0.1983	0.0192
S_{10}	0.2842	0.0267

The proximity ratios are computed using Equation 5.2.1.9 or Equation 5.2.1.10, and they are shown in Tables 5.2.2.17.

Table 5.2.2.17. Alternative's proximity ratio for arithmetic and geometric criteria

	SWAM		SWGM	
Alternatives	**Proximity Ratio**	**Rank**	**Proximity Ratio**	**Rank**
S_1	6.8753	3	3.0222	1
S_2	6.8120	4	2.0022	3
S_3	3.2105	9	0.4853	9

S_4	0	10	0	10
S_5	6.6543	5	1.7833	5
S_6	12.1902	1	2.4659	2
S_7	3.2851	7	1.0614	8
S_8	6.9396	2	1.5053	6
S_9	5.4628	6	1.2671	7
S_{10}	3.2537	8	1.8309	4

According to the value of SWAM operator, the proximity ratio for each alternative shows that the best option is S_6, and over all ranking is $S_6 > S_8 > S_1 > S_2 > S_5 > S_9 > S_7 > S_{10} > S_3 > S_4$. The maximum value, according to the proximity ratio based on the SWGM operator, is S_1, and overall ranking is $S_1 > S_6 > S_2 > S_{10} > S_5 > S_8 > S_9 > S_7 > S_3 > S_4$.

In this example, the CODAS technique is developed by incorporating the defined aggregation operators for the Neutrosophic Spherical Set (NSS) for better handling of high levels of uncertainty

5.2.3 A Framework for Fermatean Neutrosophic CODAS: Modeling and Application

To illustrate the Fermatean neutrosophic CODAS technique, the same supplier selection problem used in the Section 5.2.2 is considered.

Based on the assumption of ambiguity in the example, the judgments made by the decision-makers are compiled using the linguistic variables listed in Table 5.2.3.1 and expressed in Fermatean neutrosophic values.

Table 5.2.3.1. Terms used in linguistics and their associated Fermatean neutrosophic number

LT		(T, I, F)
Extremely High	EH	0.95, 0.90, 0.50
Very High	VH	0.90, 0.80, 0.65
High	H	0.80, 0.90, 0.79
Above Average	AA	0.70, 0.85, 0.87
Average	A	0.65, 0.80, 0.85
Below Average	BA	0.60, 0.90, 0.90
Low	L	0.50, 0.70, 0.95
Very Low	VL	0.45, 0.50, 0.95
Extremely Low	EL	0.40, 0.45, 0.90

The following tables 5.2.3.2, 5.2.2.3 and 5.2.3.4 elucidates the decisions with respect to the decision makers which are the same as those taken in the Neutrosophic Spherical CODAS method.

Table 5.2.3.2. Decisions of ĐM1

ĐM1	(K_1)	(K_2)	(K_3)	(K_4)
S_1	VH	H	A	AA
S_2	EH	VH	H	A
S_3	L	BA	VH	VL
S_4	A	VL	L	EL
S_5	BA	VL	H	H

S_6	L	M	VH	H
S_7	H	AA	AA	VL
S_8	BA	L	H	A
S_9	L	BA	H	EH
S_{10}	EH	AA	H	BA

Table 5.2.3.3. Decisions of ĐM2

ĐM2	(K_1)	(K_2)	(K_3)	(K_4)
S_1	EH	H	L	EH
S_2	BA	L	H	A
S_3	H	AA	AA	VL
S_4	VL	A	L	VH
S_5	VH	H	A	AA
S_6	EH	VH	H	A
S_7	BA	VL	H	H
S_8	L	A	VH	H
S_9	L	BA	H	VL
S_{10}	BA	VH	VL	EL

Table 5.2.3.4. Decisions of ĐM3

ĐM3	(K_1)	(K_2)	(K_3)	(K_4)
S_1	H	VH	EH	AA
S_2	BA	EH	VH	BA
S_3	BA	VL	H	H
S_4	L	A	VH	H
S_5	BA	L	H	A
S_6	L	BA	H	EH
S_7	H	AA	AA	VL
S_8	VL	M	L	VH
S_9	VH	H	M	AA
S_{10}	EH	VH	H	A

The normalized FN D-Mx shown in Tables 5.2.3.5 and 5.2.3.6 are obtained using Equations 5.1.26 and 5.1.27, respectively.

Table 5.2.3.5. Normalized FN D-Mx by for arithmetic criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.27,0.84,0.56	0.86,0.85,0.70	0.82,0.74,0.76	0.87,0.88,0.68
S_2	0.20,0.88,0.72	0.88,0.74,0.67	0.86,0.85,0.71	0.71,0.79,0.86
S_3	0.12,0.79,0.85	0.66,0.73,0.89	0.84,0.84,0.72	0.65,0.51,0.89
S_4	0.09,0.78,0.89	0.68,0.63,0.87	0.73,0.67,0.84	0.83,0.65,0.74
S_5	0.18,0.76,0.56	0.71,0.64,0.75	0.77,0.83,0.54	0.68,0.85,0.64
S_6	0.21,0.73,0.59	0.82,0.76,0.53	0.87,0.84,0.46	0.85,0.83,0.55
S_7	0.15,0.83,0.59	0.67,0.65,0.78	0.78,0.88,0.56	0.71,0.58,0.80
S_8	0.07,0.71,0.76	0.60,0.72,0.70	0.85,0.77,0.53	0.83,0.81,0.54
S_9	0.13,0.67,0.70	0.70,0.79,0.66	0.80,0.85,0.52	0.83,0.65,0.71
S_{10}	0.25,0.83,0.48	0.88,0.80,0.45	0.75,0.65,0.72	0.56,0.60,0.87

Table 5.2.3.6. Normalized FN D-Mx by for geometric criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.87,0.92,0.52	0.79,0.91,0.44	0.60,0.84,0.66	0.60,0.93,0.29
S_2	0.65,0.94,0.36	0.66,0.84,0.65	0.79,0.91,0.44	0.62,0.86,0.57
S_3	0.58,0.91,0.44	0.55,0.88,0.66	0.75,0.90,0.37	0.50,0.73,0.84
S_4	0.54,0.90,0.49	0.57,0.77,0.81	0.51,0.76,0.73	0.61,0.82,0.71
S_5	0.66,0.87,0.63	0.55,0.82,0.69	0.66,0.89,0.52	0.49,0.91,0.56
S_6	0.6,0.88,0.64	0.66,0.83,0.59	0.80,0.90,0.48	0.69,0.89,0.51
S_7	0.66,0.92,0.56	0.55,0.85,0.68	0.69,0.93,0.54	0.55,0.81,0.72
S_8	0.49,0.84,0.73	0.50,0.80,0.67	0.70,0.86,0.58	0.70,0.88,0.51
S_9	0.51,0.82,0.74	0.59,0.86,0.63	0.70,0.91,0.51	0.52,0.86,0.66
S_{10}	0.74,0.92,0.53	0.79,0.87,0.56	0.61,0.85,0.65	0.44,0.76,0.81

The decision-makers evaluated the selected criteria and weighted them by using linguistic variables. Which is given in the Table 5.2.3.7

Table 5.2.3.7. The weights assigned to each criterion

Criteria	ĐM1	ĐM2	ĐM3
(K_1)	L	BA	MI
(K_2)	AA	EI	AA
(K_3)	VL	AA	MI
(K_4)	MI	MI	BA

Based on the evaluations of DMs and Equation (5.1.26), the aggregated spherical weights are calculated which is given in the Table 5.2.3.8

Table 5.2.2.8. Aggregation of Criteria weights for arithmetic criteria

Criteria	Weights of each criterion
(K_1)	0.68,0.79,0.87
(K_2)	0.73,0.83,0.84
(K_3)	0.88,0.88,0.66
(K_4)	0.80, 0.88, 0.78

The aggregated weighted normalized FN D-Mx are formulated using Equations (5.1.16) and (5.1.17) following the assignment of criteria weights. The corresponding evaluations are presented in Tables 5.2.3.9 and 5.2.3.10.

Table 5.2.3.9. Weighted normalized FN D-Mx for arithmetic criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.69,0.92,0.90	0.92,0.94,0.90	0.95,0.93,0.84	0.94,0.96,0.86
S_2	0.69,0.94,0.92	0.93,0.90,0.89	0.96,0.95,0.81	0.88,0.94,0.93
S_3	0.68,0.91,0.95	0.83,0.90,0.96	0.96,0.95,0.83	0.86,0.9,0.94
S_4	0.68,0.91,0.96	0.83,0.88,0.95	0.93,0.92,0.89	0.92,0.91,0.88
S_5	0.68,0.89,0.90	0.85,0.88,0.91	0.94,0.95,0.74	0.87,0.95,0.85
S_6	0.69,0.89,0.91	0.90,0.91,0.87	0.96,0.95,0.71	0.93,0.95,0.83
S_7	0.68,0.92,0.91	0.83,0.88,0.92	0.94,0.96,0.75	0.88,0.90,0.91

S_8	0.68,0.88,0.93	0.80,0.90,0.90	0.96,0.93,0.74	0.92,0.94,0.82
S_9	0.68,0.87,0.92	0.84,0.92,0.89	0.94,0.95,0.73	0.92,0.91,0.87
S_{10}	0.69,0.92,0.89	0.93,0.92,0.86	0.94,0.91,0.82	0.84,0.91,0.94

Table 5.2.3.10. Weighted normalized FN D-Mx for geometric criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.60,0.74,0.45	0.58,0.75,0.37	0.53,0.73,0.44	0.48,0.81,0.22
S_2	0.44,0.75,0.32	0.49,0.69,0.54	0.70,0.80,0.29	0.49,0.76,0.45
S_3	0.39,0.73,0.39	0.41,0.73,0.56	0.66,0.79,0.24	0.40,0.65,0.66
S_4	0.37,0.71,0.43	0.41,0.64,0.68	0.46,0.67,0.49	0.49,0.72,0.55
S_5	0.45,0.69,0.55	0.40,0.68,0.58	0.58,0.79,0.35	0.39,0.80,0.44
S_6	0.41,0.71,0.56	0.48,0.69,0.49	0.71,0.79,0.32	0.55,0.79,0.39
S_7	0.45,0.73,0.49	0.41,0.71,0.58	0.61,0.81,0.36	0.44,0.71,0.56
S_8	0.33,0.67,0.64	0.37,0.67,0.57	0.62,0.75,0.38	0.56,0.77,0.40
S_9	0.35,0.65,0.65	0.43,0.71,0.53	0.62,0.80,0.33	0.42,0.75,0.52
S_{10}	0.51,0.73,0.47	0.58,0.72,0.47	0.53,0.75,0.43	0.35,0.66,0.63

SF is calculated using Equation (5.1.13) and given in Tables 5.2.3.11 and 5.2.3.12, which are based on Tables 5.2.3.9 and 5.2.3.10.

Table 5.2.3.11. SF for arithmetic criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	-0.0088	-4.1235	0.0005	-0.0004
S_2	-0.0133	4.0045	0.0003	-0.0001
S_3	-0.0194	-0.0019	0.0002	-0.0004
S_4	-0.0221	-0.0011	4.7260	4.2985
S_5	-0.0097	-0.0002	-0.0011	-0.0010
S_6	-0.0098	-4.5323	0.0017	-0.0006
S_7	-0.0104	-0.0007	-0.0025	-1.9935

S_8	-0.0155	-0.0009	0.0028	-0.0008
S_9	-0.0131	-0.0001	-0.0012	5.74605
S_{10}	-0.0079	0.0001	0.0006	-0.0008

Table 5.2.3.12. SF for geometric criteria

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	-0.0197	-0.0452	-0.0253	-0.1842
S_2	-0.0781	-0.0033	-0.0601	-0.0296
S_3	-0.0388	-0.0092	-0.0911	-0.0180
S_4	-0.0229	-0.0190	-0.0062	-0.0049
S_5	-0.0038	-0.0071	-0.0710	-0.0468
S_6	-0.0065	-0.0071	-0.0467	-0.0559
S_7	-0.0135	-0.0076	-0.0759	-0.0050
S_8	-0.0279	-0.0093	-0.0366	-0.0461
S_9	-0.0268	-0.0070	-0.0758	-0.0135
S_{10}	-0.0184	-0.0145	-0.0310	-0.0221

Find the PIS and NIS based on the equations (5.2.1.3 and 5.2.1.4) according to the score function are shown in Tables 5.2.2.13 and 5.2.2.14

Table 5.2.3.13. PIS and NIS according to FWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Best)	0.694, 0.924, 0.894	0.936, 0.926, 0.862	0.961, 0.939, 0.74	0.926, 0.918, 0.874
S^- (Worst)	0.687, 0.905, 0.969	0.831, 0.907, 0.96	0.945, 0.966, 0.753	0.944, 0.966, 0.868

Table 5.2.3.14. PIS and NIS according to FWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Best)	0.455, 0.697,	0.49, 0.696,	0.461, 0.674,	0.492, 0.723,

	0.556	0.549	0.49	0.555
S^- (Worst)	0.447, 0.753, 0.322	0.582, 0.755, 0.378	0.666, 0.796, 0.248	0.486, 0.815, 0.229

The distances between each alternative S_i, and PIS, are calculated using Equation (5.2.1.5) and listed in the Table 5.2.3.15. The distances between each alternative S_i, and NIS, are calculated using Equation (5.2.1.6) are listed in the Table 5.2.3.16.

Table 5.2.3.15. Distance to PIS and NIS for arithmetic criteria

Alternatives	$D(S_i, S^*)$	$D(S_i, S^-)$
S_1	0.0002	0.0003
S_2	0.00011	0.0001
S_3	0.00015	3.9714
S_4	0.00066	0.0004
S_5	0.0001	0.0001
S_6	1.9521	0.0001
S_7	0.00014	0.0001
S_8	0.0003	4.8273
S_9	0.00012	8.0790
S_{10}	5.9811	0.0001

Table 5.2.3.16. Distance to PIS and NIS for geometric criteria

Alternatives	$D(S_i, S^*)$	$D(S_i, S^-)$
S_1	0.1084	0.0018
S_2	0.0381	0.0025
S_3	0.0533	0.0130
S_4	0.0093	0.0103
S_5	0.0225	0.0042
S_6	0.0560	0.0035

S_7	0.0354	0.0078
S_8	0.0207	0.0062
S_9	0.0272	0.0102
S_{10}	0.0221	0.0116

The proximity ratios are computed using Equation 5.2.1.9 or Equation 5.2.1.10, and they are shown in Tables 5.2.2.17.

Table 5.2.3.17. Every alternative's proximity ratio for arithmetic and geometric criteria

	FWAM		FWGM	
Alternatives	Proximity Ratio	Rank	Proximity Ratio	Rank
S_1	11.7137	3	11.4581	1
S_2	5.7129	7	3.8908	4
S_3	7.6872	4	4.7053	3
S_4	32.8803	1	0.2038	10
S_5	5.0122	8	2.0848	7
S_6	0.7731	10	5.7281	2
S_7	7.4281	5	3.1863	5
S_8	17.4307	2	1.7331	8
S_9	6.4344	6	2.1347	6
S_{10}	2.8248	9	1.4693	9

According to the FWAM operator value, the proximity ratio for each alternative show that the best option is S_4, and over all ranking is $S_4 > S_8 > S_1 > S_3 > S_7 > S_9 > S_2 > S_5 > S_{10} > S_6$ and it is shown in Table 5.2.3.17. According to the FWGM operator value, the proximity ratio for each alternative show that the best option is S_1, and over all ranking is $S_1 > S_6 > S_3 > S_2 > S_7 > S_9 > S_5 > S_8 > S_{10} > S_4$ and it is shown in Table 5.2.3.17.

In this example, the CODAS technique is developed by incorporating our defined aggregation operator for the FNS. Additionally, the use of FNS in CODAS allows for better handling of high levels of uncertainty than the NSS.

Comparison

To analyse and compare our defined NS/ FN CODAS with neutrosophic CODAS, the same supplier selection problem used in the Section 5.2.2 is considered. For the same example, based on assumption of ambiguity, the judgements made by the decision-makers are compiled using the linguistics variables listed in Table 5.2.2.19 using neutrosophic values.

Table 5.2.3.19. Terms used in linguistics and their associated Neutrosophic Number

LT		(T, I, F)
Extremely High	EH	0.95, 0.90, 0.05
Very High	VH	0.90, 0.80, 0.10
High	H	0.80, 0.90, 0.20
Above Average	AA	0.70, 0.85, 0.30
Average	A	0.65, 0.80, 0.40
Below Average	BA	0.60, 0.90, 0.40
Low	L	0.50, 0.70, 0.50
Very Low	VL	0.45, 0.50, 0.50
Extremely Low	EL	0.40, 0.45, 0.60

Using these terms, and evaluate the same situation which is defined in Section 5.2.2 and calculated the proximity ratio for each alternative based on the arithmetic means and geometric means (Table 5.2.3.20) operators for the NS.

Table 5.2.3.20. Every alternative's proximity ratio for arithmetic and geometric criteria

	NWAM		NWGM	
Alternatives	Proximity Ratio	Rank	Proximity Ratio	Rank
S_1	7.8149	6	0.1892	9
S_2	0.3284	10	0.5188	7
S_3	9.9720	4	1.3105	4

S_4	28.636	1	0.1534	10
S_5	6.5175	8	1.4066	2
S_6	6.9720	7	0.2431	8
S_7	9.0115	5	1.1300	5
S_8	25.027	2	1.3662	3
S_9	21.910	3	1.0490	6
S_{10}	1.0340	9	2.1223	1

The comparative results are presented here based on the ranking. The comparison reveals variations in the results derived from the NS, NSS, and FNS.

Table 5.2.3.22. Weighted Arithmetic Mean

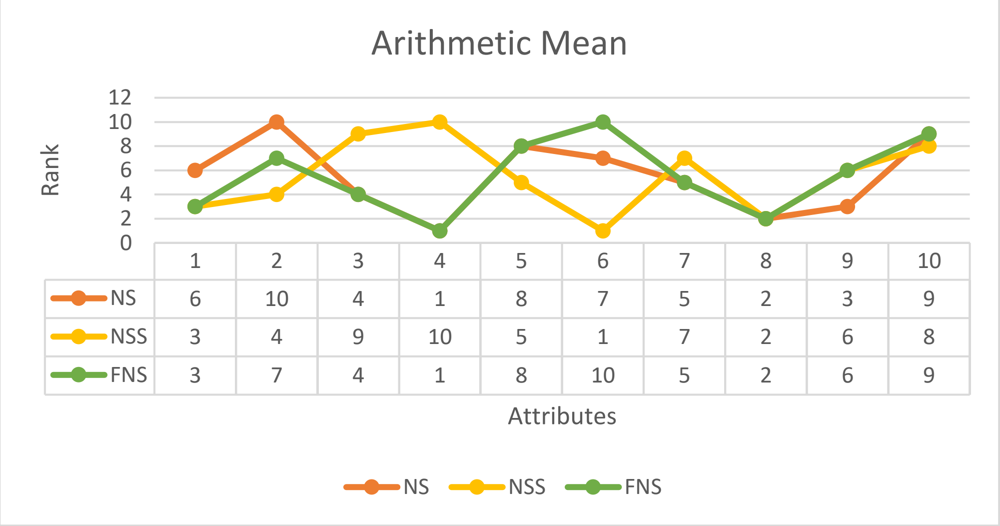
Alternatives	NS	NSS	FNS
S_1	6	3	3
S_2	10	4	7
S_3	4	9	4
S_4	1	10	1
S_5	8	5	8
S_6	7	1	10
S_7	5	7	5
S_8	2	2	2
S_9	3	6	6
S_{10}	9	8	9

Table 5.2.3.23. Weighted Geometric Mean

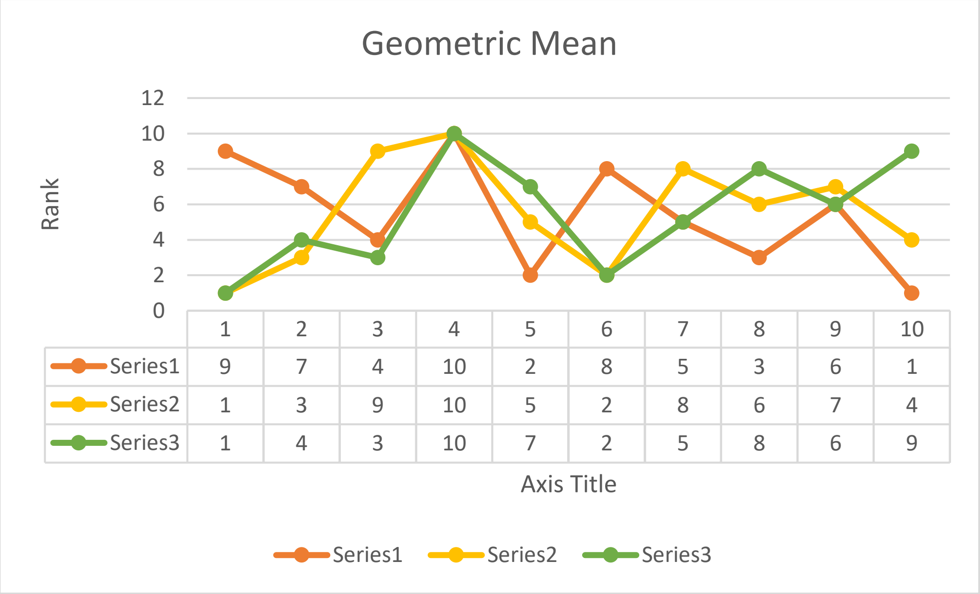
Alternatives	NS	NSS	FNS
S_1	9	1	1
S_2	7	3	4
S_3	4	9	3
S_4	10	10	10
S_5	2	5	7
S_6	8	2	2
S_7	5	8	5
S_8	3	6	8

S_9	6	7	6
S_{10}	1	4	9

A graphical representation of the ranking is also provided. This demonstrates the variations in the ranking.



	1	2	3	4	5	6	7	8	9	10
NS	6	10	4	1	8	7	5	2	3	9
NSS	3	4	9	10	5	1	7	2	6	8
FNS	3	7	4	1	8	10	5	2	6	9



	1	2	3	4	5	6	7	8	9	10
Series1	9	7	4	10	2	8	5	3	6	1
Series2	1	3	9	10	5	2	8	6	7	4
Series3	1	4	3	10	7	2	5	8	6	9

Hence, in this section to demonstrate the working methodology of NS/FN CODAS technique, a supplier selection problem has been devised assuming complex uncertain environments. To give a comparative study same linguistic variables, truth membership, indertiminacy membership were considered and different false membership values were assingned for NS, NSS, and FNS respectively. Additionally, the weights assigned to the decision-maker and the criteria evaluated by the decision-makers are also kept the same across NS, NSS, and FNS. Based on these values, the CODAS methodology has been solved and the ranks are obtained. The variation in the ranking is observed due to the fact that the NSS allow a higher membership value than NS, and FNS allow a higher membership value than both and comprehensively represent complex uncertainities.