

CHAPTER II

Some Hybrid forms of Neutrosophic Sets

(2.1) Introduction to some Neutrosophic Variants

(2.2) Fermatean Neutrosophic Cubic Set

(2.3) Fermatean Temporal Neutrosophic Set

Exploring Neutrosophic variants is essential for handling more complex uncertainties what traditional neutrosophic sets cannot offer. This chapter deals with the introduction of neutrosophic variants and incorporates their basic structures.

2.1 Introduction to some Neutrosophic variants

This section defines various neutrosophic variants based on the idea of dependency based NSs by redefining the structure of the NS. It explores an extended form of Pythagorean neutrosophic set and neutrosophic spherical set. Additionally, a graphical comparison provides a visual understanding of these concepts and their relationship.

Extended Pythagorean Neutrosophic Set

Definition 2.1.1

Let S be a non empty set. An extended Pythagorean Neutrosophic Set (PNS) A_P is of the form

$$A_P = \{ \langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle / x \in S \}$$

where $T_{A_p}(x)$, is truth membership, $I_{A_p}(x)$ is indeterminacy membership, $F_{A_p}(x)$ is false membership where $T_{A_p}(x), I_{A_p}(x), F_{A_p}(x) \in [0, \sqrt{2}]$ and

$$0 \leq [T_{A_p}(x)]^2 + [I_{A_p}(x)]^2 + [F_{A_p}(x)]^2 \leq 2.$$

where $T_{A_p}(x), I_{A_p}(x)$ and $F_{A_p}(x)$ are dependent components.

Example 2.1.2

Let $S = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set and a PNS A_p on S is denoted as follows,

$$A_p = \langle (\tilde{x}, 0.9, 0.6, 0.3), (\tilde{y}, 0.8, 0.8, 0.3), (\tilde{z}, 0.7, 0.3, 0.5) \rangle.$$

Definition 2.1.3

Let S be a non empty set, and the PNSs, A_p and B_p in S of the form $A_p = \{\langle x, T_{A_p}(x), I_{A_p}(x), F_{A_p}(x) \rangle | x \in S\}$ & $B_p = \{\langle x, T_{B_p}(x), I_{B_p}(x), F_{B_p}(x) \rangle | x \in S\}$ satisfy the following conditions:

- 1) $A_p \subseteq B_p$ iff $T_{A_p}(x) \leq T_{B_p}(x); I_{A_p}(x) \leq I_{B_p}(x); F_{A_p}(x) \geq F_{B_p}(x), \forall x \in S$.
- 2) $A_p = B_p$ iff $T_{A_p}(x) = T_{B_p}(x); I_{A_p}(x) = I_{B_p}(x); F_{A_p}(x) = F_{B_p}(x), \forall x \in S$.
- 3) $A_p \cup B_p = \{\langle x, T_{A_p \cup B_p}(x), I_{A_p \cup B_p}(x), F_{A_p \cup B_p}(x) \rangle | x \in S\}$ where,
 - i. $T_{A_p \cup B_p}(x) = \max\{T_{A_p}(x), T_{B_p}(x)\}$
 - ii. $I_{A_p \cup B_p}(x) = \max\{I_{A_p}(x), I_{B_p}(x)\}$
 - iii. $F_{A_p \cup B_p}(x) = \min\{F_{A_p}(x), F_{B_p}(x)\}$
- 4) $A_p \cap B_p = \{\langle x, T_{A_p \cap B_p}(x), I_{A_p \cap B_p}(x), F_{A_p \cap B_p}(x) \rangle | x \in S\}$ where,
 - i. $T_{A_p \cap B_p}(x) = \min\{T_{A_p}(x), T_{B_p}(x)\}$
 - ii. $I_{A_p \cap B_p}(x) = \min\{I_{A_p}(x), I_{B_p}(x)\}$
 - iii. $F_{A_p \cap B_p}(x) = \max\{F_{A_p}(x), F_{B_p}(x)\}$
- 5) Let a, b and c be three real numbers in $[0,1]$ satisfying the inequality $a^2 + b^2 + c^2 \leq 2$, where $a, b, c \in [0, \sqrt{2}]$. Then the triplet $\langle a, b, c \rangle$ is called a Pythagorean neutrosophic number.

Definition 2.1.4

Let $A_P = \{\langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle | x \in S\}$ be a PNS on S , then the complement of the set A_P may be defined as

$$(C_1) \quad \tilde{C}(A_P) = \{\langle x, F_{A_P}(x), I_{A_P}(x), T_{A_P}(x) \rangle | x \in S\}$$

$$(C_2) \quad \tilde{C}(A_P) = \{\langle x, F_{A_P}(x), \sqrt{2} - I_{A_P}(x), T_{A_P}(x) \rangle | x \in S\}$$

$$(C_3) \quad \tilde{C}(A_P) = \{\langle x, T_{A_P}(x), \sqrt{2} - I_{A_P}(x), F_{A_P}(x) \rangle | x \in S\}$$

The complement also should satisfy the PNS condition.

Example 2.1.5

Let $S = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set, and let A_P, B_P, C_P, D_P and P_P be PNSs on S ,

$$A_P = \langle (\tilde{x}, 0.9, 0.6, 0.3), (\tilde{y}, 0.8, 0.8, 0.3), (\tilde{z}, 0.7, 0.3, 0.5) \rangle.$$

$$B_P = \langle (\tilde{x}, 0.5, 0.7, 0.8), (\tilde{y}, 0.3, 0.7, 0.9), (\tilde{z}, 0.2, 0.8, 0.8) \rangle.$$

$$C_P = \langle (\tilde{x}, 0.9, 0.7, 0.3), (\tilde{y}, 0.8, 0.8, 0.3), (\tilde{z}, 0.7, 0.8, 0.5) \rangle.$$

$$D_P = \langle (\tilde{x}, 0.5, 0.6, 0.8), (\tilde{y}, 0.3, 0.7, 0.9), (\tilde{z}, 0.2, 0.8, 0.8) \rangle.$$

$$P_P = \langle (\tilde{x}, 0.4, 0.5, 0.5), (\tilde{y}, 0.7, 0.7, 0.8), (\tilde{z}, 0.5, 0.6, 0.5) \rangle.$$

$P \subseteq A$ because $T_P \leq T_A$; $I_P \leq I_A$; and $F_P \geq F_A$.

Further, $A \cup B = \{\langle T_A \vee T_B, I_A \vee I_B, F_A \wedge F_B \rangle\} = C$,

$$A \cap B = \{\langle T_A \wedge T_B, I_A \wedge I_B, F_A \vee F_B \rangle\} = D.$$

Definition 2.1.6

Let $\{A_{P_i} : i \in J\}$ be an arbitrary family of PNS in S . Then

$$a) \quad \cup A_{P_i} = \left\{ \langle x, \vee T_{A_{P_i}}(x), \vee I_{A_{P_i}}(x), \wedge F_{A_{P_i}}(x) \rangle : x \in S \right\}$$

$$b) \quad \cap A_{P_i} = \left\{ \langle x, \wedge T_{A_{P_i}}(x), \wedge I_{A_{P_i}}(x), \vee F_{A_{P_i}}(x) \rangle : x \in S \right\}$$

Definition 2.1.7

Let S be a universe. The PNSs null set and whole set in S are defined as follows:

0_{PN} is defined as:

$$0_{PN} = \{\langle x, 0, 0, \zeta \rangle | x \in S, \}$$

$\sqrt{2}_{PN}$ is defined as:

$$\sqrt{2}_{PN} = \{ \langle x, \xi, \psi, 0 \rangle | x \in S, \}$$

where $0 \leq \{ \xi = \max(T_i) \} \leq \sqrt{2}$, $0 \leq \{ \psi = \max(I_i) \} \leq \sqrt{2}$,
 $0 \leq \{ \zeta = \max(F_i) \} \leq \sqrt{2}$ with $0 \leq \xi^2 + \psi^2 + \zeta^2 \leq 2$.

Proposition 2.1.8

Let S be a universe. Then the following are true for any PNS A_P

- a) $0_{PN} \subseteq A_P, 0_{PN} \subseteq 0_{PN}$
- b) $A_P \subseteq \sqrt{2}_{PN}, \sqrt{2}_{PN} \subseteq \sqrt{2}_{PN}$

Proof:

a) $0_{PN} \subseteq A_P$

Let $A_P = \{ \langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle | x \in S \}$. From definition 2.1.7,

$$0_{PN} = \{ \langle x, 0, 0, \zeta \rangle | x \in S, \}$$

From definition 2.1.3, $0_{PN} \subseteq A_P$ implies that $0 \leq T_{A_P}(x)$, $0 \leq I_{A_P}(x)$,
 $\zeta \geq F_{A_P}(x), \forall x \in S$, where ζ is the maximum value from the Definition 2.1.7.

Hence the proof.

Similarly, the proof of $0_{PN} \subseteq 0_{PN}$ is straightforward.

b) $A_P \subseteq \sqrt{2}_{PN}$

Let $A_P = \{ \langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle | x \in S \}$. From definition 2.1.7,

$$\sqrt{2}_{PN} = \{ \langle x, \xi, \psi, 0 \rangle | x \in S, \}$$

From definition 2.1.3 $A_P \subseteq \sqrt{2}_{PN}$ implies that $T_{A_P}(x) \leq \xi$, $I_{A_P}(x) \leq \psi$,
 $F_{A_P}(x) \geq 0, \forall x \in S$ where ξ and ψ are the maximum values from the Definition 2.1.7.

Hence the proof.

Similarly, the proof of $\sqrt{2}_{PN} \subseteq \sqrt{2}_{PN}$ is straightforward.

Proposition 2.1.9

Let S be a non empty set. Let H_P, I_P, J_P and K_P are PNSs in S , then

- i. $H_P \subseteq I_P$, and $J_P \subseteq K_P$

- $\Rightarrow H_P \cup J_P \subseteq I_P \cup K_P, H_P \cap J_P \subseteq I_P \cap K_P,$
- ii. $H_P \subseteq I_P, H_P \subseteq J_P \Rightarrow H_P \subseteq I_P \cap J_P,$
- iii. $H_P \subseteq J_P, I_P \subseteq J_P \Rightarrow H_P \cup I_P \subseteq J_P,$
- iv. $H_P \subseteq I_P, I_P \subseteq J_P \Rightarrow H_P \subseteq J_P,$

Proof:

- i. Given that H_P, I_P, J_P and K_P are PNSs in S . And from Definition 2.1.3,

$$H_P \subseteq I_P \text{ iff } T_{H_P}(x) \leq T_{I_P}(x); I_{H_P}(x) \leq I_{I_P}(x); F_{H_P}(x) \geq F_{I_P}(x), \forall x \in S.$$

$$J_P \subseteq K_P \text{ iff } T_{J_P}(x) \leq T_{K_P}(x); I_{J_P}(x) \leq I_{K_P}(x); F_{J_P}(x) \geq F_{K_P}(x), \forall x \in S$$

Then

$$T_{H_P \cup J_P}(x) = \max\{T_{H_P}(x), T_{J_P}(x)\} \leq \max\{T_{I_P}(x), T_{K_P}(x)\} = T_{I_P \cup K_P}(x)$$

$$I_{H_P \cup J_P}(x) = \max\{I_{H_P}(x), I_{J_P}(x)\} \leq \max\{I_{I_P}(x), I_{K_P}(x)\} = I_{I_P \cup K_P}(x)$$

$$F_{H_P \cup J_P}(x) = \min\{F_{H_P}(x), F_{J_P}(x)\} \geq \min\{F_{I_P}(x), F_{K_P}(x)\} = F_{I_P \cup K_P}(x)$$

Also

$$T_{H_P \cap J_P}(x) = \min\{T_{H_P}(x), T_{J_P}(x)\} \geq \min\{T_{I_P}(x), T_{K_P}(x)\} = T_{I_P \cap K_P}(x)$$

$$I_{H_P \cap J_P}(x) = \min\{I_{H_P}(x), I_{J_P}(x)\} \geq \min\{I_{I_P}(x), I_{K_P}(x)\} = I_{I_P \cap K_P}(x)$$

$$F_{H_P \cap J_P}(x) = \max\{F_{H_P}(x), F_{J_P}(x)\} \leq \max\{F_{I_P}(x), F_{K_P}(x)\} = F_{I_P \cap K_P}(x)$$

- ii) $H_P \subseteq I_P, H_P \subseteq J_P \Rightarrow H_P \subseteq I_P \cap J_P,$

$$H_P \subseteq I_P \text{ iff } T_{H_P}(x) \leq T_{I_P}(x); I_{H_P}(x) \leq I_{I_P}(x); F_{H_P}(x) \geq F_{I_P}(x), \forall x \in S.$$

$$H_P \subseteq J_P \text{ iff } T_{H_P}(x) \leq T_{J_P}(x); I_{H_P}(x) \leq I_{J_P}(x); F_{H_P}(x) \geq F_{J_P}(x), \forall x \in S.$$

$$\min\{T_{H_P}(x)\} \leq \min\{T_{I_P}(x), T_{J_P}(x)\}$$

$$\min\{I_{H_P}(x)\} \leq \min\{I_{I_P}(x), I_{J_P}(x)\}$$

$$\max\{F_{H_P}(x)\} \leq \max\{F_{I_P}(x), F_{J_P}(x)\}$$

- iii) The result is similar to case (ii).

- iv) The statement is obvious by direct observation.

Definition 2.1.10

Let \tilde{X} and \tilde{Y} be two non empty universal sets and $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. If $A_P = \{\langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle: x \in \tilde{X}\}$ is a PNS in \tilde{X} , then the image of A_P under ϕ denoted by $\phi(A_P)$ is a PNS in \tilde{Y} is denoted by

$$\phi(A_P) = \{\langle y, \phi(T_{A_P})(y), \phi(I_{A_P})(y), \phi(F_{A_P})(y) \rangle: y \in \tilde{Y}\}$$

$$\phi(T_{A_P})(y) = \begin{cases} \sup_{x \in \phi^{-1}(y)} T_{A_P}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(I_{A_P})(y) = \begin{cases} \sup_{x \in \phi^{-1}(y)} I_{A_P}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(F_{A_P})(y) = \begin{cases} \inf_{x \in \phi^{-1}(y)} F_{A_P}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

Definition 2.1.11

Let \tilde{X} and \tilde{Y} be two non empty universal sets and $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. If a PNS in \tilde{Y} is $B_P = \{\langle y, T_{B_P}(y), I_{B_P}(y), F_{B_P}(y) \rangle: y \in \tilde{Y}\}$, then the pre-image of B_P under ϕ defined by $\phi^{-1}(B_P)$ is the PNS in \tilde{X} is denoted by

$$\phi^{-1}(B_P) = \{\langle x, \phi^{-1}(T_{B_P})(x), \phi^{-1}(I_{B_P})(x), \phi^{-1}(F_{B_P})(x) \rangle: x \in \tilde{X}\}$$

where

$$\phi^{-1}(T_{B_P})(x) = T_{B_P}(\phi(x)),$$

$$\phi^{-1}(I_{B_P})(x) = I_{B_P}(\phi(x)),$$

$$\phi^{-1}(F_{B_P})(x) = F_{B_P}(\phi(x)).$$

Proposition 2.1.12

Let $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. $A_{P_i} (i \in J)$ be PNSs in \tilde{X} , $B_{P_j} (j \in K)$ be PNSs in \tilde{Y} and then

- i. $A_{P_1} \subseteq A_{P_2} \Rightarrow \phi(A_{P_1}) \subseteq \phi(A_{P_2})$
- ii. $B_{P_1} \subseteq B_{P_2} \Rightarrow \phi^{-1}(B_{P_1}) \subseteq \phi^{-1}(B_{P_2})$
- iii. $A_P \subseteq \phi^{-1}(\phi(A_P))$ and if ϕ is injective, then $A_P = \phi^{-1}(\phi(A_P))$

- iv. $\phi(\phi^{-1}(B_P)) \subseteq B_P$ and if ϕ is surjective, then $\phi(\phi^{-1}(B_P)) = B_P$
- v. $\phi^{-1}(\sqrt{2}_{PN}) = \sqrt{2}_{PN}$
- vi. $\phi^{-1}(0_{PN}) = 0_{PN}$
- vii. $\phi(\sqrt{2}_{PN}) = \sqrt{2}_{PN}$ if ϕ is surjective
- viii. $\phi(0_{PN}) = 0_{PN}$

Proof:

i) $A_{P_1} \subseteq A_{P_2} \Rightarrow \phi(A_{P_1}) \subseteq \phi(A_{P_2})$

From the Definition 2.1.3, $A_{P_1} \subseteq A_{P_2}$ iff $T_{A_{P_1}}(x) \leq T_{A_{P_2}}(x); I_{A_{P_1}}(x) \leq I_{A_{P_2}}(x); F_{A_{P_1}}(x) \geq F_{A_{P_2}}(x), \forall x \in \tilde{X}$.

From the Definition 2.1.11

$$\begin{aligned} \phi(A_{P_1}) &= \left\{ \left\langle y, \phi(T_{A_{P_1}})(y), \phi(I_{A_{P_1}})(y), \phi(F_{A_{P_1}})(y) \right\rangle : y \in \tilde{Y} \right\} \\ \phi(A_{P_1})(y) &= \left\langle y, \sup_{x \in \phi^{-1}(y)} T_{A_{P_1}}(x), \sup_{x \in \phi^{-1}(y)} I_{A_{P_1}}(x), \inf_{x \in \phi^{-1}(y)} F_{A_{P_1}}(x) \right\rangle \\ \phi(A_{P_2}) &= \left\{ \left\langle y, \phi(T_{A_{P_2}})(y), \phi(I_{A_{P_2}})(y), \phi(F_{A_{P_2}})(y) \right\rangle : y \in \tilde{Y} \right\} \\ \phi(A_{P_2})(y) &= \left\langle y, \sup_{x \in \phi^{-1}(y)} T_{A_{P_2}}(x), \sup_{x \in \phi^{-1}(y)} I_{A_{P_2}}(x), \inf_{x \in \phi^{-1}(y)} F_{A_{P_2}}(x) \right\rangle \end{aligned}$$

Since $T_{A_{P_1}}(x) \leq T_{A_{P_2}}(x)$ taking supremum on both sides implies

$$\sup_{x \in \phi^{-1}(y)} T_{A_{P_1}}(x) \leq \sup_{x \in \phi^{-1}(y)} T_{A_{P_2}}(x)$$

Since $I_{A_{P_1}}(x) \leq I_{A_{P_2}}(x)$ taking supremum on both sides implies

$$\sup_{x \in \phi^{-1}(y)} I_{A_{P_1}}(x) \leq \sup_{x \in \phi^{-1}(y)} I_{A_{P_2}}(x)$$

Since $I_{A_{P_1}}(x) \leq I_{A_{P_2}}(x)$ taking infimum on both sides implies

$$\inf_{x \in \phi^{-1}(y)} F_{A_{P_1}}(x) \geq \inf_{x \in \phi^{-1}(y)} F_{A_{P_2}}(x)$$

Thus $T_{\phi(A_{P_1})}(y) \leq T_{\phi(A_{P_2})}(y), I_{\phi(A_{P_1})}(y) \leq I_{\phi(A_{P_2})}(y), F_{\phi(A_{P_1})}(y) \geq F_{\phi(A_{P_2})}(y)$

ii) If $B_{P_1} \subseteq B_{P_2}$, then $\phi^{-1}(B_{P_1}) \subseteq \phi^{-1}(B_{P_2})$

Let $x \in \tilde{X}$, and $y = \phi(x) \in \tilde{Y}$. The inverse image $\phi^{-1}(B_P)$ is defined as

$$\begin{aligned} \phi^{-1}(B_P)(x) &= \{x, \phi^{-1}(T_{B_P})(x), \phi^{-1}(I_{B_P})(x), \phi^{-1}(F_{B_P})(x)\}; x \in \tilde{X}\} \\ \phi^{-1}(T_{B_P})(x) &= T_{B_P}(\phi(x)), \\ \phi^{-1}(I_{B_P})(x) &= I_{B_P}(\phi(x)), \\ \phi^{-1}(F_{B_P})(x) &= F_{B_P}(\phi(x)) \end{aligned}$$

Given

$$B_{P_1} \subseteq B_{P_2} \text{ iff } T_{B_{P_1}}(y) \leq T_{B_{P_2}}(y); I_{B_{P_1}}(y) \leq I_{B_{P_2}}(y); F_{B_{P_1}}(y) \geq F_{B_{P_2}}(y), \forall y \in \tilde{Y}.$$

Since $y = \phi(x)$, for any $x \in \tilde{X}$.

$$\phi^{-1}(T_{B_P})(x) = T_{B_P}(y), \phi^{-1}(I_{B_P})(x) = I_{B_P}(y), \phi^{-1}(F_{B_P})(x) = F_{B_P}(y)$$

$B_{P_1} \subseteq B_{P_2}$ implies

$$\begin{aligned} \phi^{-1}(T_{B_{P_1}})(x) &\leq \phi^{-1}(T_{B_{P_2}})(x) \\ \phi^{-1}(I_{B_{P_1}})(x) &\leq \phi^{-1}(I_{B_{P_2}})(x) \\ \phi^{-1}(F_{B_{P_1}})(x) &\geq \phi^{-1}(F_{B_{P_2}})(x) \end{aligned}$$

Hence $\phi^{-1}(B_{P_1}) \subseteq \phi^{-1}(B_{P_2})$.

iii) $A_P \subseteq \phi^{-1}(\phi(A_P))$, then $A_P = \phi(\phi^{-1}(A_P))$.

Let $A_P = \{x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x)\}; x \in \tilde{X}\}$; From the Definition 2.1.10

$$\begin{aligned} \phi(A_P) &= \{y, \phi(T_{A_P})(y), \phi(I_{A_P})(y), \phi(F_{A_P})(y)\}; y \in \tilde{Y}\} \\ \phi(A_P)(y) &= \left\{ y, \sup_{x \in \phi^{-1}(y)} T_{A_P}(x), \sup_{x \in \phi^{-1}(y)} I_{A_P}(x), \inf_{x \in \phi^{-1}(y)} F_{A_P}(x) \right\} \\ \phi^{-1}(\phi(A_P)) &= \left\{ \left(x, \phi^{-1}(T_{\phi(A_P)})(x), \phi^{-1}(I_{\phi(A_P)})(x), \phi^{-1}(F_{\phi(A_P)})(x) \right); x \in \tilde{X} \right\} \end{aligned}$$

Since $x \in \tilde{X}$ and $\phi(x) \in \tilde{Y}$, and

$$= \left\{ y, \sup_{x \in \phi^{-1}(\phi(x))} T_{A_P}(x), \sup_{x \in \phi^{-1}(\phi(x))} I_{A_P}(x), \inf_{x \in \phi^{-1}(\phi(x))} F_{A_P}(x) \right\}$$

$$\phi^{-1}(T_{\phi(A_P)})(x) = \sup_{x \in \phi^{-1}(\phi(x))} T_{A_P}(x) \geq T_{A_P}(x)$$

$$\phi^{-1}(I_{\phi(A_P)})(x) = \sup_{x \in \phi^{-1}(\phi(x))} I_{A_P}(x) \geq I_{A_P}(x)$$

$$\phi^{-1}(F_{\phi(A_P)})(x) = \sup_{x \in \phi^{-1}(\phi(x))} F_{A_P}(x) \leq F_{A_P}(x)$$

Therefore $\phi^{-1}(T_{\phi(A_P)})(x) \geq T_{A_P}(x)$; $\phi^{-1}(I_{\phi(A_P)})(x) \geq I_{A_P}(x)$;

$\phi^{-1}(F_{\phi(A_P)})(x) \leq F_{A_P}(x)$. Hence $A_P \subseteq \phi^{-1}(\phi(A_P))$.

To prove: If ϕ is injective, then $A_P = \phi(\phi^{-1}(A_P))$

Assume that ϕ is injective, then $\phi(x_1) = \phi(x_2) \Rightarrow x_1 = x_2$. Given that A_P be PNS in \tilde{X} . $A_P = \{ \langle x, T_{A_P}(x), I_{A_P}(x), F_{A_P}(x) \rangle : x \in \tilde{X} \}$. Let A_P be a PNS in \tilde{Y} .

$$\phi^{-1}(A_P) = \{ \langle x, \phi^{-1}(T_{A_P})(x), \phi^{-1}(I_{A_P})(x), \phi^{-1}(F_{A_P})(x) \rangle : x \in \tilde{Y} \}$$

$$\phi^{-1}(A_P) = \{ \langle x, T_{B_P}(\phi(x)), I_{B_P}(\phi(x)), F_{B_P}(\phi(x)) \rangle : x \in \tilde{Y} \}$$

Then applying ϕ to $\phi^{-1}(A_P)$

$$\phi(\phi^{-1}(A_P)) = \{ \langle y, \phi(T_{A_P}(\phi(x))), \phi(I_{A_P}(\phi(x))), \phi(F_{A_P}(\phi(x))) \rangle : x \in \tilde{Y} \}$$

$$\phi(\phi^{-1}(A_P)) = \left\{ \left\langle x, \sup_{x \in \phi^{-1}(y)} (T_{A_P}(\phi(x))), \sup_{x \in \phi^{-1}(y)} (I_{A_P}(\phi(x))), \inf_{x \in \phi^{-1}(y)} (F_{A_P}(\phi(x))) \right\rangle : y \in \tilde{Y} \right\}$$

But since ϕ is injective, $\phi^{-1}(y)$ contains at most one element, say x , such that $\phi(x) = y$.

$$\sup_{x \in \phi^{-1}(y)} (T_{A_P}(\phi(x))) = T_{A_P}(y)$$

$$\sup_{x \in \phi^{-1}(y)} (I_{A_P}(\phi(x))) = I_{A_P}(y)$$

$$\inf_{x \in \phi^{-1}(y)} (F_{A_P}(\phi(x))) = F_{A_P}(y)$$

$$\phi(\phi^{-1}(A_P)) = \{\langle x, T_{A_P}(y), I_{A_P}(y), F_{A_P}(y) \rangle : y \in \tilde{Y}\} . \phi(\phi^{-1}(A_P)) = A_P.$$

iv) The result is similar to case (iii).

$$v) \phi^{-1}(\sqrt{2}_{PN}) = \sqrt{2}_{PN}$$

From the definition 2.1.7

$$\sqrt{2}_{PN} = \{\langle x, \xi, \psi, 0 \rangle | x \in \tilde{X}, 0 \leq \xi, \psi \leq \sqrt{2}, 0 \leq \xi^2 + \psi^2 \leq 2\}$$

Then the inverse image of

$$\phi^{-1}(B_P) = \{\langle x, \phi^{-1}(T_{B_P})(x), \phi^{-1}(I_{B_P})(x), \phi^{-1}(F_{B_P})(x) \rangle : x \in \tilde{X}\}$$

Where

$$0 \leq \{\xi = \max(T_i)\} \leq \sqrt{2} \quad , \quad 0 \leq \{\psi = \max(I_i)\} \leq \sqrt{2}, \quad 0 \leq \{\zeta = \max(F_i)\} \leq \sqrt{2}$$

with $0 \leq \xi^2 + \psi^2 + \zeta^2 \leq 2$.

$$vi) \phi^{-1}(0_{PN}) = 0_{PN}$$

From the definition 2.1.7 $0_{PN} = \{\langle x, 0, 0, \zeta \rangle | x \in \tilde{X}, 0 \leq \zeta \leq \sqrt{2}, \zeta^2 \leq 2\}$

$$\phi^{-1}(0_{PN})(x) = \{\langle x, \phi^{-1}(T_{0_{PN}})(x), \phi^{-1}(I_{0_{PN}})(x), \phi^{-1}(F_{0_{PN}})(x) \rangle : x \in \tilde{X}\}$$

$$\phi^{-1}(0_{PN})(x) = \{\langle x, 0, 0, \zeta(\phi(x)) \rangle | x \in \tilde{X}\}$$

But in 0_{PN} , for all $y \in \tilde{Y}$. $T_{0_{PN}}(y) = 0$, $I_{0_{PN}}(y) = 0$, $F_{0_{PN}}(y) = \zeta$,

where $\zeta \in [0, \sqrt{2}]$, with $0 \leq \zeta^2 \leq 2$. Thus, for any $x \in \tilde{X}$, $\phi(x) \in \tilde{Y}$.

Therefore $\zeta(\phi(x)) \in [0, \sqrt{2}]$ and $\zeta^2 \leq 2$.

vii) The result is similar to case (v).

viii) The result is similar to case (vi).

Neutrosophic Spherical Set

This section proposes the notion of neutrosophic spherical set (NSS). In this set, the square sum of the component values is extended to $\sqrt{3}$. The NSS permits higher membership values than the PNS, even when the squared sum of its component values is lower than that of the PNS. This section also verifies the basic concepts and properties of the NSS.

Definition 2.1.13

Let S be a non empty set. A Neutrosophic Spherical Set (NSS) A_S is of the form

$$A_S = \left\{ \left\langle x, \left(T_{A_S}(x), I_{A_S}(x), F_{A_S}(x) \right) \right\rangle \mid x \in S \right\} \quad (2.1.1)$$

where, $T_{A_S}(x), I_{A_S}(x), F_{A_S}(x) \in [0,1]$ and

$$0 \leq T_{A_S}^2(x) + I_{A_S}^2(x) + F_{A_S}^2(x) \leq \sqrt{3}, \quad \forall x \in S \quad (2.1.2)$$

where $T_{A_S}(x)$ is truth membership, $I_{A_S}(x)$ is indeterminacy membership, $F_{A_S}(x)$ is false membership, $T_{A_S}(x), I_{A_S}(x)$ and $F_{A_S}(x)$ are dependent components.

Example 2.1.14

Let $S = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set and a NSS A_S on S is denoted as

$$A_S = \langle (\tilde{x}, 0.9, 0.8, 0.5), (\tilde{y}, 0.6, 0.8, 0.6), (\tilde{z}, 0.7, 0.9, 0.8) \rangle.$$

Definition 2.1.15

Let S be a non empty set, and the NSSs

$$A_S = \left\{ \left\langle x, \left(T_{A_S}(x), I_{A_S}(x), F_{A_S}(x) \right) \right\rangle \mid x \in S \right\} \& B_S = \left\{ \left\langle x, \left(T_{B_S}(x), I_{B_S}(x), F_{B_S}(x) \right) \right\rangle \mid x \in S \right\}$$

satisfy the following conditions:

1. $A_S \subseteq B_S$ iff $T_{A_S}(x) \leq T_{B_S}(x); I_{A_S}(x) \leq I_{B_S}(x); F_{A_S}(x) \geq F_{B_S}(x) \forall x \in S$
2. $A_S = B_S$ iff $T_{A_S}(x) = T_{B_S}(x); I_{A_S}(x) = I_{B_S}(x); F_{A_S}(x) = F_{B_S}(x) \forall x \in S$
3. $A_S \cap B_S = \left\{ \left\langle x, \left(T_{A_S \cap B_S}(x), I_{A_S \cap B_S}(x), F_{A_S \cap B_S}(x) \right) \right\rangle \mid x \in S \right\}$ where,
 - i. $T_{A_S \cap B_S}(x) = \min\{T_{A_S}(x), T_{B_S}(x)\}$
 - ii. $I_{A_S \cap B_S}(x) = \min\{I_{A_S}(x), I_{B_S}(x)\}$

- iii. $F_{A_S \cap B_S}(x) = \max\{F_{A_S}(x), F_{B_S}(x)\}$
4. $A_S \cup B_S = \{\langle x, T_{A_S \cup B_S}(x), I_{A_S \cup B_S}(x), F_{A_S \cup B_S}(x) \rangle | x \in S\}$ where,
- i. $T_{A_S \cup B_S}(x) = \max\{T_{A_S}(x), T_{B_S}(x)\}$
- ii. $I_{A_S \cup B_S}(x) = \max\{I_{A_S}(x), I_{B_S}(x)\}$
- iii. $F_{A_S \cup B_S}(x) = \min\{F_{A_S}(x), F_{B_S}(x)\}$
5. Let a, b and c be three real numbers in $[0,1]$ satisfying the inequality $a^2 + b^2 + c^2 \leq \sqrt{3}$, where $a, b, c \in [0,1]$. Then the triplet $\langle a, b, c \rangle$ is called a neutrosophic spherical number.

Definition 2.1.16

Let $A_S = \{\langle x, T_{A_S}(x), I_{A_S}(x), F_{A_S}(x) \rangle | x \in S\}$ be a NSS on S , then the complement of the set A_S can be defined as

$$(C_1) \tilde{C}(A_S) = \{\langle x, F_{A_S}(x), I_{A_S}(x), T_{A_S}(x) \rangle | x \in S\}$$

$$(C_2) \tilde{C}(A_S) = \{\langle x, F_{A_S}(x), 1 - I_{A_S}(x), T_{A_S}(x) \rangle | x \in S\}$$

$$(C_3) \tilde{C}(A_S) = \{\langle x, 1 - T_{A_S}(x), 1 - I_{A_S}(x), 1 - F_{A_S}(x) \rangle | x \in S\}$$

$$(C_4) \tilde{C}(A_S) = \{\langle x, 1 - T_{A_S}(x), I_{A_S}(x), 1 - F_{A_S}(x) \rangle | x \in S\}$$

Example 2.1.17

Let $S = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ and let A_S, B_S, C_S, D_S and P_S be NSSs.

$$A_S = \langle (\tilde{x}, 0.9, 0.8, 0.5), (\tilde{y}, 0.6, 0.8, 0.6), (\tilde{z}, 0.7, 0.9, 0.8) \rangle.$$

$$B_S = \langle (\tilde{x}, 0.5, 0.7, 0.3), (\tilde{y}, 0.3, 0.9, 0.8), (\tilde{z}, 0.6, 0.9, 0.6) \rangle.$$

$$C_S = \langle (\tilde{x}, 0.9, 0.8, 0.3), (\tilde{y}, 0.6, 0.9, 0.6), (\tilde{z}, 0.7, 0.9, 0.6) \rangle.$$

$$D_S = \langle (\tilde{x}, 0.5, 0.7, 0.5), (\tilde{y}, 0.3, 0.8, 0.8), (\tilde{z}, 0.6, 0.9, 0.8) \rangle.$$

$$P_S = \langle (\tilde{x}, 0.5, 0.7, 0.4), (\tilde{y}, 0.6, 0.7, 0.6), (\tilde{z}, 0.6, 0.6, 0.9) \rangle.$$

$P_S \subseteq C_S$ because $T_{P_S} \leq T_{C_S}; I_{P_S} \leq I_{C_S};$ and $F_{P_S} \geq F_{C_S}$.

Further, $A_S \cup B_S = \{\langle T_{\tilde{A}_S} \vee T_{\tilde{B}_S}, I_{\tilde{A}_S} \vee I_{\tilde{B}_S}, F_{\tilde{A}_S} \wedge F_{\tilde{B}_S} \rangle\} = C_S,$

$A_S \cap B_S = \{\langle T_{\tilde{A}_S} \wedge T_{\tilde{B}_S}, I_{\tilde{A}_S} \wedge I_{\tilde{B}_S}, F_{\tilde{A}_S} \vee F_{\tilde{B}_S} \rangle\} = D_S$

Definition 2.1.18

Assume that $\{A_{S_i}; i \in J\}$ is an arbitrary family of NSS in S . Then

$$a) \cap A_{S_i} = \left\{ \left\langle x, \wedge T_{A_{S_i}}(x), \wedge I_{A_{S_i}}(x), \vee F_{A_{S_i}}(x) \right\rangle : x \in S \right\}$$

$$b) \cup A_{S_i} = \left\{ \left\langle x, \vee T_{A_{S_i}}(x), \vee I_{A_{S_i}}(x), \wedge F_{A_{S_i}}(x) \right\rangle : x \in S \right\}$$

Definition 2.1.19

Let S be a universe. The NSSs null set and whole set in S are defined as follows:

0_{NS} is defined as:

$$0_{NS} = \{ \langle x, 0, 0, \zeta \rangle | x \in S, \}$$

1_{NS} is defined as:

$$1_{NS} = \{ \langle x, \xi, \psi, 0 \rangle | x \in S, \}$$

where $0 \leq \{ \xi = \max(T_i) \} \leq 1$, $0 \leq \{ \psi = \max(I_i) \} \leq 1$, $0 \leq \{ \zeta = \max(F_i) \} \leq 1$ with $0 \leq \xi^2 + \psi^2 + \zeta^2 \leq \sqrt{3}$.

Proposition 2.1.20

The following are true for any NSS A_S :

- a) $0_{NS} \subseteq A_S, 0_{NS} \subseteq 0_{NS}$
- b) $A_S \subseteq 1_{NS}, 1_{NS} \subseteq 1_{NS}$

Proof: This proof is similar to Proposition 2.1.8 and is based on Definition 2.1.19.

Proposition 2.1.21

Let W_S, X_S, Y_S and Z_S are NSSs in S , then

- i. $W_S \subseteq X_S$, and $Y_S \subseteq Z_S$
 $\Rightarrow W_S \cup Y_S \subseteq X_S \cup Z_S, W_S \cap Y_S \subseteq X_S \cap Z_S$
- ii. $W_S \subseteq X_S, W_S \subseteq Y_S \Rightarrow W_S \subseteq X_S \cap Y_S$
- iii. $W_S \subseteq Y_S, X_S \subseteq Y_S \Rightarrow W_S \cup X_S \subseteq Y_S$
- iv. $W_S \subseteq X_S, X_S \subseteq Y_S \Rightarrow W_S \subseteq Y_S$

Proof: This proof is similar to Proposition 2.1.9 and is based on Definition 2.1.15.

Definition 2.1.22

Let \tilde{X} and \tilde{Y} be two non empty universal sets and $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. If $A_S = \{(x, T_{A_S}(x), I_{A_S}(x), F_{A_S}(x)): x \in \tilde{X}\}$ is a NSS in \tilde{X} , then the image of A_S under ϕ defined by (A_S) is the NSS in \tilde{Y} is denoted by

$$\begin{aligned} \phi(A_S) &= \{(y, \phi(T_{A_S})(y), \phi(I_{A_S})(y), \phi(F_{A_S})(y)): y \in \tilde{Y}\} \\ \phi(T_{A_S})(y) &= \begin{cases} \sup_{x \in \phi^{-1}(y)} T_{A_S}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ \phi(I_{A_S})(y) &= \begin{cases} \sup_{x \in \phi^{-1}(y)} I_{A_S}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ \phi(F_{A_S})(y) &= \begin{cases} \inf_{x \in \phi^{-1}(y)} F_{A_S}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise,} \end{cases} \end{aligned}$$

Definition 2.1.23

Let \tilde{X} and \tilde{Y} two non empty universal sets and $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. If $B_S = \{(y, T_{B_S}(y), I_{B_S}(y), F_{B_S}(y)): y \in \tilde{Y}\}$ is a NSS in \tilde{Y} , then the pre-image of B_S under ϕ defined by $\phi^{-1}(B_S)$ is the NSS in \tilde{X} is denoted by

$$\phi^{-1}(B_S) = \{(x, \phi^{-1}(T_{B_S})(x), \phi^{-1}(I_{B_S})(x), \phi^{-1}(F_{B_S})(x)): x \in \tilde{X}\}$$

Where

$$\phi^{-1}(T_{B_S})(x) = T_{B_S}(\phi(x)), \phi^{-1}(I_{B_S})(x) = I_{B_S}(\phi(x)), \phi^{-1}(F_{B_S})(x) = F_{B_S}(\phi(x))$$

Proposition 2.1.24

Let $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. $A_{S_i} (i \in J)$ be NSSs in \tilde{X} , $B_{S_j} (j \in K)$ be PNSs in \tilde{Y} and then

- i. $A_{S_1} \subseteq A_{S_2} \Rightarrow \phi(A_{S_1}) \subseteq \phi(A_{S_2})$
- ii. $B_{S_1} \subseteq B_{S_2} \Rightarrow \phi^{-1}(B_{S_1}) \subseteq \phi^{-1}(B_{S_2})$
- iii. $A_S \subseteq \phi^{-1}(\phi(A_S))$ and if ϕ is injective, then $A_S = \phi^{-1}(\phi(A_S))$
- iv. $\phi(\phi^{-1}(B_S)) \subseteq B_S$ and if ϕ is surjective, then $\phi(\phi^{-1}(B_S)) = B_S$
- v. $\phi^{-1}(1_{NS}) = 1_{NS}$
- vi. $\phi^{-1}(0_{NS}) = 0_{NS}$

- vii. $\phi(1_{NS}) = 1_{NS}$ if ϕ is surjective
- viii. $\phi(0_{NS}) = 0_{NS}$

Proof: This proof is similar to Proposition 2.1.12 and is based on Definition 2.1.15, Definition 2.1.22 and Definition 2.1.23

Definition 2.1.25 [4]

Consider a non empty set S . A Fermatean Neutrosophic set (FNS) A_F is of the form

$$A_F = \{\langle x, (T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)) \rangle | x \in S\}$$

where $T_{A_F}(x), I_{A_F}(x), F_{A_F}(x) \in [0,1]$ and

$$0 \leq T_{A_F}^3(x) + I_{A_F}^3(x) + F_{A_F}^3(x) \leq 2 \forall x \in S.$$

where $T_{A_F}(x)$ is truth membership, $I_{A_F}(x)$ is indeterminacy membership, $F_{A_F}(x)$ is false membership. $T_{A_F}(x), I_{A_F}(x)$ are independent components and $F_{A_F}(x)$ is dependent component with respect to $T_{A_F}(x)$ that is $0 \leq T_{A_F}^3(x) + F_{A_F}^3(x) \leq 1$, $0 \leq I_{A_F}^3(x) \leq 1$.

Example 2.1.26

Let $S = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set and a FNS A_p on S is denoted as

$$A_p = \langle (\tilde{x}, 0.85, 0.7, 0.7), (\tilde{y}, 0.9, 0.8, 0.6), (\tilde{z}, 0.5, 0.8, 0.9) \rangle.$$

Definition 2.1.27

Let S be a non empty set, and the FNSs A_F and B_F is of the form $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x) \rangle | x \in S\}$, $B_F = \{\langle x, T_{B_F}(x), I_{B_F}(x), F_{B_F}(x) \rangle | x \in S\}$ satisfy the following conditions:

1. $A_F \subseteq B_F$ iff $T_{A_F}(x) \leq T_{B_F}(x); I_{A_F}(x) \leq I_{B_F}(x); F_{A_F}(x) \geq F_{B_F}(x)$
2. $A_F = B_F$ iff $T_{A_F}(x) = T_{B_F}(x); I_{A_F}(x) = I_{B_F}(x); F_{A_F}(x) = F_{B_F}(x) \forall x \in S$
3. $A_F \cap B_F = \{\langle x, T_{A_F \cap B_F}(x), I_{A_F \cap B_F}(x), F_{A_F \cap B_F}(x) \rangle | x \in S\}$ where,
 - i. $T_{A_F \cap B_F}(x) = \min\{T_{A_F}(x), T_{B_F}(x)\}$

- ii. $I_{A_F \cap B_F}(x) = \min\{I_{A_F}(x), I_{B_F}(x)\}$
- iii. $F_{A_F \cap B_F}(x) = \max\{F_{A_F}(x), F_{B_F}(x)\}$
- 4. $A_F \cup B_F = \{\langle x, T_{A_F \cup B_F}(x), I_{A_F \cup B_F}(x), F_{A_F \cup B_F}(x) \rangle | x \in S\}$ where,
 - i. $T_{A_F \cup B_F}(x) = \max\{T_{A_F}(x), T_{B_F}(x)\}$
 - ii. $I_{A_F \cup B_F}(x) = \max\{I_{A_F}(x), I_{B_F}(x)\}$
 - iii. $F_{A_F \cup B_F}(x) = \min\{F_{A_F}(x), F_{B_F}(x)\}$
- 5. Let a, b and c be three real numbers in $[0,1]$ satisfying the inequality $a^3 + b^3 + c^3 \leq 2$, where $a, b, c \in [0,1]$. Then the triplet $\langle a, b, c \rangle$ is called a Fermatean neutrosophic number.

Definition 2.1.28

Let $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x) \rangle | x \in S\}$ be a FNS on S , then the complement of the set A_F , can be defined as

$$(C_1) C(A_F) = \{\langle x, F_{A_F}(x), 1 - I_{A_F}(x), T_{A_F}(x) \rangle | x \in S\}$$

$$(C_2) C(A_F) = \{\langle x, 1 - T_{A_F}(x), I_{A_F}(x), 1 - F_{A_F}(x) \rangle | x \in S\}$$

$$(C_3) C(A_F) = \{\langle x, F_{A_F}(x), I_{A_F}(x), T_{A_F}(x) \rangle | x \in S\}$$

$$(C_4) C(A_F) = \{\langle x, 1 - T_{A_F}(x), 1 - I_{A_F}(x), 1 - F_{A_F}(x) \rangle | x \in S\}$$

Example 2.1.29

Let $S = \{\tilde{x}, \tilde{y}, z\}$ and let A_F, B_F, C_F, D_F and P_F are FNSs.

$$A_F = \langle (\tilde{x}, 0.85, 0.9, 0.7), (\tilde{y}, 0.9, 0.8, 0.6), (\tilde{z}, 0.5, 0.8, 0.9) \rangle.$$

$$B_F = \langle (\tilde{x}, 0.8, 0.9, 0.7), (\tilde{y}, 0.6, 0.9, 0.8), (\tilde{z}, 0.9, 0.7, 0.6) \rangle.$$

$$C_F = \langle (\tilde{x}, 0.85, 0.9, 0.7), (\tilde{y}, 0.9, 0.9, 0.6), (\tilde{z}, 0.9, 0.8, 0.6) \rangle.$$

$$D_F = \langle (\tilde{x}, 0.8, 0.9, 0.7), (\tilde{y}, 0.6, 0.8, 0.8), (\tilde{z}, 0.5, 0.7, 0.9) \rangle.$$

$$P_F = \langle (\tilde{x}, 0.8, 0.8, 0.7), (\tilde{y}, 0.85, 0.7, 0.8), (\tilde{z}, 0.9, 0.8, 0.7) \rangle.$$

$P_F \subseteq C_F$ because $T_{P_F} \leq T_{C_F}$; $I_{P_F} \leq I_{C_F}$; and $F_{P_F} \geq F_{C_F}$. Further,

$$A_F \cup B_F = \{\langle T_{A_F} \vee T_{B_F}, I_{A_F} \vee I_{B_F}, F_{A_F} \wedge F_{B_F} \rangle\} = C_F \text{ and}$$

$$A_F \cap B_F = \{\langle T_{A_F} \wedge T_{B_F}, I_{A_F} \wedge I_{B_F}, F_{A_F} \vee F_{B_F} \rangle\} = D_F$$

Definition 2.1.30

Let $\{A_{F_i}: i \in J\}$ be an arbitrary family of FNS in S . Then

- a) $\cap A_{F_i} = \left\{ \langle x, \wedge T_{A_{F_i}}(x), \wedge I_{A_{F_i}}(x), \vee F_{A_{F_i}}(x) \rangle : x \in S \right\}$
- b) $\cup A_{F_i} = \left\{ \langle x, \vee T_{A_{F_i}}(x), \vee I_{A_{F_i}}(x), \wedge F_{A_{F_i}}(x) \rangle : x \in S \right\}$

Definition 2.1.31

Let S be a universal non empty set. The FNS 0_{FN} is an empty set and 1_{FN} is a whole set in S are defined as follows:

0_{FN} is defined as

$$0_{FN} = \{ \langle x, 0, 0, 1 \rangle : x \in S \}$$

1_{FN} is defined as

$$1_{FN} = \{ \langle x, 1, 1, 0 \rangle : x \in S \}$$

where 1 and 0 represent the constant maps that assign every element of S to 1 and 0, respectively.

Proposition 2.1.32

The following are true for any FNS A_F :

- a) $0_{FN} \subseteq A_F, 0_{FN} \subseteq 0_{FN}$
- b) $A_F \subseteq 1_{FN}, 1_{FN} \subseteq 1_{FN}$

Proof: This proof is similar to Proposition 2.1.8 and is based on Definition 2.1.31.

Proposition 2.1.33

Let A_F, B_F, C_F and D_F are FNSs in X , then

- i. $A_F \subseteq B_F$, and $C_F \subseteq D_F$
 $\Rightarrow A_F \cup C_F \subseteq B_F \cup D_F, A_F \cap C_F \subseteq B_F \cap D_F.$
- ii. $A_F \subseteq B_F, A_F \subseteq C_F \Rightarrow A_F \subseteq B_F \cap C_F,$
- iii. $A_F \subseteq C_F, B_F \subseteq C_F \Rightarrow A_F \cup B_F \subseteq C_F,$
- iv. $A_F \subseteq B_F, B_F \subseteq C_F \Rightarrow A_F \subseteq C_F,$

$$v. \overline{1_{FN}} = 0_{FN} \text{ and } \overline{0_{FN}} = 1_{FN}.$$

Proof: This proof is similar to Proposition 2.1.9 and is based on Definition 2.1.27 and Definition 2.1.31.

Definition 2.1.34

Let X and Y be two non empty universal sets and $\phi: X \rightarrow Y$ be a function. If $A_F = \{\langle x, T_{A_F}(x), I_{A_F}(x), F_{A_F}(x) \rangle: x \in X\}$ is a FNS in X , then the image of A_F under ϕ defined by $\phi(A_F)$ is the FNS in Y denoted by

$$\begin{aligned} \phi(A_F) &= \{\langle y, \phi(T_{A_F})(y), \phi(I_{A_F})(y), \phi(F_{A_F})(y) \rangle: y \in Y\} \\ \phi(T_{A_F})(y) &= \begin{cases} \sup_{x \in \phi^{-1}(y)} T_{A_F}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ \phi(I_{A_F})(y) &= \begin{cases} \sup_{x \in \phi^{-1}(y)} I_{A_F}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ \phi(F_{A_F})(y) &= \begin{cases} \inf_{x \in \phi^{-1}(y)} F_{A_F}(x) & \text{if } \phi^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise,} \end{cases} \end{aligned}$$

Definition 2.1.35

Let X and Y two non empty universal sets and $\phi: X \rightarrow Y$ be a function. If $B_F = \{\langle y, T_{B_F}(y), I_{B_F}(y), F_{B_F}(y) \rangle: y \in Y\}$ is a FNS in Y , then the pre-image of B_F under ϕ defined by $\phi^{-1}(B)$ is the FNS in X is denoted by

$$\phi^{-1}(B_F) = \{\langle x, \phi^{-1}(T_{B_F})(x), \phi^{-1}(I_{B_F})(x), \phi^{-1}(F_{B_F})(x) \rangle: x \in X\}$$

where

$$\phi^{-1}(T_{B_F})(x) = T_{B_F}(\phi(x)), \phi^{-1}(I_{B_F})(x) = I_{B_F}(\phi(x)), \phi^{-1}(F_{B_F})(x) = F_{B_F}(\phi(x))$$

Proposition 2.1.36

Let $\phi: \tilde{X} \rightarrow \tilde{Y}$ be a function. $A_{F_i} (i \in J)$ be FNSs in \tilde{X} , $B_{F_j} (j \in K)$ be FNSs in \tilde{Y} and then

$$i. A_{F_1} \subseteq A_{F_2} \Rightarrow \phi(A_{F_1}) \subseteq \phi(A_{F_2})$$

- ii. $B_{F_1} \subseteq B_{F_2} \Rightarrow \phi^{-1}(B_{F_1}) \subseteq \phi^{-1}(B_{F_2})$
- iii. $A_F \subseteq \phi^{-1}(\phi(A_F))$ and if ϕ is injective, then $A_F = \phi(\phi^{-1}(A_F))$
- iv. $\phi(\phi^{-1}(B_F)) \subseteq B_F$ and if ϕ is surjective, then $\phi(\phi^{-1}(B_F)) = B_F$
- v. $\phi^{-1}(1_{FN}) = 1_{FN}$
- vi. $\phi^{-1}(0_{FN}) = 0_{FN}$
- vii. $\phi(1_{FN}) = 1_{FN}$ if ϕ is surjective
- viii. $\phi(0_{FN}) = 0_{FN}$

Proof: This proof is similar to Proposition 2.1.12 and is based on Definition 2.1.34 and Definition 2.1.35.

Comparison between Neutrosophic variants

A graphical comparison of neutrosophic set and their variants, focusing on the case of dependence, has been presented. The PNS extends the range from 0 to $\sqrt{2}$, enabling the assignment of values greater than 1, up to $\sqrt{2}$. Therefore, it is useful in situations where there is data that needs to be assigned member values greater than 1. In such cases, the PNS is more appropriate. However, in this set, it is difficult to manage $\sqrt{2}$ in some places such as define the universal set (whole set) and the null set. To address this issue, an attempt is made to reduce the range of values from $[0, \sqrt{2}]$ to $[0, 1]$, while still allowing higher membership values in the case of dependence.

In another scenario, the NSS is introduced as a generalized framework that effectively handles indeterminacy, unlike fuzzy sets and their extensions, which assume dependency among membership values and cannot address indeterminacy. In NSS, the square sum of truth, indeterminacy, and falsity values ranges from 0 to $\sqrt{3}$, allowing for higher individual membership values, even when the sum is lower compared to PNS.

Hence in general, difficulties arise when the maximum values are more than 1 while defining certain properties such as the universal set, null set, complement, and their related characteristics.

From the observation, the FNS is a generalized framework that encompasses all fuzzy and neutrosophic variants. The FNS allows higher values of truth, indeterminacy, and falsity membership. Although the individual membership values do not exceed 1, the cube sum is considered, which allows the set to cover broader regions than other models. Additionally, it is easy to define the universal set, null set, complement, and their related characteristics within this framework.

A graphical comparison of neutrosophic sets and their variants is given below, along with a comparison table of PNS, NSS, and FNS.

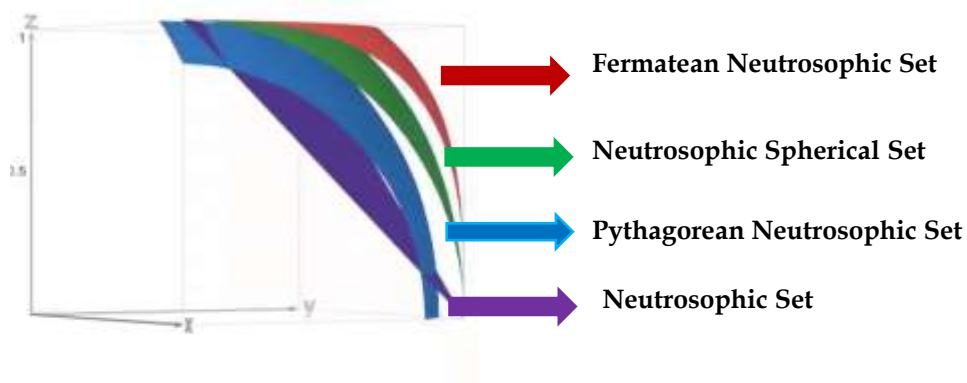
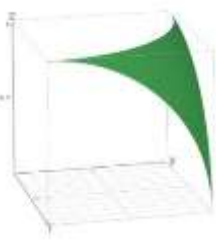
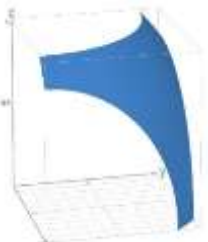



Table 2.1.1. Tabular representation of comparison between the Neutrosophic variants, focusing on the case of dependency

Properties	Pythagorean Neutrosophic Sets	Neutrosophic Spherical Sets	Fermatean Neutrosophic Sets
Membership Function	Truth Membership (T): $T \in [0, \sqrt{2}]$	Truth Membership (T): $T \in [0, 1]$	Truth Membership (T): $T \in [0, 1]$

	Indeterminacy Membership (I): $I \in [0, \sqrt{2}]$	Indeterminacy Membership (I): $I \in [0, 1]$	Indeterminacy Membership (I): $I \in [0, 1]$
	Falsity Membership (F): $F \in [0, \sqrt{2}]$	Falsity Membership (F): $F \in [0, 1]$	Falsity Membership (F): $F \in [0, 1]$
Independence of Component	The values of T , I and F are dependent of each other.	The values of T , I and F are dependent of each other.	The values of T , and F are dependent of each other. I is independent.
	$0 \leq T^2 + I^2 + F^2 \leq 2$	$0 \leq T^2 + I^2 + F^2 \leq \sqrt{3}$	$0 \leq T^3 + I^3 + F^3 \leq 2$
Mathematical Structure	Use the square sum of truth, indeterminacy, and falsity components, ranging from 0 to $\sqrt{2}$	Use the square sum of truth, indeterminacy, and falsity components, ranging from 0 to 1.	Use the cubic sum condition $0 \leq$ $T^3 + I^3 + F^3 \leq 2$.
Handling of Uncertainty	Suitable for general cases of uncertainty, indeterminacy, and inconsistency when all the components are dependent, and the range exceeds more than one.	Suitable for general cases of uncertainty, indeterminacy, and inconsistency when all the components are dependent, but the values of components have higher values.	It is particularly effective in scenarios where uncertainty is high.
Necessary and Sufficient condition	The Pythagorean neutrosophic set does not satisfy the	The neutrosophic spherical set does not satisfy the Fermatean	The Fermatean neutrosophic set satisfies the

	Fermatean neutrosophic set condition.	neutrosophic set condition.	neutrosophic spherical set condition.
Graphical Representation			

2.2 Fermatean Neutrosophic Cubic set

A cubic set is one of the essential theories for handling both exact and non exact values, which can be represented through intervals. Also, it handles different types of uncertainty simultaneously. In this section, the idea of cubic set is imposed to FNS by introducing interval-valued Fermatean neutrosophic sets.

Definition 2.2.1

Let X be a non empty set. An interval-valued Fermatean neutrosophic set (IVFNS) A_{IF} in X is denoted by

$$A_{IF} = \{x: [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)], [I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)], [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)] / x \in X\}$$

where

$$T_{A_{IF}}^-(x), I_{A_{IF}}^-(x), F_{A_{IF}}^-(x) \in [0,1], 0 \leq [T_{A_{IF}}^-(x)]^3 + [I_{A_{IF}}^-(x)]^3 + [F_{A_{IF}}^-(x)]^3 \leq 2$$

$$T_{A_{IF}}^+(x), I_{A_{IF}}^+(x), F_{A_{IF}}^+(x) \in [0,1], 0 \leq [T_{A_{IF}}^+(x)]^3 + [I_{A_{IF}}^+(x)]^3 + [F_{A_{IF}}^+(x)]^3 \leq 2$$

where,

$$I_{A_{IF}}^-(x) = \min\{(1, 1 - [T_{A_{IF}}^-(x)] - [F_{A_{IF}}^-(x)])\},$$

$$I_{A_{IF}}^+(x) = \max\{(1, 1 - [T_{A_{IF}}^+(x)] - [F_{A_{IF}}^+(x)])\}.$$

Definition 2.2.2

Let X be a non empty set, and the IVFNSs, A_{IF} and B_{IF} in X of the form

$$A_{IF} = \{x: [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)], [I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)], [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)] / x \in X\} \quad \text{and}$$

$$B_{IF} = \{x: [T_{B_{IF}}^-(x), T_{B_{IF}}^+(x)], [I_{B_{IF}}^-(x), I_{B_{IF}}^+(x)], [F_{B_{IF}}^-(x), F_{B_{IF}}^+(x)] / x \in X\} \quad \text{satisfy the}$$

following conditions:

$$A_{IF} \subseteq B_{IF} \text{ iff } T_{A_{IF}}^-(x) \leq T_{B_{IF}}^-(x); I_{A_{IF}}^-(x) \leq I_{B_{IF}}^-(x); F_{A_{IF}}^-(x) \geq F_{B_{IF}}^-(x), \forall x \in X.$$

$$T_{A_{IF}}^+(x) \leq T_{B_{IF}}^+(x); I_{A_{IF}}^+(x) \leq I_{B_{IF}}^+(x); F_{A_{IF}}^+(x) \geq F_{B_{IF}}^+(x), \forall x \in X.$$

$$A_{IF} = B_{IF} \text{ iff } T_{A_{IF}}^-(x) = T_{B_{IF}}^-(x); I_{A_{IF}}^-(x) = I_{B_{IF}}^-(x); F_{A_{IF}}^-(x) = F_{B_{IF}}^-(x), \forall x \in X.$$

$$T_{A_{IF}}^+(x) = T_{B_{IF}}^+(x); I_{A_{IF}}^+(x) = I_{B_{IF}}^+(x); F_{A_{IF}}^+(x) = F_{B_{IF}}^+(x), \forall x \in X.$$

$$A_{IF} \cup B_{IF} = \{ \langle x, T_{A_{IF} \cup B_{IF}}(x), I_{A_{IF} \cup B_{IF}}(x), F_{A_{IF} \cup B_{IF}}(x) \rangle | x \in X \} \text{ where,}$$

$$T_{A_{IF} \cup B_{IF}}(x) = [\max\{T_{A_{IF}}^-(x), T_{B_{IF}}^-(x)\}, \max\{T_{A_{IF}}^+(x), T_{B_{IF}}^+(x)\}]$$

$$I_{A_{IF} \cup B_{IF}}(x) = [\max\{I_{A_{IF}}^-(x), I_{B_{IF}}^-(x)\}, \max\{I_{A_{IF}}^+(x), I_{B_{IF}}^+(x)\}]$$

$$F_{A_{IF} \cup B_{IF}}(x) = [\min\{F_{A_{IF}}^-(x), F_{B_{IF}}^-(x)\}, \min\{F_{A_{IF}}^+(x), F_{B_{IF}}^+(x)\}]$$

$$A_{IF} \cap B_{IF} = \{ \langle x, T_{A_{IF} \cap B_{IF}}(x), I_{A_{IF} \cap B_{IF}}(x), F_{A_{IF} \cap B_{IF}}(x) \rangle | x \in X \} \text{ where,}$$

$$T_{A_{IF} \cap B_{IF}}(x) = [\min\{T_{A_{IF}}^-(x), T_{B_{IF}}^-(x)\}, \min\{T_{A_{IF}}^+(x), T_{B_{IF}}^+(x)\}]$$

$$I_{A_{IF} \cap B_{IF}}(x) = [\min\{I_{A_{IF}}^-(x), I_{B_{IF}}^-(x)\}, \min\{I_{A_{IF}}^+(x), I_{B_{IF}}^+(x)\}]$$

$$F_{A_{IF} \cap B_{IF}}(x) = [\max\{F_{A_{IF}}^-(x), F_{B_{IF}}^-(x)\}, \max\{F_{A_{IF}}^+(x), F_{B_{IF}}^+(x)\}]$$

Definition 2.2.3

$$\text{Let } A_{IF} = \{ x: [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)], [I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)], [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)] / x \in X \}$$

be a IVFNS on X , then the complement of the set A_{IF} may be defined as

$$\tilde{C}(A_{IF}) = \left\{ \left\langle \begin{array}{l} x, [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)], \\ [1 - I_{A_{IF}}^-(x), 1 - I_{A_{IF}}^+(x)], \\ [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)](x) \end{array} \right\rangle | x \in X \right\}$$

Definition 2.2.4

Let X be a non empty set. A Fermatean neutrosophic cubic set (FNCS) in ζ_F is denoted by

$$\zeta_F = \{ \langle x, A_{IF}(x), \lambda_F(x) \rangle / x \in X \}$$

where $A_{IF}(x)$ is an IVFNS in X and $\lambda_F(x)$ is a FNS in X .

Example 2.2.5

For $X = \{p, q, r\}$, the pair $\zeta_F = (A_{IF}, \lambda_F)$ is an FNCS in X is represented Table 2.2.1.

Table 2.2.1. Tabular representation of $\zeta_F = (A_{IF}, \lambda_F)$

X	A_{IF}	λ_F
p	$([0.4,0.6], [0,1], [0.6,0.7])$	$(0.6,0.9,0.7)$
q	$([0.6,0.7], [0.2,1], [0.2,0.5])$	$(0.7,0.8,0.5)$
r	$([0.2,0.5], [0.6,1], [0.2,0.4])$	$(0.5,1,0.4)$

Definition 2.2.6

Let $\zeta_F = \{x, A_{IF}(x), \lambda_F(x) \mid x \in X\}$ be a FNCS on X , then the complement of the set ζ_F may be defined as

$$\tilde{C}(\zeta_F) = \{x, \tilde{C}(A_{IF}), \tilde{C}(\lambda_F(x)) \mid x \in X\}$$

where

$$\tilde{C}(A_{IF}) = \left\{ \left(\begin{array}{l} x, [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)], \\ [1 - I_{A_{IF}}^-(x), 1 - I_{A_{IF}}^+(x)], \\ [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)](x) \end{array} \right) \mid x \in X \right\}, \quad \tilde{C}(\lambda_F) = \left\{ \left(\begin{array}{l} x, F_{\lambda_F}(x), \\ 1 - I_{\lambda_F}(x), \\ T_{\lambda_F}(x) \end{array} \right) \mid x \in S \right\}$$

Definition 2.2.7

Let X be a non empty set. A FNCS $\zeta_F = (A_{IF}, \lambda_F)$ in X is said to be

- i. **Truth-Internal (T-internal)** if the following inequality is true and determined by

$$(T_{A_{IF}}^-(x) \leq T_{\lambda_F}(x) \leq T_{A_{IF}}^+(x)), (\forall x \in X) \quad (2.2.1)$$

- ii. **Indeterminacy-internal (I-internal)** if the following inequality is true and determined by

$$(I_{A_{IF}}^-(x) \leq I_{\lambda_F}(x) \leq I_{A_{IF}}^+(x)), (\forall x \in X) \quad (2.2.2)$$

- iii. **Falsity-internal (F-internal)** if the following inequality is true and determined by

$$(F_{A_{IF}}^-(x) \leq F_{\lambda_F}(x) \leq F_{A_{IF}}^+(x)), (\forall x \in X) \quad (2.2.3)$$

If a FNCS $\zeta_F = (A_{IF}, \lambda_F)$ in X satisfies the above inequalities (equations 2.2.1, 2.2.2, 2.2.3) then $\zeta_F = (A_{IF}, \lambda_F)$ is an internal FNCS in X .

Example 2.2.8

For $X = \{p, q, r\}$, the pair $\zeta_F = (A_{IF}, \lambda_F)$ is an internal FNCS in X is represented Table 2.2.2.

Table 2.2.2. Tabular representation of $\zeta_F = (A_{IF}, \lambda_F)$

X	$A_F(x)$	$\lambda_F(x)$
p	$([0.4,0.6], [0,1], [0.6,0.7])$	$(0.5,0.8,0.65)$
q	$([0.6,0.7], [0.2,1], [0.2,0.5])$	$(0.7,0.6,0.5)$
r	$([0.2,0.5], [0.6,1], [0.2,0.4])$	$(0.4,0.85,0.35)$

Definition 2.2.9

Let X be a non empty set. A FNCS $\zeta_F = (A_{IF}, \lambda_F)$ in X is said to be

- i. **Truth-external (T-external)** if the following inequality is true and determined by

$$T_{\lambda_F}(x) \notin (T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)), (\forall x \in X) \tag{2.2.4}$$

- ii. **Indeterminacy-external (I-external)** if the following inequality is true and determined by

$$I_{\lambda_F}(x) \notin (I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)), (\forall x \in X) \tag{2.2.5}$$

- iii. **Falsity-external (F-external)** if the following inequality is true and determined by

$$F_{\lambda_F}(x) \notin (F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)), (\forall x \in X) \tag{2.2.6}$$

If a FNCS $\zeta_F = (A_{IF}, \lambda_F)$ in X satisfies the above inequalities (equations 2.2.4, 2.2.5, 2.2.6) the $\zeta_F = (A_{IF}, \lambda_F)$ is an external FNCS in X .

Example 2.2.10

For $X = \{p, q, r\}$, the pair $\zeta_F = (A_{IF}, \lambda_F)$ is an external FNCS in X is represented in Table 2.2.3

Table 2.2.3. Tabular representation of $\zeta_F = (A_{IF}, \lambda_F)$

X	$A_{IF}(x)$	$\lambda_F(x)$
p	$([0.4,0.6], [0,1], [0.6,0.7])$	$(0.65,1.1,0.8)$
q	$([0.6,0.7], [0.2,1], [0.2,0.5])$	$(0.55,0.1,0.7)$
r	$([0.2,0.5], [0.6,1], [0.2,0.4])$	$(1,0.5,0.45)$

Proposition 2.2.11

Let $\zeta_F = (A_{IF}, \lambda_F)$ be an internal FNCS in X that is non empty, then there exists $x \in X$ such that $T_{\lambda_F}(x) \in (T_{A_{IF}}^-(x), T_{A_{IF}}^+(x))$, $I_{\lambda_F}(x) \in (I_{A_{IF}}^-(x), I_{A_{IF}}^+(x))$, $F_{\lambda_F}(x) \in (F_{A_{IF}}^-(x), F_{A_{IF}}^+(x))$.

Proof: From Definition 2.2.9, the proof is straightforward.

Proposition 2.2.12

Let $\zeta_F = (A_{IF}, \lambda_F)$ be an FNCS in a non empty set X , if $\zeta_F = (A_{IF}, \lambda_F)$ is both T-internal and T-external then

$$(T_{\lambda_F}(x) \in \{T_{A_{IF}}^-(x) | x \in X\} \cup \{T_{A_{IF}}^+(x) | x \in X\}) \quad (2.2.7)$$

Proof:

The equations (2.2.1) and (2.2.4) imply that $T_{A_{IF}}^-(x) \leq T_{\lambda_F}(x) \leq T_{A_{IF}}^+(x)$ and $T_{\lambda_F}(x) \notin (T_{A_{IF}}^-(x), T_{A_{IF}}^+(x))$, $\forall x \in X$.

It follows that $T_{\lambda_F}(x) = T_{A_{IF}}^-(x)$ or $T_{\lambda_F}(x) = T_{A_{IF}}^+(x)$ so that

$$T_{\lambda_F}(x) \in \{T_{A_{IF}}^-(x) | x \in X\} \cup \{T_{A_{IF}}^+(x) | x \in X\}$$

Proposition 2.2.13

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non empty set X , if $\zeta_F = (A_{IF}, \lambda_F)$ is both I-internal and I-external then

$$(I_{\lambda_F}(x) \in \{I_{A_{IF}}^-(x) | x \in X\} \cup \{I_{A_{IF}}^+(x) | x \in X\}) \quad (2.2.8)$$

Proof: The proof is similar to Proposition 2.2.12.

Proposition 2.2.14

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non-empty set X , if $\zeta_F = (A_{IF}, \lambda_F)$ is both F-internal and F-external then

$$(F_{\lambda_F}(x) \in \{F_{A_{IF}}^-(x), x \in X\} \cup \{F_{A_{IF}}^+(x), x \in X\}) \quad (2.2.9)$$

Proof: The proof is similar to Proposition 2.2.12.

Definition 2.2.15

Let $\mathbb{Q}_F = (A_{IF}, \lambda_F)$ and $\mathbb{R}_F = (B_{IF}, \psi_F)$ be a FNCSs in a non empty set X where

$$A_{IF} = \{x: [T_{A_{IF}(x)}^-, T_{A_{IF}(x)}^+][I_{A_{IF}(x)}^-, I_{A_{IF}(x)}^+][F_{A_{IF}(x)}^-, F_{A_{IF}(x)}^+]/x \in X\}$$

$$\lambda_F: \{(x, T_{\lambda_F}(x), I_{\lambda_F}(x), F_{\lambda_F}(x))/x \in X\}$$

$$B_{IF} = \{x: [T_{B_{IF}(x)}^-, T_{B_{IF}(x)}^+][I_{B_{IF}(x)}^-, I_{B_{IF}(x)}^+][F_{B_{IF}(x)}^-, F_{B_{IF}(x)}^+]/x \in X\}$$

$$\psi_F: \{(x, T_{\psi_F}(x), I_{\psi_F}(x), F_{\psi_F}(x))/x \in X\}$$

The equality, P-order, and R-order are defined as follows:

- i. **Equality** $\mathbb{Q}_F = \mathbb{R}_F \Leftrightarrow A_{IF} = B_{IF}$ and $\lambda_F = \psi_F$
- ii. **P-order** $\mathbb{Q}_F \subseteq_P \mathbb{R}_F \Leftrightarrow A_{IF} \subseteq B_{IF}$ and $\lambda_F \leq \psi_F$
- iii. **R-order** $\mathbb{Q}_F \subseteq_R \mathbb{R}_F \Leftrightarrow A_{IF} \subseteq B_{IF}$ and $\lambda_F \geq \psi_F$

Definition 2.2.16

For any FNCS $\zeta_F = (A_{IF}, \lambda_F)$ in a non empty set X , where

$$A_{IF} = \{x: [T_{A_{IF}(x)}^-, T_{A_{IF}(x)}^+], [I_{A_{IF}(x)}^-, I_{A_{IF}(x)}^+], [F_{A_{IF}(x)}^-, F_{A_{IF}(x)}^+]/x \in X\}$$

$\lambda_{F_i}: \{(x, T_{\lambda_{F_i}}(x), I_{\lambda_{F_i}}(x), F_{\lambda_{F_i}}(x))/x \in X\}$ for $i \in J$ and J is any index set, and define

- i. $\bigcup_{i \in J} \zeta_{F_i} = (\bigcup_{i \in J} A_{IF_i}, \bigvee_{i \in J} \lambda_{F_i})$ (*P-Union*)
- ii. $\bigcap_{i \in J} \zeta_{F_i} = (\bigcap_{i \in J} A_{IF_i}, \bigwedge_{i \in J} \lambda_{F_i})$ (*P-Intersection*)
- iii. $\bigcup_{i \in J} \zeta_{F_i} = (\bigcup_{i \in J} A_{IF_i}, \bigwedge_{i \in J} \lambda_{F_i})$ (*R-Union*)
- iv. $\bigcap_{i \in J} \zeta_{F_i} = (\bigcap_{i \in J} A_{IF_i}, \bigvee_{i \in J} \lambda_{F_i})$ (*R-Intersection*)

where

$$\bigcup_{i \in J} A_{IF_i} = \left\{ \left(\begin{array}{l} x, \left(\bigcup_{i \in J} [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)] \right) (x), \\ \left(\bigcup_{i \in J} [I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)] \right) (x), \\ \left(\bigcup_{i \in J} [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)] \right) (x) \end{array} \right) / x \in X \right\}$$

$$\bigvee_{i \in J} \lambda_{F_i} = \left\{ \left(x, \left(\bigvee_{i \in J} T_{\lambda_{F_i}} \right) (x), \left(\bigvee_{i \in J} I_{\lambda_{F_i}} \right) (x), \left(\bigvee_{i \in J} F_{\lambda_{F_i}} \right) (x) \right) / x \in X \right\}$$

$$\bigcap_{i \in J} A_{IF_i} = \left\{ \left(\begin{array}{l} x, \left(\bigcap_{i \in J} [T_{A_{IF}}^-(x), T_{A_{IF}}^+(x)] \right) (x), \\ \left(\bigcap_{i \in J} [I_{A_{IF}}^-(x), I_{A_{IF}}^+(x)] \right) (x), \\ \left(\bigcap_{i \in J} [F_{A_{IF}}^-(x), F_{A_{IF}}^+(x)] \right) (x) \end{array} \right) / x \in X \right\}$$

$$\bigwedge_{i \in J} \lambda_{F_i} = \left\{ \left(x, \left(\bigwedge_{i \in J} T_{\lambda_{F_i}} \right) (x), \left(\bigwedge_{i \in J} I_{\lambda_{F_i}} \right) (x), \left(\bigwedge_{i \in J} F_{\lambda_{F_i}} \right) (x) \right) / x \in X \right\}$$

Proposition 2.2.17

For any FNCSs $\mathbb{N}_F = (A_{IF}, \lambda_F)$, $\mathbb{Q}_F = (B_{IF}, \psi_F)$, $\mathbb{R}_F = (D_{IF}, \phi_F)$, $\mathbb{Z}_F = (E_{IF}, \Omega_F)$ in a non empty set X , then

- 1) If $\mathbb{N}_F \subseteq_P \mathbb{Q}_F$ and $\mathbb{Q}_F \subseteq_P \mathbb{R}_F$ then $\mathbb{N}_F \subseteq_P \mathbb{R}_F$.
- 2) If $\mathbb{N}_F \subseteq_P \mathbb{Q}_F$ then $\mathbb{Q}_F^c \subseteq_P \mathbb{N}_F^c$.
- 3) If $\mathbb{N}_F \subseteq_P \mathbb{Q}_F$ and $\mathbb{N}_F \subseteq_P \mathbb{R}_F$ then $\mathbb{N}_F \subseteq_P \mathbb{Q}_F \cap_P \mathbb{R}_F$.
- 4) If $\mathbb{N}_F \subseteq_P \mathbb{Q}_F$ and $\mathbb{R}_F \subseteq_P \mathbb{Q}_F$ then $\mathbb{N}_F \cup_P \mathbb{R}_F \subseteq_P \mathbb{Q}_F$.
- 5) If $\mathbb{N}_F \subseteq_P \mathbb{Q}_F$ and $\mathbb{R}_F \subseteq_P \mathbb{Z}_F$ then $\mathbb{N}_F \cup_P \mathbb{R}_F \subseteq_P \mathbb{Q}_F \cup_P \mathbb{Z}_F$ and $\mathbb{N}_F \cap_P \mathbb{R}_F \subseteq_P \mathbb{Q}_F \cap_P \mathbb{Z}_F$
- 6) If $\mathbb{N}_F \subseteq_R \mathbb{Q}_F$ and $\mathbb{Q}_F \subseteq_R \mathbb{R}_F$ then $\mathbb{N}_F \subseteq_R \mathbb{R}_F$.
- 7) If $\mathbb{N}_F \subseteq_R \mathbb{Q}_F$ then $\mathbb{Q}_F^c \subseteq_R \mathbb{N}_F^c$.
- 8) If $\mathbb{N}_F \subseteq_R \mathbb{Q}_F$ and $\mathbb{N}_F \subseteq_R \mathbb{R}_F$ then $\mathbb{N}_F \subseteq_R \mathbb{Q}_F \cap_R \mathbb{R}_F$.
- 9) If $\mathbb{N}_F \subseteq_R \mathbb{Q}_F$ and $\mathbb{R}_F \subseteq_R \mathbb{Q}_F$ then $\mathbb{N}_F \cup_R \mathbb{R}_F \subseteq_R \mathbb{Q}_F$.
- 10) If $\mathbb{N}_F \subseteq_R \mathbb{Q}_F$ and $\mathbb{R}_F \subseteq_R \mathbb{Z}_F$ then $\mathbb{N}_F \cup_R \mathbb{R}_F \subseteq_R \mathbb{Q}_F \cup_R \mathbb{Z}_F$ and $\mathbb{N}_F \cap_R \mathbb{R}_F \subseteq_R \mathbb{Q}_F \cap_R \mathbb{Z}_F$.

Proof: The proof is straightforward.

Theorem 2.2.18

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non empty set X . If $\zeta_F = (A_{IF}, \lambda_F)$ is I-internal (I-external) then the complement $\zeta_{F_i}^c = (A_{IF_i}^c, \lambda_{F_i}^c)$ of $\zeta_F = (A_{IF}, \lambda_F)$ is an I-internal (I-external) FNCS in X .

Proof:

If $\zeta_F = (A_{IF}, \lambda_F)$ is an I-internal (resp. I-external) FNCS in X then $I_{A_{IF}}^-(x) \leq I_{\lambda_F}(x) \leq I_{A_{IF}}^+(x)$, (resp. $I_{\lambda_F}(x) \notin (I_{A_{IF}}^-(x), I_{A_{IF}}^+(x))$) for all $x \in X$.

It follows that $1 - I_{A_{IF}}^-(x) \leq 1 - I_{\lambda_F}(x) \leq 1 - I_{A_{IF}}^+(x)$, (resp. $1 - I_{\lambda_F}(x) \notin (1 - I_{A_{IF}}^+(x), 1 - I_{A_{IF}}^-(x))$).

$\therefore \zeta_{F_i}^c = (A_{IF_i}^c, \lambda_{F_i}^c)$ is an I-internal (I-external) FNCS in a non-empty set X .

Theorem 2.2.19

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non empty set X . If $\zeta_F = (A_{IF}, \lambda_F)$ is T-internal (T-external) then the complement $\zeta_F^c = (A_{IF}^c, \lambda_F^c)$ of $\zeta_F = (A_{IF}, \lambda_F)$ is T-internal (T-external) FNCS in X .

Proof: The proof is similar to Theorem 2.2.18.

Theorem 2.2.20

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non empty set X . If $\zeta_F = (A_{IF}, \lambda_F)$ is F-internal (F-external) then the complement $\zeta_F^c = (A_{IF}^c, \lambda_F^c)$ of $\zeta_F = (A_{IF}, \lambda_F)$ is an F-internal (F-external) FNCS in X .

Proof: The proof is similar to Theorem 2.2.18.

Corollary 2.2.21

Let $\zeta_F = (A_{IF}, \lambda_F)$ be a FNCS in a non empty set X . If $\zeta_F = (A_{IF}, \lambda_F)$ is internal (external) then the complement $\zeta_F^c = (A_{IF}^c, \lambda_F^c)$ of $\zeta_F = (A_{IF}, \lambda_F)$ is an internal (external) FNCS in X .

Theorem 2.2.22

If $\zeta_{F_i} = ((A_{IF_i}, \lambda_{F_i})/i \in J)$ is a family of F-internal FNCSs in a non empty set X , then the P-union and P-intersection of $\{\zeta_{F_i} = (A_{IF_i}, \lambda_{F_i})/i \in J\}$ are F-internal FNCSs in X .

Proof:

Since $\zeta_{F_i} = (A_{IF_i}, \lambda_{F_i})$ is a family of F-internal FNCS in a non empty set X , then $F_{A_{IF_i}}^-(x) \leq F_{\lambda_{F_i}}(x) \leq F_{A_{IF_i}}^+(x) \forall i \in J$.

It follows that $(\bigcup_{i \in J} F_{A_{IF_i}})^-(x) \leq (\bigvee_{i \in J} F_{\lambda_{F_i}})(x) \leq (\bigcup_{i \in J} F_{A_{IF_i}})^+(x)$ and $(\bigcap_{i \in J} F_{A_{IF_i}})^-(x) \leq (\bigwedge_{i \in J} F_{\lambda_{F_i}})(x) \leq (\bigcap_{i \in J} F_{A_{IF_i}})^+(x)$

$\therefore \cup_{i \in J} F_{\zeta_{F_i}} = (\cup_{i \in J} A_{IF_i}, \forall_{i \in J} \lambda_{F_i})$ and $\cap_{i \in J} C_{F_i} = (\cap_{i \in J} A_{IF_i}, \wedge_{i \in J} \lambda_{F_i})$ are F-internal FNCSs in X .

Theorem 2.2.23

If $\zeta_{F_i} = ((A_{IF_i}, \lambda_{F_i})/i \in J)$ is a family of T-internal FNCSs in a non-empty set X , then the P-union and P-intersection of $\{\zeta_{F_i} = (A_{IF_i}, \lambda_{F_i})/i \in J\}$ are T-internal FNCSs in X .

Proof: The proof is similar to Theorem 2.2.22.

Theorem 2.2.24

If $\zeta_{F_i} = ((A_{IF_i}, \lambda_{F_i})/i \in J)$ is a family of I-internal FNCSs in a non empty set X , then the P-union and P-intersection of $\{\zeta_{F_i} = (A_{IF_i}, \lambda_{F_i})/i \in J\}$ are I-internal FNCSs in X .

Proof: The proof is similar to Theorem 2.2.22.

Corollary 2.2.25

If $\zeta_{F_i} = ((A_{IF_i}, \lambda_{F_i})/i \in J)$ is a family of Internal FNCSs in a non empty set, then the P-union and P-intersection of $\{\zeta_{F_i} = (A_{IF_i}, \lambda_{F_i})/i \in J\}$ are internal FNCSs in X .

Note: The following examples shows that P-union and P-intersection of F-external (I-external and T-external) FNCSs may not be F-external (I-external and T-external) FNCSs.

Example 2.2.26

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be FNCSs in $[0,1]$ where

$$A_{IF} = \{(x, (0.6,0.8), (0.1,1), (0.3,0.5))/x \in [0,1]\},$$

$$\lambda_F = \{(x, 0.7,0.9,0.9)/x \in [0,1]\}$$

$$B_{IF} = \{(x, (0.4,0.6), (0,1), (0.6,0.8))/x \in [0,1]\},$$

$$\psi_F = \{(x, 0.6,0.9,0.2)/x \in [0,1]\}$$

Then $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ are F-external FNCSs in $[0,1]$.

$C_F \cup_P N_F = (A_{IF} \cup B_{IF}, \lambda_F \vee \psi_F)$ of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is given as follows

$$A_{IF} \cup B_{IF} = \{(x, (0.6,0.8), (0.1,1), (0.6,0.8))/x \in [0,1]\}$$

$\lambda_F \vee \psi_F = \{(x, 0.7,0.9,0.9)/x \in [0,1]\}$ is not a F-external FNCSs in $[0,1]$.

Since $(F_{\lambda_F} \vee F_{\psi_F})(x) = 0.9 \notin (0.6,0.8)$

$$= (F_{A_{IF}} \cup F_{B_{IF}})^-(x), (F_{A_{IF}} \cup F_{B_{IF}})^+(x)$$

$\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \wedge \psi_F)$ of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is given as follows

$$A_{IF} \cap B_{IF} = \{(x, (0.4,0.6), (0,1), (0.3,0.5))/x \in [0,1]\}$$

$\lambda_F \wedge \psi_F = \{(x, 0.6,0.9,0.3)/x \in [0,1]\}$ is not an F-external FNCSs in $[0,1]$.

$$(F_{\lambda_F} \wedge F_{\psi_F})(x) = 0.2 \notin (0.3,0.5) = (F_{A_P} \cap F_{B_P})^-(x), (F_{A_P} \cap F_{B_P})^+(x)$$

Example 2.2.27

For $X = \{p, q, r\}$, let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be FNCSs in X representations in Table 2.2.4 and 2.2.5, respectively.

Table 2.2.4. Tabular representation of $\zeta_F = (A_F, \lambda_F)$

X	$A_{IF}(x)$	$\lambda_F(x)$
p	$([0.6, 0.8], [0.2, 1], [0.2, 0.4])$	$(0.5, 0.1, 0.4)$
q	$([0.4, 0.7], [0.4, 1], [0.2, 0.5])$	$(0.3, 0.3, 0.4)$
r	$([0.5, 0.6], [0.1, 1], [0.4, 0.6])$	$(0.7, 0, 0.5)$

Table 2.2.5. Tabular representation of $\mathbb{N}_F = (B_F, \psi_F)$

X	$B_{IF}(x)$	$\psi_F(x)$
p	$([0.2, 0.5], [0.5, 1], [0.3, 0.4])$	$(0.7, 0.4, 0.65)$
q	$([0.6, 0.7], [0.2, 1], [0.2, 0.5])$	$(0.4, 0.1, 0.5)$
r	$([0.2, 0.5], [0.6, 1], [0.2, 0.4])$	$(0.6, 0.5, 0.4)$

Table 2.2.6. Tabular representation of $\zeta_F \cup_P \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \vee \psi_F)$

X	$(A_{IF} \cup B_{IF})(x)$	$(\lambda_F \vee \psi_F)(x)$
p	$([0.6, 0.8], [0.5, 1], [0.3, 0.4])$	$(0.7, 0.4, 0.65)$
q	$([0.6, 0.7], [0.4, 1], [0.2, 0.5])$	$(0.4, 0.3, 0.5)$
r	$([0.5, 0.6], [0.6, 1], [0.4, 0.6])$	$(0.7, 0.5, 0.5)$

Table 2.2.7. Tabular representation of $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \wedge \psi_F)$

X	$(A_{IF} \cap B_{IF})(x)$	$(\lambda_F \wedge \psi_F)(x)$
p	$([0.2, 0.5], [0.2, 1], [0.2, 0.4])$	$(0.5, 0.1, 0.4)$
q	$([0.4, 0.7], [0.2, 1], [0.2, 0.5])$	$(0.3, 0.1, 0.4)$
r	$([0.2, 0.5], [0.1, 1], [0.2, 0.4])$	$(0.6, 0, 0.4)$

Then $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ are both T-external and I-external FNCSs in X . Observe that the tabular depiction of $\zeta_F \cup_P \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \vee \psi_F)$ and $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \wedge \psi_F)$ are given by Tables 2.2.5 and 2.2.6 respectively.

Then $\zeta_F \cup_P \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \vee \psi_F)$ is neither an I-external FNCSs nor and T-external FNCSs in X since any

$$(I_{\lambda_F} \vee I_{\psi_F})(q) = 0.4 \notin (0.5, 1) = \left((I_{A_{IF}} \cup I_{B_{IF}})^-(q), (I_{A_{IF}} \cup I_{B_{IF}})^+(q) \right)$$

and $(T_{\lambda_F} \vee T_{\psi_F})(r) = 0.4 \notin (0.6, 0.7) = ((T_{A_{IF}} \cup T_{B_{IF}})^-(r), (T_{A_{IF}} \cup T_{B_{IF}})^+(r))$.

Note: R-union and R-intersection of T-internal (I-internal and F-internal) FNCSs may not be T-internal (I-internal and F-internal) FNCSs as seen in the following examples.

Example 2.2.28

Let $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be FNCSs in $[0, 1]$ where

$$A_{IF} = \{(x, (0.4, 0.5), (0.4, 1), (0.2, 0.5))/x \in [0, 1]\},$$

$$\lambda_F = \{(x, 0.4, 0.6, 0.4)/x \in [0, 1]\}$$

$$B_{IF} = \{(x, (0.5, 0.6), (0.1, 1), (0.4, 0.6))/x \in [0, 1]\},$$

$$\psi_F = \{(x, 0.6, 0.3, 0.5)/x \in [0, 1]\}$$

Then $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ are T-internal FNCSs in $[0, 1]$, and $\zeta_F \cup_R \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ with

$$A_{IF} \cup B_{IF} = \{(x, [0.5, 0.6], [0.4, 1], [0.4, 0.6])/x \in [0, 1]\},$$

$$\lambda_F \wedge \psi_F = \{(x, 0.4, 0.3, 0.4)/x \in [0, 1]\}.$$

Note that $(T_{\lambda_F} \wedge T_{\psi_F})(x) = 0.4 \notin [0.5, 0.7] = (T_{A_F} \cup T_{B_F})^-(x)$

And $(I_{\lambda_F} \wedge I_{\psi_F})(x) = 0.3 \notin [0.4, 1] = (I_{A_F} \cup I_{B_F})^-(x)$.

Hence $\zeta_F \cup_R \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ is neither a T-internal FNCS nor an I-internal FNCS in $[0, 1]$. But $\zeta_F \cup_R \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ is an F-internal FNCS in $[0, 1]$.

Also, the R-intersection $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \vee \psi_F)$ with

$$A_{IF} \cap B_{IF} = \{(x, [0.4, 0.5], [0.1, 1], [0.2, 0.5])/x \in [0, 1]\},$$

$$\lambda_F \vee \psi_F = \{(x, 0.6, 0.6, 0.6)/x \in [0, 1]\}.$$

Since $(I_{A_{IF}} \cap I_{B_{IF}})^-(x) \leq (I_{\lambda_F} \vee I_{\psi_F})(x) \leq (I_{A_{IF}} \cap I_{B_{IF}})^+(x)$ for all $x \in [0,1]$, $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \vee \psi_F)$ is a I-internal FNCS in $[0,1]$. But neither a T-internal FNCS nor F-internal FNCS in $[0,1]$.

Example 2.2.29

Let $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be FNCSs in $[0,1]$ where

$$A_{IF} = \{(x, (0.2,0.5), (0.6,1), (0.2,0.4))/x \in [0,1]\},$$

$$\lambda_F = \{(x, 0.6,0.6,0.5)/x \in [0,1]\}$$

$$B_{IF} = \{(x, (0.4,0.7), (0.4,1), (0.2,0.5))/x \in [0,1]\},$$

$$\psi_F = \{(x, 0.3,0.5,0.6)/x \in [0,1]\}$$

Then $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ are I-internal FNCSs in $[0,1]$, and $\zeta_F \cup_R \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ with

$$A_{IF} \cup B_{IF} = \{(x, [0.4,0.7], [0.6,1], [0.2,0.5])/x \in [0,1]\},$$

$$\lambda_F \wedge \psi_F = \{(x, 0.3,0.5,0.5)/x \in [0,1]\}$$

Since $(I_{\lambda} \wedge I_{\psi})(x) = 0.5 \in [0.6,1] = (I_A \cup I_B)^-(x)$, and $\zeta_F \cup_R \mathbb{N}_F$ is not an I-internal FNCS in $[0,1]$.

Also, the R-intersection $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \vee \psi_F)$ with

$$A_{IF} \cap B_{IF} = \{(x, [0.2,0.5], [0.4,0.5], [0.2,0.4])/x \in [0,1]\},$$

$$\lambda_F \vee \psi_F = \{(x, 0.6, 0.6, 0.6)/x \in [0,1]\}$$
 and it is not an I-internal FNCSs in $[0,1]$.

Example 2.2.30

Let $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be FNCSs in $[0,1]$ where

$$A_{IF} = \{(x, (0.4,0.8), (0.3,1), (0.3,0.6))/x \in [0,1]\},$$

$$\lambda_F = \{(x, 0.9,0.3,0.4)/x \in [0,1]\}$$

$$B_{IF} = \{(x, (0.3,0.6), (0.2,1), (0.5,0.9))/x \in [0,1]\},$$

$$\psi_F = \{(x, 0.7,0.9,0.7)/x \in [0,1]\}$$

Then $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ are F-internal FNCSs in $[0,1]$, and $\zeta_F \cup_R \mathbb{N}_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ with

$$A_{IF} \cup B_{IF} = \{(x, [0.4,0.8], [0.3,1], [0.5,0.9])/x \in [0,1]\},$$

$\lambda_F \wedge \psi_F = \{(x, 0.7,0.3,0.4)/x \in [0,1]\}$ which is not an F-internal FNCSs in $[0,1]$.

R-intersection $\zeta_F \cap_P \mathbb{N}_F = (A_{IF} \cap B_{IF}, \lambda_F \vee \psi_F)$ of $\zeta_F = (A_{IF}, \lambda_F)$, and $\mathbb{N}_F = (B_{IF}, \psi_F)$ which is given as follows:

$$A_{IF} \cap B_{IF} = \{(x, [0.3,0.6], [0.2,1], [0.3,0.6])/x \in R\},$$

$\lambda_F \vee \psi_F = \{(x, 0.9,0.9,0.7)/x \in R\}$ and it is not an F-internal FNCSs in $[0,1]$.

Theorem 2.2.31

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be T-internal FNCSs in a non empty set X such that,

$$\left(\max\{T_{A_{IF}}^-(x), T_{B_{IF}}^-(x)\} \leq (T_{\lambda_F} \wedge T_{\psi_F})(x) \right), (\forall x \in X) \quad (2.2.10)$$

Then the R-union of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is a T-internal FNCSs in a non-empty set X .

Proof:

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be a T-internal FNCSs in a non empty set X , that meet the requirements (equation 2.2.10).

Then $T_{A_{IF}}^-(x) \leq T_{\lambda_F}(x) \leq T_{A_{IF}}^+(x)$ and $T_{B_{IF}}^-(x) \leq T_{\psi_F}(x) \leq T_{B_{IF}}^+(x)$.

so, $(T_{\lambda_F} \wedge T_{\psi_F})(x) \leq (T_{A_{IF}} \cup T_{B_{IF}})^+(x)$.

It follows from equation (2.2.10) that,

$$(T_{A_{IF}} \cup T_{B_{IF}})^-(x) = \max\{T_{A_{IF}}^-(x), T_{B_{IF}}^-(x)\} \leq (T_{\lambda_F} \wedge T_{\psi_F})(x) \leq (T_{A_{IF}} \cup T_{B_{IF}})^+(x)$$

Hence $C_F \cup_R B_F = (A_{IF} \cup B_{IF}, \lambda_F \wedge \psi_F)$ is a T-internal FNCSs in X .

Theorem 2.2.32

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be I-internal FNCSs in a non empty set X such that

$$\left(\max\{I_{A_{IF}}^-(x), I_{B_{IF}}^-(x)\} \leq (I_{\lambda_F} \wedge I_{\psi_F})(x) \right), (\forall x \in X) \quad (2.2.11)$$

Then the R-union of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is a I-internal FNCS in X .

Proof: The proof is similar to Theorem 2.2.28.

Theorem 2.2.33

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be F-internal FNCSs in a non empty set X such that,

$$\left(\max\{F_{A_{IF}}^-(x), F_{B_{IF}}^-(x)\} \leq (F_{\lambda_F} \wedge F_{\psi_F})(x) \right), (\forall x \in X) \quad (2.2.12)$$

Then the R-union of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is a F-internal FNCS in X .

Proof: The proof is similar to Theorem 2.2.28.

Corollary 2.2.34

If two internal FNCSs $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ satisfy equation (2.2.10 to 2.2.12 then the R-union of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is an internal FNCSs. Given the conditions for the R-intersection of two T-internal (I-internal and F-internal) FNCSs to be T-internal (I-internal and F-internal) FNCSs.

Theorem 2.2.35

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be I-internal FNCSs in a non empty set X such that,

$$\left((I_{\lambda_F} \vee I_{\psi_F})(x) \leq \min\{I_{A_{IF}}^+(x), I_{B_{IF}}^+(x)\} \right), (\forall x \in X) \quad (2.2.13)$$

Then the R-intersection of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is an I-internal FNCS in X .

Proof:

Assume that the equation 2.2.13 is valid. Then $I_{A_{IF}}^-(x) \leq I_{\lambda_F}(x) \leq I_{A_{IF}}^+(x)$ and $I_{B_{IF}}^-(x) \leq I_{\psi_F}(x) \leq I_{B_{IF}}^+(x)$ for all X . It follows from equation 2.2.13 that

$$\left((I_{A_{IF}} \cap I_{B_{IF}})^-(x) \leq (I_{\lambda_F} \cap I_{\psi_F})(x) \leq \min\{I_{A_{IF}}^+(x), I_{B_{IF}}^+(x)\} \leq (I_{A_{IF}} \cap I_{B_{IF}})^+(x) \right)$$

for all $x \in X$.

$\therefore A_{IF} \cap_R B_{IF} = (A_{IF} \cap B_{IF}, \lambda_F \vee \psi_F)$ is a I-internal FNCS in X .

Theorem 2.2.36

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be T-internal FNCSs in a non empty set X such that,

$$\left((T_{\lambda_F} \vee T_{\psi_F})(x) \leq \min\{T_{A_{IF}}^+(x), T_{B_{IF}}^+(x)\} \right), (\forall x \in X) \quad (2.2.14)$$

Then the R-intersection of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is a T-internal FNCS in X .

Proof: The proof is similar to Theorem 2.2.32.

Theorem 2.2.37

Let $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ be F-internal FNCSs in a non empty set X such that,

$$\left((F_{\lambda_F} \vee F_{\psi_F})(x) \leq \min\{F_{A_{IF}}^+(x), F_{B_{IF}}^+(x)\} \right), (\forall x \in X) \quad (2.2.15)$$

Then the R-intersection of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is a F-internal FNCS in X .

Proof: The proof is similar to Theorem 2.2.32.

Corollary 2.2.38

If two internal FNCSs $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ satisfy equations 2.2.13 to 2.2.15 then the R-intersection of $\zeta_F = (A_{IF}, \lambda_F)$ and $\mathbb{N}_F = (B_{IF}, \psi_F)$ is an internal FNCSs.

2.3 Fermatean Temporal Neutrosophic set

A temporal set is one in which the components are linked to specific time moment. This section integrates, time moment with FNS and introduces Fermatean temporal neutrosophic set. This concept will be helpful in addressing real-time problems in temporal databases, time-series analysis, and temporal information-based issues.

Definition 2.3.1

Let a , b and c be three real numbers in $[0,1]$ satisfying the inequality $a^3 + b^3 + c^3 \leq 2$. Then the triplet $\langle a, b, c \rangle$ is called a Fermatean neutrosophic number. Let $\langle \tilde{a}_1, \tilde{b}_1, \tilde{c}_1 \rangle$ and $\langle \tilde{a}_2, \tilde{b}_2, \tilde{c}_2 \rangle$ be two Fermatean neutrosophic number. Then define

- i. $\langle \tilde{a}_1, \tilde{b}_1, \tilde{c}_1 \rangle \leq \langle \tilde{a}_2, \tilde{b}_2, \tilde{c}_2 \rangle \Leftrightarrow \tilde{a}_1 \leq \tilde{a}_2, \tilde{b}_1 \leq \tilde{b}_2$ and $\tilde{c}_1 \geq \tilde{c}_2$.
- ii. $\langle \tilde{a}_1, \tilde{b}_1, \tilde{c}_1 \rangle = \langle \tilde{a}_2, \tilde{b}_2, \tilde{c}_2 \rangle \Leftrightarrow \tilde{a}_1 = \tilde{a}_2, \tilde{b}_1 = \tilde{b}_2$ and $\tilde{c}_1 = \tilde{c}_2$.
- iii. If $\{\langle \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \rangle; i \in J\}$ is a family of Fermatean neutrosophic triplets, then $\vee \langle \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \rangle = \langle \vee \tilde{a}_i, \vee \tilde{b}_i, \wedge \tilde{c}_i \rangle$ and $\wedge \langle \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \rangle = \langle \wedge \tilde{a}_i, \wedge \tilde{b}_i, \vee \tilde{c}_i \rangle$.
- iv. The complement of $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ is defined by $\overline{\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle} = \langle \tilde{c}, 1 - \tilde{b}, \tilde{a} \rangle$.

Definition 2.3.2

Let U be a universe and T be a non-empty set. Then the elements of T “time moments”. Based on the definition of FNS, a Fermatean Temporal Neutrosophic Set (FTNS) is defined as the following:

$$A(T) = \{(x, T_A(x, t), I_A(x, t), F_A(x, t)) : (x, t) \in U \times T\}$$

where

- a. $A(T) \subseteq U$ is a fixed set.
- b. $T_A^3(x, t) + I_A^3(x, t) + F_A^3(x, t) \leq 2$ for every $(x, t) \in U \times T$
- c. $T_A(x, t), I_A(x, t)$ and $F_A(x, t)$ are the degrees of truth, indeterminacy, and falsity-membership functions respectively, of the element $x \in U$ at the time moment

$t \in T$. $T_A(x, t), I_A(x, t)$ are independent components and $F_A(x, t)$ is dependent component with respect to $T_A(x, t)$.

Definition 2.3.3

Let

$$A(T') = \{(x, T_A(x, t), I_A(x, t), F_A(x, t)): (x, t) \in X \times T'\}$$

$$B(T'') = \{(x, T_B(x, t), I_B(x, t), F_B(x, t)): (x, t) \in X \times T''\}$$

where T' and T'' have finite number of distinct time-elements. Then,

$$A(T') \cap B(T'') = \left\{ x, \begin{pmatrix} \min(\bar{T}_A(x, t), \bar{T}_B(x, t)), \\ \min(\bar{I}_A(x, t), \bar{I}_B(x, t)), \\ \max(\bar{F}_A(x, t), \bar{F}_B(x, t)). \end{pmatrix} \right\} : (x, t) \in X \times (T' \cup T'')$$

$$A(T') \cup B(T'') = \left\{ x, \begin{pmatrix} \max(\bar{T}_A(x, t), \bar{T}_B(x, t)), \\ \max(\bar{I}_A(x, t), \bar{I}_B(x, t)), \\ \min(\bar{F}_A(x, t), \bar{F}_B(x, t)) \end{pmatrix} \right\} : (x, t) \in X \times (T' \cup T'')$$

Also, subsets of FTNS defined as the follows:

$$A(T') \subseteq B(T'') \Leftrightarrow \bar{T}_A(x, t) \leq \bar{T}_B(x, t), \bar{I}_A(x, t) \leq \bar{I}_B(x, t) \text{ and } \bar{F}_A(x, t) \geq \bar{F}_B(x, t)$$

for every $(x, t) \in X \times (T' \cup T'')$ where

$$\bar{T}_A(x, t) = \begin{cases} T_A(x, t), & \text{if } t \in T' \\ 0 & \text{if } t \in T'' - T' \end{cases}$$

$$\bar{T}_B(x, t) = \begin{cases} T_B(x, t), & \text{if } t \in T'' \\ 0 & \text{if } t \in T' - T'' \end{cases}$$

$$\bar{I}_A(x, t) = \begin{cases} I_A(x, t), & \text{if } t \in T' \\ 0 & \text{if } t \in T'' - T' \end{cases}$$

$$\bar{I}_B(x, t) = \begin{cases} I_B(x, t), & \text{if } t \in T'' \\ 0 & \text{if } t \in T' - T'' \end{cases}$$

$$\bar{F}_A(x, t) = \begin{cases} F_A(x, t), & \text{if } t \in T' \\ 1 & \text{if } t \in T'' - T' \end{cases}$$

$$\bar{F}_B(x, t) = \begin{cases} F_B(x, t), & \text{if } t \in T'' \\ 1 & \text{if } t \in T' - T'' \end{cases}$$

It is obviously seen that $\bar{T}_A(x, t) = T_A(x, t)$, $\bar{T}_B(x, t) = T_B(x, t)$, $\bar{I}_A(x, t) = I_A(x, t)$, $\bar{I}_B(x, t) = I_B(x, t)$, $\bar{F}_A(x, t) = F_A(x, t)$, $\bar{F}_B(x, t) = F_B(x, t)$ when $T' = T''$.

Let J be an arbitrary index set. Then, define that $T = \cup_{i \in J} T_i$ where T_i is a time set for each $i \in J$.

The definition of union and intersection of FTNSs family $F = \{A_i(T_i) = (x, T_{A_i}(x, t), I_{A_i}(x, t), F_{A_i}(x, t)) : (x, t) \in X \times T_i, i \in J\}$ as follows:

$$\bigcup_{i \in J} A(T_i) = \left\{ \left(\begin{array}{l} x, \max_{i \in J} (\bar{T}_{A_i}(x, t)), \\ \max_{i \in J} (\bar{I}_{A_i}(x, t)), \\ \min_{i \in J} (\bar{F}_{A_i}(x, t)) \end{array} \right) \right\} : (x, t) \in X \times T$$

$$\bigcap_{i \in J} A(T_i) = \left\{ \left(\begin{array}{l} x, \min_{i \in J} (\bar{T}_{A_i}(x, t)), \\ \min_{i \in J} (\bar{I}_{A_i}(x, t)), \\ \max_{i \in J} (\bar{F}_{A_i}(x, t)) \end{array} \right) \right\} : (x, t) \in X \times T$$

where

$$\bar{T}_{A_i}(x, t) = \begin{cases} T_{A_i}(x, t), & \text{if } t \in T_i \\ 0 & \text{if } t \in T - T_i \end{cases}$$

$$\bar{I}_{A_i}(x, t) = \begin{cases} I_{A_i}(x, t), & \text{if } t \in T_i \\ 0 & \text{if } t \in T - T_i \end{cases}$$

$$\bar{F}_{A_i}(x, t) = \begin{cases} F_{A_i}(x, t), & \text{if } t \in T_i \\ 1 & \text{if } t \in T - T_i \end{cases}$$

Definition 2.3.4

Let

$$A(T') = \{(x, T_A(x, t), I_A(x, t), F_A(x, t)) : (x, t) \in X \times T'\}$$

$$B(T'') = \{(x, T_B(x, t), I_B(x, t), F_B(x, t)) : (x, t) \in X \times T''\}$$

where T' and T'' have a finite number of distinct time elements. Then, the definitions of instant intersection, and instant union of FTNSs are defined as follows:

$$A(T') \cap_{t_0} B(T'') = \left\{ x, \begin{pmatrix} \min(\bar{T}_A(x, t_0), \bar{T}_B(x, t_0)), \\ \min(\bar{I}_A(x, t_0), \bar{I}_B(x, t_0)), \\ \max(\bar{F}_A(x, t_0), \bar{F}_B(x, t_0)) \end{pmatrix} \right\} : (x, t_0) \in X \times (T' \cup T'')$$

$$A(T') \cup_{t_0} B(T'') = \left\{ x, \begin{pmatrix} \max(\bar{T}_A(x, t_0), \bar{T}_B(x, t_0)), \\ \max(\bar{I}_A(x, t_0), \bar{I}_B(x, t_0)), \\ \min(\bar{F}_A(x, t_0), \bar{F}_B(x, t_0)) \end{pmatrix} \right\} : (x, t_0) \in X \times (T' \cup T'')$$

Also, from the definition of subset in FNS theory, subsets of FTNS can be defined as follows:

$$A(T') \subseteq_{t_0} B(T'') \Leftrightarrow \bar{T}_A(x, t_0) \leq \bar{T}_B(x, t_0), \quad \bar{I}_A(x, t_0) \leq \bar{I}_B(x, t_0) \quad \text{and} \\ \bar{F}_A(x, t_0) \geq \bar{F}_B(x, t_0) \text{ for every } (x, t_0) \in X \times (T' \cup T'') \text{ where}$$

$$\bar{T}_A(x, t_0) = \begin{cases} T_A(x, t_0), & \text{if } t_0 \in T' \\ 0 & \text{if } t_0 \in T'' - T' \end{cases}$$

$$\bar{T}_B(x, t_0) = \begin{cases} T_B(x, t_0), & \text{if } t_0 \in T'' \\ 0 & \text{if } t_0 \in T' - T'' \end{cases}$$

$$\bar{I}_A(x, t_0) = \begin{cases} I_A(x, t_0), & \text{if } t_0 \in T' \\ 0 & \text{if } t_0 \in T'' - T' \end{cases}$$

$$\bar{I}_B(x, t_0) = \begin{cases} I_B(x, t_0), & \text{if } t_0 \in T'' \\ 0 & \text{if } t_0 \in T' - T'' \end{cases}$$

$$\bar{F}_A(x, t_0) = \begin{cases} F_A(x, t_0), & \text{if } t_0 \in T' \\ 1 & \text{if } t_0 \in T'' - T' \end{cases}$$

$$\bar{F}_B(x, t_0) = \begin{cases} F_B(x, t_0), & \text{if } t_0 \in T'' \\ 1 & \text{if } t_0 \in T' - T'' \end{cases}$$

Let J be an arbitrary index set. Then define that $T = \cup_{i \in J} T_i$ where T_i is a time set for each $i \in J$.

The definition of union and intersection of FTNSs family $F_{t_0} = \{A_i(T_i) = (x, T_{A_i}(x, t_0), I_{A_i}(x, t_0), F_{A_i}(x, t_0)) : (x, t_0) \in X \times T_i, i \in J\}$ as follows:

$$\bigcap_{i \in J}^{t_0} A(T_i) = \left\{ \left(x, \max_{i \in J} (\bar{T}_{A_i}(x, t_0)), \max_{i \in J} (\bar{I}_{A_i}(x, t_0)), \min_{i \in J} (\bar{F}_{A_i}(x, t_0)) \right) \right\} : (x, t_0) \in X \times T$$

$$\bigcap_{i \in J}^{t_0} A(T_i) = \left\{ \left(x, \min_{i \in J} (\bar{T}_{A_i}(x, t_0)), \min_{i \in J} (\bar{I}_{A_i}(x, t_0)), \max_{i \in J} (\bar{F}_{A_i}(x, t_0)) \right) \right\} : (x, t_0) \in X \times T$$

where

$$\bar{T}_{A_i}(x, t_0) = \begin{cases} T_{A_i}(x, t_0), & \text{if } t_0 \in T_i \\ 0 & \text{if } t_0 \in T - T_i \end{cases}$$

$$\bar{I}_{A_i}(x, t_0) = \begin{cases} I_{A_i}(x, t_0), & \text{if } t_0 \in T_i \\ 0 & \text{if } t_0 \in T - T_i \end{cases}$$

$$\bar{F}_{A_i}(x, t_0) = \begin{cases} F_{A_i}(x, t_0), & \text{if } t_0 \in T_i \\ 1 & \text{if } t_0 \in T - T_i \end{cases}$$

In fact, these FTNS operators can be seen as FNS operators over FTNSs, since they are defined for a single time moment.

Definition 2.3.5

0_{FN}^t and $1_{FN}^t \in FTNS^{(X, T)}$ are defined as:

$$0_{FN}^t = \left\{ \langle x, 0, 0, 1 \rangle, \langle x, 0, 0, 0 \rangle \right\} : (x, t) \in X \times T \text{ and}$$

$$1_{FN}^t = \left\{ \langle x, 1, 0, 0 \rangle, \langle x, 1, 1, 0 \rangle \right\} : (x, t) \in X \times T$$

for each time moment t , i.e.

$$T_{0_{FN}^t}(x, t) = 0, I_{0_{FN}^t}(x, t) = 0, F_{0_{FN}^t}(x, t) = 1 \text{ or}$$

$$T_{0_{FN}^t}(x, t) = 0, I_{0_{FN}^t}(x, t) = 0, F_{0_{FN}^t}(x, t) = 0$$

and $T_{1_{FN}^t}(x, t) = 1, I_{1_{FN}^t}(x, t) = 0, F_{1_{FN}^t}(x, t) = 0$ or

$$T_{1_{FN}^t}(x, t) = 1, I_{1_{FN}^t}(x, t) = 1, F_{1_{FN}^t}(x, t) = 0 \text{ for each } (x, t) \in X \times T .$$

Definition 2.3.6

$0_N^{t_0}$ and $1_N^{t_0} \in TNS^{(X,T)}$ are defined as:

$$0_N^{t_0} = \left\{ \left(x, T_{0_N^{t_0}}(x, t), I_{0_N^{t_0}}(x, t), F_{0_N^{t_0}}(x, t) \right) : (x, t) \in X \times T \right\}$$

$$1_N^{t_0} = \left\{ \left(x, T_{1_N^{t_0}}(x, t), I_{1_N^{t_0}}(x, t), F_{1_N^{t_0}}(x, t) \right) : (x, t) \in X \times T \right\}$$

for individual time moment $t_0 \in T$, i.e.

$$T_{0_N^{t_0}}(x, t_0) = 0, \quad I_{0_N^{t_0}}(x, t_0) = 0, \quad F_{0_N^{t_0}}(x, t_0) = 1 \quad \text{or}$$

$$T_{0_N^{t_0}}(x, t_0) = 0, \quad I_{0_N^{t_0}}(x, t_0) = 0, \quad F_{0_N^{t_0}}(x, t_0) = 0$$

and $T_{1_N^{t_0}}(x, t_0) = 1, I_{1_N^{t_0}}(x, t_0) = 0, F_{1_N^{t_0}}(x, t_0) = 0$ or $T_{1_N^{t_0}}(x, t_0) = 1, I_{1_N^{t_0}}(x, t_0) = 1,$

$F_{1_N^{t_0}}(x, t_0) = 0$ for each $(x, t_0) \in X \times \{t_0\}$.