Chapter 2

λ_g^δ-Closed Sets in Topological Spaces

2.1 Introduction

Immediately after the notion of δ-closed sets in topological spaces was introduced by Velicko in 1968, several authors started to extend this concept via various types of generalizations. As an outcome of these generalizations, various forms of closed sets and interesting separation axioms have come into existence. In 2004, Georgiou discussed a unique type of generalization of δ-closed sets namely (Λ, δ)-closed sets. This chapter deals with yet another notion namely λ_g^δ-closed sets and their properties.

λ_g^δ-closed sets are weaker than regular closed and δ-closed sets but stronger than δg^*-closed, gδ-closed, δgs-closed and gδs-closed sets. Meanwhile, λ_g^δ-closed sets stand isolated from closed, g-closed, α-closed, pre-closed, semi-closed and δg-closed sets.

The notions of λ_g^δ-open set, λ_g^δ-closure operator, λ_g^δ-interior operator are developed and their properties are analyzed. The family of all λ_g^δ-open sets happen to form an Alexandrov space. Subsequently, λ_g^δ-closure operator satisfies the Kuratowski closure axioms.

2.2 λ_g^δ-closed sets and their properties

Definition 2.2.1. A subset A of a topological space (X, τ) is called ***λ_g^δ-closed set*** if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is (Λ, δ)-open in X. The family of all λ_g^δ-closed sets of (X, τ) is denoted by $\lambda_g^\delta C(X, \tau)$.

Example 2.2.2. In the real line $\Re$ with standard topology, the following facts are observed:

(1) all intervals are Λ_δ-sets.

(2) (Λ, δ)-open sets are of the form:

- (i) $(-\infty, a) \cup (a, \infty)$
- (ii) $(-\infty, a] \cup [b, \infty)$
- (iii) $(-\infty, a) \cup (b, \infty)$
- (iv) $(-\infty, a] \cup (b, \infty)$
- (v) $(-\infty, a) \cup [b, \infty)$

(3) all singletons and closed intervals in $\Re$ are all λ_g^δ-closed.

Proposition 2.2.3. For a topological space (X, τ), the following conditions are valid.

(i) Every δ-closed set is λ_g^δ-closed but not conversely.

(ii) Every λ_g^δ-closed set is $g\delta$-closed but not conversely.

(iii) Every λ_g^δ-closed set is δgs-closed but not conversely.

(iv) Every λ_g^δ-closed set is $g\delta s$-closed but not conversely.

(v) Every regular closed set is a λ_g^δ-closed set but not conversely.

Proof. (i) Let A be a δ-closed set and U be a (Λ, δ)-open set containing A. Since A is δ-closed, $cl_\delta(A) = A$. Therefore $cl_\delta(A) = A \subseteq U$ and hence A is λ_g^δ-closed.

(ii) Let A be a λ_g^δ-closed set and U be a δ-open set containing A. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq U$. As $cl(A) \subseteq cl_\delta(A) \subseteq U$, A is $g\delta$-closed by Definition 2.2.1.

(iii) Let A be a λ_g^δ-closed set and U be a δ-open set containing A. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq U$. As δ-$scl(A) \subseteq cl_\delta(A) \subseteq U$, A is δgs-closed by Definition 2.2.1.

(iv) Let A be a λ_g^δ-closed set and U be a δ-open set containing A. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq U$. As $scl(A) \subseteq cl_\delta(A) \subseteq U$, A is $g\delta s$-closed by Definition 2.2.1.

(v) Since every regular closed is δ-closed, the proof follows from (i).

□

Remark 2.2.4. The converse of the above statements is not true in general. This is justified by the following Example.

Example 2.2.5. (i) Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\}$. Take $A = \{a\}$ then A is λ_g^δ-closed but not δ-closed as $\delta C(X, \tau) = \{X, \phi\}$.

(ii) Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Take $A = \{b\}$ then A is $g\delta$-closed but not λ_g^δ-closed.

(iii) Let X, τ and A be defined as in (ii) then A is δgs-closed but not λ_g^δ-closed.

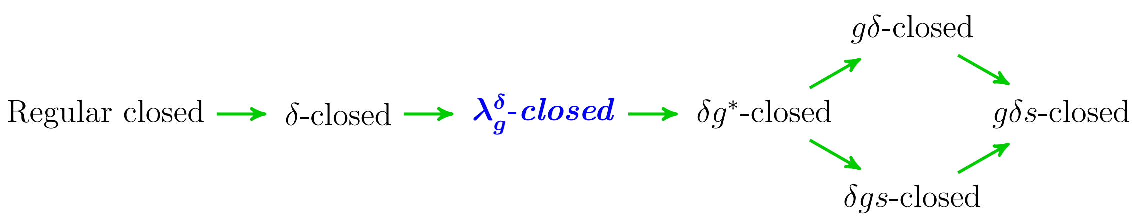
(iv) Let X, τ and A be defined as in (ii) then A is $g\delta s$-closed but not λ_g^δ-closed.

Proposition 2.2.6. Every λ_g^δ-closed set is δg^*-closed.

Proof. Let A be a λ_g^δ-closed set and U be a δ-open set containing A. As every δ-open set is (Λ, δ)-open[Georgiou, 2004], U is (Λ, δ)-open in X. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq U$. Therefore A is δg^*-closed. □

Remark 2.2.7. The newly defined family of λ_g^δ-closed sets properly fits between the family of δ-closed sets and δg^*-closed sets as observed from the following figure.

Figure 2.1: **Dependance Relationship**



Remark 2.2.8. Closedness (resp. g-closedness, α-closedness, semi-closedness, pre-closedness, δg-closedness) is independent of λ_g^δ-closedness as observed from the following Examples.

Example 2.2.9. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Take $A = \{b\}$ then A is closed(resp. g-closed, α-closed, semi-closed, pre-closed and δg-closed) but not λ_g^δ-closed in (X, τ).

Example 2.2.10. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Take $A = \{a\}$ then A is λ_g^δ-closed but not closed(resp. g-closed, α-closed, semi-closed, pre-closed and δg-closed) in (X, τ).

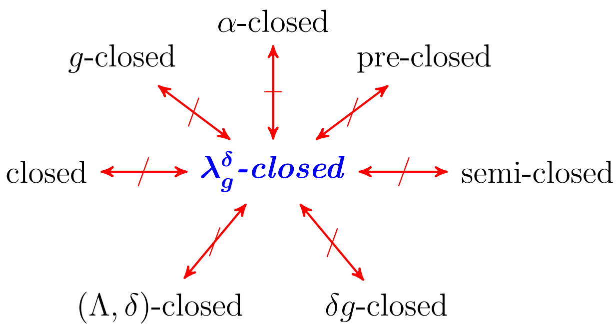
Remark 2.2.11. λ_g^δ-closed sets and (Λ, δ)-closed sets are independent of each other as seen from the following examples.

Example 2.2.12. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Take $A = \{a\}$ then A is λ_g^δ-closed but not (Λ, δ)-closed.

Example 2.2.13. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Take $A = \{a, b\}$ then A is (Λ, δ)-closed but not λ_g^δ-closed.

Remark 2.2.14. The following figure displays the independent relationship among the sets.

Figure 2.2: **Independence Relationship**



Remark 2.2.15. Λ_g-closed sets and λ_g^δ-closed sets are independent of each other. This can be observed from the following Examples.

Example 2.2.16. Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Take $A = \{b\}$ then A is Λ_g-closed but not λ_g^δ-closed.

Example 2.2.17. Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$. Take $A = \{a, c\}$ then A is λ_g^δ-closed but not Λ_g-closed.

Remark 2.2.18. Λ-g-closed sets and λ_g^δ-closed sets are independent of each other. This can be observed from the following Examples.

Example 2.2.19. Let X and τ be defined as in Example 2.2.16. Take $A = \{a\}$ then A is Λ-g-closed but not λ_g^δ-closed.

Example 2.2.20. Let X and τ be defined as in Example 2.2.17. Take $A = \{a, c\}$ then A is λ_g^δ-closed but not Λ-g-closed.

Remark 2.2.21. αg-closed sets and λ_g^δ-closed sets are independent of each other. This can be observed from the following Examples.

Example 2.2.22. Let X and τ be defined as in Example 2.2.16. Take $A = \{b\}$ then A is αg-closed but not λ_g^δ-closed.

Example 2.2.23. Let X and τ be defined as in Example 2.2.16. Take $A = \{a, d\}$ then A is λ_g^δ-closed but not αg-closed.

Theorem 2.2.24. *Let A be a λ_g^δ-closed set in (X,τ). Then $cl_\delta(A) \setminus A$ does not contain any non-empty (Λ,δ)-closed set.*

Proof. Suppose that A is λ_g^δ-closed and let F be (Λ,δ)-closed set contained in $cl_\delta(A) \setminus A$. Now $X \setminus F$ is a (Λ,δ)-open set in X such that $A \subseteq (X \setminus F)$. Since A is a λ_g^δ-closed, $cl_\delta(A) \subseteq (X \setminus F)$. Thus $F \subseteq (X \setminus (cl_\delta(A)))$. Also $F \subseteq cl_\delta(A) \setminus A$. Therefore $F \subseteq (cl_\delta(A))^c \cap cl_\delta(A) = \phi$. Hence $F = \phi$. □

Corollary 2.2.25. Let A be a λ_g^δ-closed set in (X,τ). Then $cl_\delta(A) \setminus A$ does not contain any non-empty δ-closed set.

Proof. Follows from the fact that every δ-closed set is (Λ,δ)-closed. □

Proposition 2.2.26. If A is a (Λ,δ)-open set and a λ_g^δ-closed set of X then A is a δ-closed set of X.

Proof. Since A is (Λ,δ)-open and λ_g^δ-closed, $cl_\delta(A) \subseteq A$. Hence A is δ-closed. □

Corollary 2.2.27. If a subset A of (X,τ) is both δ-open and λ_g^δ-closed then it is δ-closed.

Proof. It follows as in Corollary 2.2.25. □

Theorem 2.2.28. *If every subset of X is λ_g^δ-closed then every δ-open set of X is δ-closed.*

Proof. Let A be δ-open then$A$ is (Λ,δ)-open. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq A$. Hence A is δ-closed. □

Theorem 2.2.29. *If A is a λ_g^δ-closed and (Λ,δ)-open and F is δ-closed in X, then $A \cap F$ is δ-closed.*

Proof. Since A is λ_g^δ-closed and (Λ,δ)-open, A is δ-closed by Proposition 2.2.26. Since F is δ-closed in X, $A \cap F$ is δ-closed in X. □

Proposition 2.2.30. If A is a λ_g^δ-closed set in X and $A \subseteq B \subseteq cl_\delta(A)$, then B is also a λ_g^δ-closed set.

Proof. Let U be a (Λ, δ)-open set of X containing B. Then $A \subseteq U$. Since A is λ_g^δ-closed, $cl_\delta(A) \subseteq U$. Also since $B \subseteq cl_\delta(A), cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$. Hence $cl_\delta(B) \subseteq U$ and therefore B is λ_g^δ-closed. □

Theorem 2.2.31. *Let A be a λ_g^δ-closed set of X. Then A is δ-closed iff $cl_\delta(A) \setminus A$ is (Λ, δ)-closed.*

Proof. *Necessity:* Let A be a δ-closed subset of X. Then $cl_\delta(A) = A$ and so $cl_\delta(A) \setminus A = \phi$, which is (Λ, δ)-closed.
Sufficiency: Let $cl_\delta(A) \setminus A$ be (Λ, δ)-closed. Since A is λ_g^δ-closed, by Theorem 2.2.24, we have $cl_\delta(A) \setminus A$ does not contain a non-empty (Λ, δ)-closed set which implies $cl_\delta(A) \setminus A = \phi$. Therefore $cl_\delta(A) = A$ and hence A is δ-closed. □

Theorem 2.2.32. *If every λ_g^δ-closed set is δ-closed then every singleton is (Λ, δ)-closed or δ-open.*

Proof. Let every λ_g^δ-closed set be δ-closed. Suppose $\{x\}$ is not (Λ, δ)-closed then $X \setminus \{x\}$ is not (Λ, δ)-open. Thus the only (Λ, δ)-open set containing $X \setminus \{x\}$ is X. Hence $cl_\delta\{X \setminus \{x\}\} \subseteq X$ which implies $X \setminus \{x\}$ is λ_g^δ-closed. By hypothesis, $X \setminus \{x\}$ is δ-closed implying $\{x\}$ is δ-open. □

Theorem 2.2.33. *For any topological space (X, τ) with $x \in X$, $X \setminus \{x\}$ is λ_g^δ-closed or (Λ, δ)-open in (X, τ).*

Proof. Suppose $X \setminus \{x\}$ is not (Λ, δ)-open. Then X is the only (Λ, δ)-open set containing $X \setminus \{x\}$. Then $cl_\delta(X \setminus \{x\}) = X$. Hence $X \setminus \{x\}$ is λ_g^δ-closed in (X, τ). □

Definition 2.2.34. For a subset A of a topological space (X, τ), ***(Λ, δ)-closure of*** A (briefly $(\Lambda, \delta)cl(A)$ is defined is defined to be the intersection of all (Λ, δ)-closed sets containing A.

Theorem 2.2.35. *If A is λ_g^δ-closed then $(\Lambda, \delta)cl\{x\} \cap A \neq \phi$, for all $x \in cl_\delta\{A\}$.*

Proof. Suppose $(\Lambda, \delta)cl\{x\} \cap A = \phi$, for some $x \in cl_\delta\{A\}$. Then $X \setminus (\Lambda, \delta)cl\{x\}$ is a (Λ, δ)-open set containing A. As $x \in cl_\delta\{A\}$ and $x \notin X \setminus (\Lambda, \delta)cl\{x\}$, $x \in cl_\delta\{A\} \setminus [X \setminus$

$(\Lambda,\delta)cl(\{x\})]$. Hence $cl_\delta\{A\} \not\subseteq X \setminus cl_\delta\{x\}$, which is a contradiction to the fact that A is λ_g^δ-closed. Thus $(\Lambda,\delta)cl\{x\} \cap A \neq \phi$, for all $x \in cl_\delta\{A\}$. □

Remark 2.2.36. The converse of the above theorem need not be true as observed from the following example.

Example 2.2.37. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. Then $\lambda_g^\delta C(X,\tau) = \{X, \phi, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$ and $\delta C(X,\tau) = \{X, \phi, \{d\}, \{c,d\}, \{a,b,d\}$. Take $A = \{a\}$. Then $cl_\delta(A) = \{a,b,d\}$. We have, for $a \in cl_\delta(A), cl_\delta(\{a\}) \cap A \neq \phi$. For $b \in cl_\delta(A), cl_\delta(\{b\}) \cap A \neq \phi$. But $A = \{a\}$ is not λ_g^δ-closed in (X,τ).

Theorem 2.2.38. *Let A be λ_g^δ-closed. If A is regular open then $pInt(A)$ and $sCl(A)$ are λ_g^δ-closed.*

Proof. Let A be regular open. Then $A = int(cl(A))$ and every regular open set is (Λ,δ)-open. Further since A is λ_g^δ-closed, $cl_\delta A \subseteq A$. Now, $pInt(A) = A \cup int(cl(A)) = A$ and $sCl(A) = A \cap int(cl(A)) = A$. Therefore $cl_\delta\{pInt(A)\} = cl_\delta\{A\} \subseteq A$ and $cl_\delta\{scl(A)\} = cl_\delta\{A\} \subseteq A$ which proves that $pInt(A)$ and $sCl(A)$ are λ_g^δ-closed. □

Theorem 2.2.39. *Finite union of λ_g^δ-closed sets is λ_g^δ-closed.*

Proof. Let $\{A_n\}$ be a finite family of λ_g^δ-closed sets and let $\bigcup_1^n\{A_n\} \subseteq U$, where U is a (Λ,δ)-open set. Then $cl_\delta(\bigcup_1^n\{A_n\}) = \bigcup_1^n(cl_\delta(A_n)) \subseteq U$. Hence finite union of λ_g^δ-closed sets is λ_g^δ-closed. □

Remark 2.2.40. Finite intersection of λ_g^δ-closed sets need not be λ_g^δ-closed as observed from the following example.

Example 2.2.41. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$ then $\lambda_g^\delta C(X,\tau) = \{X, \phi, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$. Take $A = \{a,c\}$ and $B = \{a,d\}$ which are both λ_g^δ-closed but $A \cap B = \{a\}$ is not λ_g^δ-closed.

Remark 2.2.42. Difference of two λ_g^δ-closed sets need not be a λ_g^δ-closed set as observed from the following example.

Example 2.2.43. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ then $\lambda_g^\delta C(X, \tau) = \{X, \phi, \{c\}, \{b, c\}, \{a, c\}\}$. The difference of the λ_g^δ-closed sets {b, c} and {c} is {b} which is not λ_g^δ-closed sets in (X, τ).

2.3 Characterizations of λ_g^δ-closed sets

Remark 2.3.1. Although δg-closed sets and λ_g^δ-closed sets are independent of each other, we obtain the following relationship between them in a $T_{3/4}$-space.

Proposition 2.3.2. In $T_{3/4}$-space, every δg-closed set is λ_g^δ-closed.

Proof. Let (X, τ) be a $T_{3/4}$-space and let A be a δg-closed set in (X, τ). In $T_{3/4}$ -space, every δg-closed set is δ-closed [Dontchev, 2000]. Therefore A is δ-closed in (X, τ). By Proposition 2.2.3(i), every δ-closed set is λ_g^δ-closed and hence A is λ_g^δ-closed in (X, τ). □

Theorem 2.3.3. *In an almost weakly Hausdorff space (X, τ), every λ_g^δ-closed subset of X is δ-closed.*

Proof. Let $A \subseteq X$ be λ_g^δ-closed and let $x \in cl_\delta(A)$. Since (X, τ) is an almost weakly Hausdorff space, every singleton is δ-closed or δ-open.

case (a) **:** $\{x\}$ is δ-open. Since $x \in cl_\delta(A), \{x\} \cap A \neq \phi$ which implies $x \in A$. Hence A is δ-closed.

case (b) **:** $\{x\}$ is δ-closed. If $x \in A$ then $x \notin cl_\delta(A) \setminus A$ which is a contradiction to Theorem 2.2.29. Thus $x \in A$ or equivalently A is δ-closed. □

Proposition 2.3.4. In a topological space (X, τ) which is a partition space as well as a $T_{3/4}$-space, every subset is λ_g^δ-closed.

Proof. Follows from the Definition of a partition space and Proposition 2.3.2. □

Theorem 2.3.5. *Let A be a subset of a topological space (X, τ) which is semi-regular and $T_{3/4}$. Then* *(i) A is g-closed* $\Rightarrow$ *A is* λ_g^δ*-closed.*

(ii) A is closed $\Rightarrow$ *A is* λ_g^δ*-closed.*

Proof. (i) In a semi-regular space, g-closed and δg-closed sets are equivalent. Further in a $T_{3/4}$-space, every δg-closed set is λ_g^δ-closed.

(ii) In a semi-regular space, g-closed sets are λ_g^δ-closed. In general closed sets are g-closed and thus the result follows.

□

Theorem 2.3.6. *In a topological space* (X, τ) *which is semi regular, the following conditions are equivalent.*

(i) A is Λ_g*-closed.*

(ii) λ_g^δ*-closed.*

Proof. In a semi-regular space, δ-open sets and closed sets coincide. Therefore (i) and (ii) are equivalent. □

Proposition 2.3.7. In a topological space (X, τ) which is weakly Hausdorff, every singleton is λ_g^δ-closed.

Proof. From the Definition of weakly Hausdorff and Proposition 2.2.3(i). □

Proposition 2.3.8. In a topological space (X, τ) which is semi-regular, every λ_g^δ-closed set is Λ_g-closed.

Proof. In a semi-regular space, Λ_g-closed sets and λ_g^δ-closed sets are equivalent. In general, every Λ_g-closed set is a Λ-g-closed set [Caldas, 2008]. Therefore the result follows.

□

Proposition 2.3.9. In a topological space (X, τ) which is semi-regular as well as T_1, every λ_g^δ-closed set is closed.

Proof. In a semi-regular space, Λ_g-closed sets and λ_g^δ-closed sets are equivalent. In a T_1-space, every Λ_g-closed set is closed [Caldas, 2008]. Therefore every λ_g^δ-closed set is closed. □

Result 2.3.10. (Caldas, 2008) Every Λ_g-closed set is a Λ-g-closed set.

Proposition 2.3.11. In a topological space (X, τ) which is semi-regular as well as T_0, every λ_g^δ-closed set is λ-closed.

Proof. In a semi-regular space, Λ_g-closed sets and λ_g^δ-closed sets are equivalent. In a T_0-space, every Λ-g-closed set is λ-closed. Further the result follows from Result 2.3.10. □

Proposition 2.3.12. In a topological space (X, τ) which is pointwise semi-regular, every closed singleton of X is λ_g^δ-closed.

Proof. Follows from the definition of pointwise semi-regular space. □

Proposition 2.3.13. In a ${}_\delta T_{3/4}$ space, every $g\delta s$-closed set is λ_g^δ-closed.

Proof. Follows from the definition of ${}_\delta T_{3/4}$ space and Proposition 2.2.3(i). □

Remark 2.3.14. If (X, τ) is almost weakly Hausdorff then the g-closed sets of (X, τ_s) are δ-closed and hence λ_g^δ-closed.

Lemma 2.3.15. If a topological space (X, τ) is almost weakly Hausdorff then every λ_g^δ-closed subset of X is closed.

Proof. Let A be λ_g^δ-closed in X. Since (X, τ) is almost weakly Hausdorff, every $g\delta$-closed set is closed in X. Since every λ_g^δ-closed set is $g\delta$-closed the result follows. □

2.4 λ_g^δ-Open Sets

Definition 2.4.1. A subset A of a topological space (X, τ) is called ***λ_g^δ-open*** if its complement $X \setminus A$ is λ_g^δ-closed in (X, τ). The collection of all λ_g^δ-open sets in (X, τ) is denoted by $\lambda_g^\delta O(X, \tau)$.

Lemma 2.4.2 (Velicko, 1968)**.** For a subset A of (X,τ), $cl_\delta(X \setminus A) = X \setminus int_\delta(A)$.

Theorem 2.4.3. *A subset A of a topological space X is λ_g^δ-open iff $G \subseteq int_\delta(A)$ whenever $G \subseteq A$ and G is (Λ,δ)-closed.*

Proof. *Necessity:* Assume that A is λ_g^δ-open. Then $X \setminus A$ is λ_g^δ-closed. Let G be a (Λ,δ)-closed set in X such that $G \subseteq A$. Then $X \setminus G$ is (Λ,δ)-open in X such that $(X \setminus A) \subseteq (X \setminus G)$. Since $X \setminus A$ is λ_g^δ-closed, $cl_\delta(X \setminus A) \subseteq X \setminus G$, equivalently $G \subseteq cl_\delta(X \setminus A) \Rightarrow G \subseteq int_\delta(A)$.
Sufficiency: Assume that $G \subseteq int_\delta(A)$, whenever $G \subseteq A$ and G is (Λ,δ)-closed in X. Let $(X \setminus A) \subseteq F$, where F is (Λ,δ)-open. Then $(X \setminus F) \subseteq A$. By criteria, $(X \setminus F) \subseteq int_\delta(A) \Rightarrow cl_\delta(X \setminus A) \subseteq F$, by Lemma 2.4.2. Thus $X \setminus A$ is λ_g^δ-closed and hence A is λ_g^δ-open. □

Proposition 2.4.4. If $int_\delta(A) \subseteq B \subseteq A$ and A is λ_g^δ-open in X, then B is λ_g^δ-open in X.

Proof. It follows from Lemma 2.4.2 and Proposition 2.2.30. □

Theorem 2.4.5. *If A is λ_g^δ-open in X then $G = X$ whenever G is (Λ,δ)-open and $int_\delta(A) \cup (X \setminus A) \subseteq G$.*

Proof. Let A be λ_g^δ-open set and G be (Λ,δ)-open and $int_\delta(A) \cup (X \setminus A) \subseteq G$. This implies $(X \setminus G) \subseteq (X \setminus (int_\delta(A) \cup (X \setminus A))) = (X \setminus (int_\delta(A))) \cap A = (X \setminus (int_\delta(A))) \setminus (X \setminus A) = cl_\delta(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is λ_g^δ-closed and $X \setminus G$ is (Λ,δ)-closed, $X \setminus G = \phi$ and hence $G = X$, follows from by Theorem 2.2.24. □

Example 2.4.6. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \phi, \{c\}, \{a,b\}, \{a,b,c\}\}$. Take $A = \{d\}$ then $X \setminus A = \{a,b,c\}$ and $int_\delta A = \phi$. Therefore $(X \setminus A) \cup int_\delta A = \{a,b,c\} \subseteq$ (Λ,δ)-open set $G = X$. Here $G = X$ but $A = \{d\}$ is not λ_g^δ-open.

Theorem 2.4.8. *For subsets A and B of a topological space (X, τ), where B is λ_g^δ-open and $int_\delta(B) \subseteq A$, $A \cap B$ is λ_g^δ-open.*

Proof. Let B be λ_g^δ-open in X and $int_\delta(B) \subseteq A$. Now $int_\delta(B) \subseteq B$. This implies $int_\delta(B) \subseteq A \cap B \subseteq B$. Then by Proposition 2.4.4, $A \cap B$ is λ_g^δ-open. □

Proposition 2.4.9. For each $x \in X$, either $\{x\}$ is (Λ, δ)-closed or $\{x\}$ is λ_g^δ-open in X. That is, for any topological space X, $X = (\Lambda, \delta)C(X, \tau) \cup \lambda_g^\delta O(X, \tau)$.

Proof. Suppose that $\{x\}$ is not (Λ, δ)-closed then $X \setminus \{x\}$ is not (Λ, δ)-open and the only (Λ, δ)-open set containing $X \setminus \{x\}$ is the space X itself. That is, $X \setminus \{x\} \subseteq X$. Therefore, $cl_\delta(X \setminus \{x\}) \subseteq X$ and so $X \setminus \{x\}$ is λ_g^δ-closed and hence $\{x\}$ is λ_g^δ-open. □

Theorem 2.4.10. *Finite intersection of λ_g^δ-open sets is a λ_g^δ-open set.*

Proof. Follows from Theorem 2.2.39. □

Remark 2.4.11. Finite union of λ_g^δ-open sets need not be a λ_g^δ-open set as observed from the following example.

Example 2.4.12. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ then $\{c\}$ and $\{d\}$ are λ_g^δ-open sets but their union $\{c, d\}$ is not a λ_g^δ-open set.

Theorem 2.4.13. *In an almost weakly Hausdorff space (X, τ), every singleton is λ_g^δ-closed or λ_g^δ-open.*

Proof. In an almost weakly Hausdorff space, every singleton is δ-closed or δ-open. Since every δ-closed(resp. δ-open) is λ_g^δ-closed(resp. λ_g^δ-open)[Proposition 2.2.3 (i)], the result follows. □

Proposition 2.4.14. In a δ-door space, every subset is either λ_g^δ-open or λ_g^δ-closed.

Proof. From the Definition of δ-door space and Proposition 2.2.3(i), the proof follows. □

2.5 λ_g^δ-closure and λ_g^δ-interior

Definition 2.5.1. The ***λ_g^δ-closure of A*** (briefly $\lambda_g^\delta cl(A)$) in a topological space (X, τ) is defined to be the intersection of all λ_g^δ-closed sets containing A.

Proposition 2.5.2. If a subset A of (X, τ) is λ_g^δ-closed in X then $\lambda_g^\delta cl(A) = A$ but not conversely.

Proof. Let A be a λ_g^δ-closed set in X. By Definition 2.5.1, $\lambda_g^\delta cl(A) = \cap\{F \subseteq X \mid A \subseteq F, F \in \lambda_g^\delta C(X, \tau)\}$. Since A is λ_g^δ-closed, the smallest set containing A is A and hence $\lambda_g^\delta cl(A) = A$. □

Example 2.5.3. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$. Then $\lambda_g^\delta C(X, \tau) = \{X, \phi, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Take $A = \{a\}$ then $\lambda_g^\delta cl(A) = \{a, c\} \cap \{a, d\} \cap \{a, b, c\} \cap \{a, b, d\} \cap \{a, c, d\} = \{a\} = A$ but A is not λ_g^δ-closed in X.

Proposition 2.5.4. For a topological space X with subsets A and B, the following conditions are valid:

(i) $\lambda_g^\delta cl(\phi) = \phi$ and $\lambda_g^\delta cl(X) = X$.

(ii) If $A \subseteq B$, then $\lambda_g^\delta cl(A) \subseteq \lambda_g^\delta cl(B)$.

(iii) $A \subseteq \lambda_g^\delta cl(A)$.

(iv) $\lambda_g^\delta cl(A \cup B) = \lambda_g^\delta cl(A) \cup \lambda_g^\delta cl(B)$.

(v) $\lambda_g^\delta cl(A \cap B) \subseteq \lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(B)$.

(vi) $\lambda_g^\delta cl(\lambda_g^\delta cl(A)) = \lambda_g^\delta cl(A)$.

(vii) For $A \subseteq X$, $\lambda_g^\delta cl(A) \subseteq cl_\delta(A)$.

Proof. (i), (ii), (iii) and (vi) follow from Definition 2.5.1. (vii) is obvious from Proposition 2.2.3(i).

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (ii) we have $\lambda_g^\delta cl(A) \subseteq \lambda_g^\delta cl(A \cup B)$ and $\lambda_g^\delta cl(B) \subseteq \lambda_g^\delta cl(A \cup B)$. Hence $\lambda_g^\delta cl(A) \cup \lambda_g^\delta cl(B) \subseteq \lambda_g^\delta cl(A \cup B)$. Let $x \notin \lambda_g^\delta cl(A) \cup \lambda_g^\delta cl(B)$ then $x \notin \lambda_g^\delta cl(A)$ and $x \notin \lambda_g^\delta cl(B)$. Therefore, there exist λ_g^δ-closed sets U and V in X such that $A \subseteq U$ and $B \subseteq V$ where $x \notin U$ and $x \notin V$. Therefore $A \cup B \subseteq U \cup V$ and $x \notin U \cup V$. By Theorem 2.2.39, $U \cup V$ is λ_g^δ-closed and hence $x \notin \lambda_g^\delta cl(A \cup B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (ii) we have $\lambda_g^\delta cl(A \cap B) \subseteq \lambda_g^\delta cl(A)$ and $\lambda_g^\delta cl(A \cap B) \subseteq \lambda_g^\delta cl(B)$. Hence $\lambda_g^\delta cl(A \cap B) \subseteq \lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(B)$. The reverse inequality does not hold good as observed from the following Example.

□

Example 2.5.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ then $\lambda_g^\delta C(X, \tau) = \{X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $A = \{a\}, B = \{b\}$ and $A \cap B = \phi$ which implies $\lambda_g^\delta cl(A \cap B) = \phi$, $\lambda_g^\delta cl(A) = \{a, d\}$ and $\lambda_g^\delta cl(B) = \{b, d\}$. Then $\lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(B) = \{d\}$ but $\lambda_g^\delta cl(A \cap B) = \phi$. Hence $\lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(B) \nsubseteq \lambda_g^\delta cl(A \cap B)$.

Theorem 2.5.6. *Let $A \subseteq X$. Then $x \in \lambda_g^\delta cl(A)$ iff $U \cap A \neq \phi$, for every λ_g^δ-open set U containing x.*

Proof. Let $x \in \lambda_g^\delta cl(A)$. Suppose there exists a λ_g^δ-open set U containing x such that $U \cap A = \phi$ then $A \subseteq X \setminus U$, where $X \setminus U$ is λ_g^δ-closed. This implies $\lambda_g^\delta cl(A) \subseteq X \setminus U$ and hence $x \notin \lambda_g^\delta cl(A)$, a contradiction. Conversely, suppose $x \notin \lambda_g^\delta cl(A)$. Then there exists a λ_g^δ-closed set F containing A such that $x \notin F$ so that $X \setminus F$ is a λ_g^δ-open set containing x. Therefore $(X \setminus F) \cap A = \phi$, a contradiction to the hypothesis. □

Remark 2.5.7. Proposition 2.5.4 reveals that λ_g^δ-closure is a Closure operator as well as a Kuratowski closure operator.

Definition 2.5.8. The ***λ_g^δ- interior of A*** (briefly $\lambda_g^\delta int(A)$) in a topological space (X, τ) is defined to be the union of all λ_g^δ-open sets contained in A.

Proposition 2.5.9. If a subset A of (X, τ) is λ_g^δ-open then $\lambda_g^\delta int(A) = A$ but not conversely.

Proof. Let A be a λ_g^δ-open set in X. By Definition 2.5.8, $\lambda_g^\delta int(A) = \cup\{G \subseteq X \mid G \subseteq A, G \in \lambda_g^\delta O(X,\tau)\}$. Since A is λ_g^δ-open, it is the largest λ_g^δ-open set contained in itself is A. Therefore $\lambda_g^\delta int(A) = A$. □

Example 2.5.10. Let X and τ be defined as in Example 2.4.12. Then $\lambda_g^\delta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}\}$. Take $A = \{c,d\}$ then $\lambda_g^\delta int(A) = \{c\} \cup \{d\} = \{c,d\} = A$. But A is not λ_g^δ-open in (X,τ).

Proposition 2.5.11. For a topological space X with subsets A and B, the following conditions are valid:

(i) $\lambda_g^\delta int(\phi) = \phi$ and $\lambda_g^\delta int(X) = X$.

(ii) If $A \subseteq B$, then $\lambda_g^\delta int(A) \subseteq \lambda_g^\delta int(B)$.

(iii) $\lambda_g^\delta int(A) \subseteq A$.

(iv) $\lambda_g^\delta int(A \cup B) \supseteq \lambda_g^\delta int(A) \cup \lambda_g^\delta int(B)$.

(v) $\lambda_g^\delta int(A \cap B) = \lambda_g^\delta int(A) \cap \lambda_g^\delta int(B)$.

(vi) $\lambda_g^\delta int(\lambda_g^\delta int(A)) = \lambda_g^\delta int(A)$.

(vii) For $A \subseteq X$, $\lambda_g^\delta int(A) \subseteq int_\delta(A)$.

Proof. Easy verification is omitted. □

Remark 2.5.12. Reverse inequality of (iv) in Proposition 2.5.11 need not be true as seen from the following example.

Example 2.5.13. Let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}$. Take $A = \{a,d\}$ and $B = \{b,c\}$ then $A \cup B = X$. We have, $\lambda_g^\delta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. Now, $\lambda_g^\delta int(A) = \{a\}$, $\lambda_g^\delta int(B) = \{b,c\}$ and $\lambda_g^\delta int(A \cup B) = X$. Therefore $\lambda_g^\delta int(A \cup B) \not\supseteq \lambda_g^\delta int(A) \cup \lambda_g^\delta int(B)$.

Lemma 2.5.14. For a subset A of (X,τ), $\lambda_g^\delta cl(X \setminus U) = X \setminus (\lambda_g^\delta int(U))$.