Chapter 2

δP_S-open sets in topological spaces

2.1 Introduction

In 1968, the class of δ-open subsets of a topological space was first introduced by Velicko. This class of sets plays an important role in the study of various properties in topological spaces. Since then many authors used this class to define new classes of sets in topological spaces. In 1993, Raychaudhuri and Mukherjee introduced and investigated a class of sets called δ-preopen. Khalaf and Asaad introduced a new concept called P_S-open sets in topological spaces. This class of sets lies strictly between the classes of δ-open and preopen sets. Combining the concepts of δ-preopen and P_S-open sets, a new class of sets called δP_S-open sets is introduced. The behaviour of δP_S-open sets in various spaces such as locally indiscrete, hyperconnected, extremally disconnected, semi-T_1, s-regular spaces are discussed and various interesting results are obtained.

2.2 δP_S-Open Sets

In this section, a new class of open sets called δP_S-open sets is defined and some relations between δP_S-open sets and some other existing open sets are analyzed.

Definition 2.2.1. A subset A of a space X is called a ***δP_S*-open set** if A is a δ- preopen set and for each $x \in A$, there exists a semi-closed set F such that $x \in F \subseteq A$.

The family of all δP_S-open subsets of a topological space (X, τ) is denoted by $\delta P_S O(X, \tau)$ or $\delta P_S O(X)$.

Proposition 2.2.2. A subset A of a space X is δP_S-open if and only if A is a δ-preopen set and A is a union of semi-closed sets.

Proof: From the Definition 2.2.1, a δP_S-open subset A of X is a δ preopen subset.

For every $x \in A$ there exists a semi-closed set F_x such that $x \in F_x \subseteq A$

Hence $A = \bigcup_{x \in A}\{x\} \subseteq \bigcup F_x \subseteq A$, which will imply $A = \bigcup_{x \in A} F_x$, a union of semi-closed sets.

Remark 2.2.3. A δ-preopen set need not be a δP_{S}-open set. This can be seen from the following example.

Example 2.2.4. Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}\}$. Then $\delta PO(X) = P(X)$ and $SC(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$ and $\delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then $\{a\} \in \delta PO(X)$, but $\{a\} \notin \delta P_S O(X)$.

Remark 2.2.5. Union of semi-closed sets need not be a δP_S-open set.

Example 2.2.6. Let X = { a,b,c,d} with the topology
$\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then
$SC(X) = \{X, \emptyset, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\}$ and $\delta PO(X) = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ then
$\delta P_S O(X) = \{X, \emptyset, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Then $\{\{b\}, \{d\}\} \in SC(X)$ but their union $\{b, d\} \notin \delta P_S O(X)$.

Proposition 2.2.7. Any union of δP_S-open sets is a δP_S-open set.

Proof: Let $\{A_\alpha\}$ be a collection of δP_S-open sets. Consider $A = \cup A_\alpha$.

From the Lemma 1.1.24, any union of δ-preopen sets is δ-preopen we get A is δ-preopen.

Now $x \in A$, then $x \in A_\alpha$, for some α and since A_α is a δP_S-open set, there exists a semi-closed set F_x such that $x \in A_\alpha \subseteq F_x \subseteq \cup A_\alpha = A$

$\therefore x \in F_x \subseteq A$

Thus, A is a δP_S-open set.

The following example shows that the intersection of two δP_S-open sets need not be δP_S-open set in general.

Example 2.2.8. Let $X = (0,1)$. If A is the set of rational numbers in X and B is the set of irrational numbers in X together with the singleton set $\{1/2\}$. Then $A \in PO(X) \subseteq \delta PO(X)$. Since X is a T_1-space, every singleton set is closed and hence is semi-closed, then A ∈ δP_SO(X) and $B \in PO(X) \cap SC(X) \subseteq \delta PO(X) \cap SC(X)$, then $B \in \delta P_S O(X)$. But $A \cap B = \{1/2\} \notin \delta P_S O(X)$.

From the above example we notice that the family of all δP_S-open sets need not be a topology on X.

Definition 2.2.9. If (X, τ) is said to have property P' if the δ-closure is preserved under finite intersection or equivalently, if the δ-closure of intersection of any two subsets equals the intersection of their δ-closures.

Lemma 2.2.10. If a space X has the property P', then the intersection of any two δ-preopen sets is δ-preopen, as a consequence of this, $\delta PO(X,\tau)$ is a topology for X and it is finer than τ.

Proof: Let A and B be δ-preopen subsets

$\therefore A \subseteq Int(\delta Cl\, A)$ and $B \subseteq Int\,(\delta Cl\, B)$

$\Rightarrow A \cap B \subseteq Int\,(\delta\, Cl\, A) \cap Int\,(\delta\, Cl\, B)$

$\subseteq Int\,(\delta\, Cl\, A \cap \delta Cl\, B)$

$\subseteq Int(\delta Cl(A \cap B))$ [$\because X$ has property P']

$\Rightarrow A \cap B$ is δ-preopen.

Proposition 2.2.11. If (X,τ) possesses property P' mentioned in Definition 2.2.9, then $\delta P_S O(X,\tau)$ forms a topology.

Proof: By Proposition 2.2.7, arbitrary union of δP_S-open sets is δP_S-open.

Let A and B are δP_S-open sets, Then A and B are δ-preopen and A$\cap B$ is δ-preopen from Lemma 2.2.10.

For each $x \in A \cap B$ there exist semi closed sets F_A & F_B such that

$x \in F_A \subseteq A \; and \; x \in F_B \subseteq B$

$\Longrightarrow x \in F_A \cap F_B \subseteq A \cap B \Rightarrow x \in F \subseteq A \cap B$, where F is semi-closed. [put $F = F_A \cap F_B$]

Hence $A \cap B \in \delta P_S O(X)$.

$\therefore A \cap B$ is $\delta P_S O(X,\tau)$ forms a topology.

$\therefore \delta P_S O(X,\tau_S)$ forms a topology.

Proposition 2.2.12. If A and B are δP_S-open subsets of a topological space (X,τ) and if the family of all δ-preopen sets in X forms a topology on X, then $A \cap B$ is a δP_S-open set and hence the family of δP_S-open sets forms a topology on X.

Proposition 2.2.13. Every P_S-open set is a δP_S-open set.

Proof: Let A be a P_S-open set. By definition A is preopen.

By Lemma 1.1.21(a), A is δ-preopen. ⟶ (1)

Moreover, since A is P_S-open, we get, for each $x \in A$ there exists a semi-closed set F such that

$x \in F \subseteq A$ ⟶ (2)

From (1) & (2) we get, A is δP_S-open, from Definition 2.2.1

Proposition 2.2.14: A δ-open set is a δP_S-open set.

Proof: From Proposition 1.1.31, A δ-open set is P_S-open set.

Now by above property, a P_S-open set is a δP_S-open set. Now by Proposition 2.2.13, a P_S-open set is δP_S-open set.

Hence a δ-open set is δP_S-open set.

Proposition 2.2.15. Every regular open set is a δP_S-open set (ie)$RO(X) \subseteq \delta P_S O(X)$.

Proof: Let A be regular open by a Corollary 1.1.34, A is semi-closed and preopen.

Now A is preopen $\Rightarrow$ A is δ-preopen [By Lemma 1.1.21(a)].

Since A is semiclosed $\Rightarrow$ for each $x \in A$ there exists the semi-closed set A itself, such that $x \in A \subseteq A$

Hence A is in $\delta P_S O(X)$.

$\therefore RO(X) \subseteq \delta P_S O(X)$

The converse of Proposition 2.2.15 is not true in general.

Example 2.2.16. Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}$. Then $\{b\} \in \delta P_S O(X)$ but $\{b\} \notin RO(X)$.

Proposition 2.2.17. Each clopen set is δP_S-open.

Proof: The proof follows from Lemma 1.1.23 and Proposition 2.2.15.

Proposition 2.2.18. Each θ-open set is δP_S-open set.

Proof: Every θ-open is regular open from Remark 1.1.22**.** Every regular open set is δP_S-open by Proposition 2.2.15. Hence every θ-open set is a δP_S- open set.

Remark 2.2.19. From all the above Propositions we have the following figure:

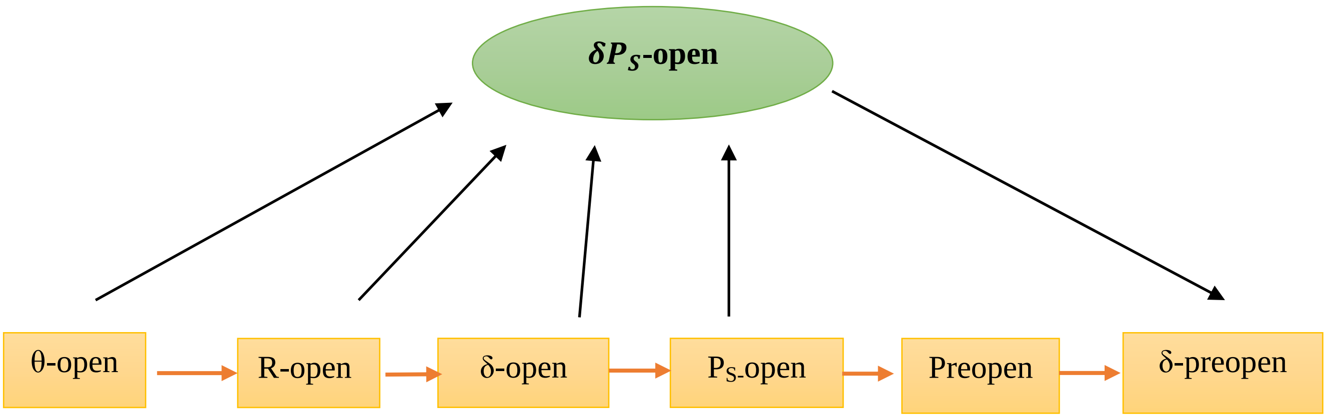

Figure 2.1

Remark 2.2.20. δP_S-open sets are independent with open sets.

Example 2.2.21. Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}\}$. Then $\{a\}$ is open but not δP_S-open and $\{c\}$ is δP_S-open but not open.

Theorem 2.2.22. Let (X, τ) be a space and $x \in X$. Then

a) If $\{x\} \in \delta P_S O(X)$, then $\{x\} \in SC(X)$.

b) $\{x\} \in \delta P_S O(X)$ if and only if $\{x\} \in RO(X)$.

Proof. (a) Let $\{x\} \in \delta P_S O(X)$. Then there exists a semi -closed set F such that $x \in F \subseteq \{x\}, \Rightarrow \{x\}$ is semi-closed. Then $\{x\} \in SC(X)$.

(b) Let $\{x\} \in \delta P_S O(X) \Rightarrow \{x\}$ is semi-closed by (a). (i.e.,) $IntCl\{x\} \subseteq \{x\} \Rightarrow \{x\}$ is clopen $\Rightarrow \{x\} \in RO(X)$.

Proposition 2.2.23. If a space X is semi-T_1, then $\delta P_S O(X) = \delta PO(X)$.

Proof: Let $A \subseteq X$ and $A \in \delta PO(X)$. If $A = \emptyset$, then $A \in \delta P_S O(X)$. If $A \neq \emptyset$, then for each $x \in A$, by Lemma 1.1.10. $\{x\}$ is semi-closed set, since X is semi- T_1.

Now $x \in \{x\} \subseteq A$. Therefore $A \in \delta P_S O(X)$ by Proposition 2.2.2. Hence $\delta PO(X) \subseteq \delta P_S O(X)$.

But $\delta P_S O(X) \subseteq \delta PO(X)$ in general, by Proposition 2.2.2. Therefore, $\delta P_S O(X) = \delta PO(X)$.

Remark 2.2.24. δP_S-open sets are independent with α-open sets.

Example 2.2.25. Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}\}$. Then $\{a\}$ is open but not δP_S-open and $\{b\}$ is δP_S-open but not open.

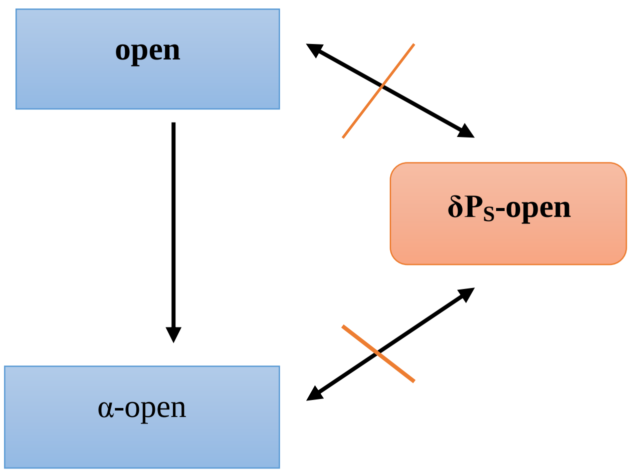

Figure 2.2

Lemma 2.2.26. In a hyperconnected space,

a) $\delta O(X) = \{X, \emptyset\}$

b) $\delta PO(X) = \mathcal{P}(X)$

Proof: (a) If $A \neq \emptyset$ and $A \in \delta SO(X)$, for all $x \in A$, there exists an open set G such that

$x \in G \subseteq Int\, Cl(G) \subseteq A \longrightarrow$ (1)

Since (X, τ) is hyperconnected, for all $a \in \tau$, we get $Cl(a) = X \longrightarrow$ (2)

From (1) & (2) we get, $x \in a \subseteq Int\ Cl(G) = Int\ X = X \subseteq A$.

$$\therefore A = X$$

Hence $\delta O(X, \tau) = \{X, \emptyset\}$

(b) For any subset A, $A \subseteq X = Int\ X = Int(\delta Cl(A)$ $[\because \delta C(A) = \{X, \emptyset\}]$

$$\therefore A \text{ is } \delta\text{-preopen.}$$

$\therefore \delta PO(X) = \mathcal{P}(X)$, the power set of X.

Proposition 2.2.27. In a hyperconnected space, $SC(X) \subseteq \delta P_S O(X)$

Proof: Let X be hyperconnected. Then by Lemma 2.2.26(b) any subset is δ-preopen.

Let $A \in SC(X)$. Now A is δ-preopen and semi-closed. Hence $A \in \delta P_S O(X)$.

Proposition 2.2.28. If $\delta P_S O(X) = \{X, \emptyset\}$ then (X, τ) is hyperconnected.

Proof: Suppose that $\delta P_S O(X) = \{X, \emptyset\}$. Since $RO(X) \subseteq \delta P_S O(X)$ by Proposition 2.2.15, then $RO(X) = \{X, \emptyset\}$. By Lemma 1.1.13, we have (X, τ) is a hyperconnected space.

The converse is not true in general it can be seen from the following example.

Example 2.2.29. Let $X = \{a, b, c\}, \tau = X, \emptyset, \{a\}\}$. Then (X, τ) is hyperconnected, since $Cl\{a\} = X$. By Lemma 1.1.13, $RO(X, \tau) = \{X, \emptyset\}$. But $\delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\} \neq \{X, \emptyset\}$.

Remark 2.2.30. It is to be noted that in the case of P_S-open sets, (X, τ) is hyperconnected if and only if $P_S O(X) = \{X, \emptyset\}$ which is not true in the case of δP_S-open sets.

Proposition 2.2.31. In a locally indiscrete space, $\delta P_S O(X) = \tau$.

Proof: Let (X, τ) be a locally indiscrete space. Let $U \subseteq X$, such that $U \in \tau$. By Definition 1.1.8 of locally indiscrete space every open set in X is closed, then $Int\ Cl(U) = U$ which implies that $U \in RO(X)$. By Proposition 2.2.15, $U \in \delta P_S O(X)$. Thus $\tau \subseteq \delta P_S O(X)$.

Conversely, take $V \in \delta P_S O(X)$. Then V is δ-preopen and for each $x \in V$ and there exists a semi-closed set F such that $x \in F \subseteq V$. By Lemma 1.1.14(b), F is open making V open. Thus $\delta P_S O(X) \subseteq \tau$.

Proposition 2.2.32. In a locally indiscrete space, $SC(X) \subseteq \delta P_S O(X)$.

Proof: The proof is similar to that of Proposition 2.2.27.

Corollary 2.2.33. In a locally indiscrete space, a singleton {x} is semi-closed if and only if $\{x\} \subseteq \delta P_S O(X)$.

Proof: If {x] is semi-closed then by Proposition 2.2.32, $\{x\} \in \delta P_S O(X)$.

Conversely, if $\{x\} \in \delta P_S O(X)$, there exists a semi-closed set F such that $x \in F \subseteq \{x\}$ which implies $\{x\} = F$. Hence $\{x\}$ is semi-closed. Thus, $\{x\} \in SC(X)$.

Proposition 2.2.34. If $A \in \beta O(X) \cap P_S O(X)$, then $A \in \delta P_S O(X)$.

Proof. Let A ∈ βO(X)∩P_SO(X). Then δ-preopen sets are the same as preopen sets from Theorem 1.1.19(b) Then δP_S-open sets are identical with P_S-open sets. Hence $A \in \delta P_S O(X)$.

Proposition 2.2.35. If a topological space (X, τ) is s-regular, then $\tau \subseteq \delta P_S O(X)$.

Proof: Let $A \subseteq X$ such that $A \in \tau$.

If $A = \emptyset$, then $A \in \delta P_S O(X)$

If $A \neq \emptyset$, since X is s-regular, then by Definition 1.1.4 for each $x \in A$, there exists $U \in SO(X)$ such that $x \in U \subseteq sCl(U) \subseteq A$. Thus, we have $x \in sCl(U) \subseteq A$. Since $A \in \tau$, we get $A \in PO(X)$ which implies $A \in \delta PO(X)$.

Moreover, for all $x \in A$ there exist a semi-closed set $sCl(U)$ such that $x \in sCl(U) \subseteq A$.

Hence $A \in \delta P_S O(X)$. Thus $\tau \subseteq \delta P_S O(X)$.

Theorem 2.2.36. For any topological space (X,τ), we have:

a) If τ (resp., $\delta PO(X)$) is indiscrete, then $\delta P_S O(X)$ is also indiscrete.

b) If $\delta P_S O(X)$ is discrete, then $\delta PO(X)$ is discrete.

Proof: (a) **case (i)** Let τ be indiscrete then $\tau = \{X, \emptyset\}$, since $\delta O(X) \subseteq \tau$ τ, we get $\delta O(X) = \{X, \emptyset\}$ and $SO(X) = \{X, \emptyset\}$. Hence if A is δ-preopen and for each x ∈ A there is no semiclosed set except X containing x and contained in A. Hence $\delta P_S O(X) = \{X, \emptyset\} \Rightarrow \delta P_S O(X)$ is indiscrete.

case (ii) Even if $\delta PO(X)$ is indiscrete, then the result follows as in case-i, since $\delta P_S O(X) \subseteq \delta PO(X)$.

(b) Follows from the fact that $\delta P_S O(X) \subseteq \delta PO(X)$

Proposition 2.2.37. If τ is discrete if and only if $P_S O(X)$ is discrete.

Proof: The proof follows from $\tau = \mathcal{P}(X) \Leftrightarrow \delta PO(X) = SC(X) = \mathcal{P}(X)$, the power set of X.

Corollary 2.2.38. For any subset $A \subseteq X$. The following conditions are equivalent:

a) A is clopen.

b) A is δ-open and δ-closed.

c) A is P_S-open and δ-closed.

d) A is α-open and δ-closed.

e) A is δP_S-open and δ-closed.

f) A is δ-preopen and δ-closed

Proof. $(a) \Rightarrow (b)$ By Theorem 1.1.32

$(b) \Rightarrow (c)$ By Remark 1.1.22.

$(c) \Rightarrow (d)$ From (c) we get A is P_S-open and δ-closed. Now by Corollary 1.1.33.

A is α-open

$\therefore (c) \Rightarrow (d)$

$(d) \Rightarrow (e)$ From (d) A is α-open $\Rightarrow A$ is δ-preopen ⟶ (1)

A is δ-closed $\Rightarrow A$ is semi-closed. ⟶ (2)

(1) & (2) $\Rightarrow A$ is δP_S-open.

$\therefore (d) \Rightarrow (e)$

$(e) \Rightarrow (f)$ A is δP_S-open ⇒ δ-preopen by Definition 2.2.1

$\therefore (e) \Rightarrow (f)$

$(f) \Rightarrow (a)$ Now A is δ-preopen $\Rightarrow A \subseteq Int(\delta clA)$

But A is δ-closed also $\therefore$ $A \subseteq Int\ A \Rightarrow A$ is open and

Moreover A is δ-closed $\Rightarrow A$ is closed

Hence A is clopen.

$\therefore (f) \Rightarrow (a)$

Corollary 2.2.39. For a semi-regular space, the following conditions are equivalent:

a) A is regular open.
b) A is P_S-open and semi-closed.
c) A is open and semi-closed.
d) A is α-open and semi-closed.
e) A is δP_S-open and semi-closed.
f) A is δ-preopen and semi-closed.

Proof. In a semi-regular space, δ-closed sets coincide with closed sets. So $\delta PO(X) = PO(X)$ and $\delta P_S O(X) = P_S O(X)$.

Hence the result follows from Theorem 1.1.35.

Corollary 2.2.40. For any topological space, the following statements are equivalent:

a) A is regular open.
b) A is δ-open and δ-semiregular.
c) A is δ-open and δ-semi-θ-closed.
d) A is δ-open and δ-semi-closed.
e) A is α-open and δ-semi-closed
f) A is δP_S-open and δ-semi-closed.

g) A is δ-preopen and δ-semi-closed.

h) A is a-open and e*closed.

Proof. The proof of $a \Rightarrow b, b \Rightarrow c, c \Rightarrow d, d \Rightarrow e$ follow from Theorem 1.1.34 and from Corollary 2.2.39

Proposition 2.2.41: For any space $(X,\tau), \delta P_S O(X,\tau) = P_S O(X,\tau_s)$

Proof: By Lemma 1.1.21(b), $A \in \delta PO(X,\tau) \Leftrightarrow A \in PO(X,\tau_s)$

Proposition 2.2.42. A subset A of a space (X, τ) is δP_S-open if and only if for each x ∈ A, there exists an δP_S-open set B such that x ∈ B ⊆ A.

Proof: If A is an δP_S-open subset in the space (X, τ), then for each x ∈ A, putting A = B, which is δP_S-open containing x such that $x \in B_x \subseteq A$. Conversely. Suppose that for each x ∈ A, there exists a δP_S-open set B_x such that $x \in B_x \subseteq A$. So, $A = \cup_{x \in A}\{x\} \subseteq \cup B_x \subseteq A$ where $B_x \in \delta P_S O(X)$ for each x. Therefore, by Proposition 2.2.7, A is δP_S-open.

Proposition 2.2.43. Let X be a topological space, and A, B ⊆ X. If A ∈ $\delta P_S O(X)$ and B is both α-open and semi-closed, then $A \cap B \in \delta P_S O(X)$.

Proof: Let $A \in \delta P_S O(X)$ and B be α-open, then A is δP-open by Definition 2.2.1. Then by Lemma 1.1.25, $A \cap B \in \delta PO(X)$. Now let $x \in A$ and there exists a semi-closed set F such that $x \in F \subseteq A$. Since B is s-closed, $F \cap B$ is semi-closed and hence $x \in F \cap B \subseteq A \cap B$.

Thus $A \cap B$ is δP_S-open in X.

Proposition 2.2.44. For any topological space, if $A \in \delta PO(X)$ either $A \in \eta O(X) \cup S\theta(X)$, then $A \in \delta P_S O(X)$.

Proof: Let $A \in \eta O(X)$ and $A \in \delta PO(X)$. If $A = \emptyset$, then $A \in \delta P_S O(X)$. If $A \neq \emptyset$, Since $A \in \eta O(X)$, then $A = \cup F_\alpha$, where $F_\alpha \in \delta C(X)$, for each α. Since $\delta C(X) \subseteq SC(X)$, then $F_\alpha \in SC(X)$, for each α. Since $A \in \delta PO(X)$. Then by Proposition 2.2.2, $A \in \delta P_S O(X)$. Suppose that $A \in S\theta O(X)$ and $A \in \delta PO(X)$. If $X = \emptyset$, then $A \in \delta P_S O(X)$. If $A \neq \emptyset$, Since $A \in S\theta O(X)$, then for each x ∈ A, there exists $U \in SO(X)$ such that $x \in U \subseteq sCl(U) \subseteq A$ implies that $x \in sClU \subseteq A$ and $A \in \delta PO(X)$. Therefore, by Definition 2.2.1, $A \in \delta P_S O(X)$

Corollary 2.2.45. For any subset A of a space X. If $A \in S\theta O(X) \cap \delta PO(X)$, then $A \in \delta P_S O(X)$.

Proof. Follows from the Proposition 2.2.44, and the fact that $\theta SO(X) \subseteq S\theta O(X)$ or $\theta SO(X) \subseteq \eta O(X)$ [Proposition 1.1.35].

Proposition 2.2.46. Let (X,τ) be any extremally disconnected space. If $A \in \theta SO(X)$, then $A \in \delta P_S O(X)$.

Proof. Let $A \in \theta SO(X)$. If A = $\emptyset$, then $A \in \delta P_S O(X)$. If $A \neq \emptyset$. Since X is extremally disconnected, then by Theorem 1.1.12, $\theta SO(X) = \delta O(X)$. Hence $A \in \delta O(X)$. But $\delta O(X) \subseteq \delta P_S O(X)$ by Proposition 2.2.14. Therefore, $A \in \delta P_S O(X)$.

2.3 Subspace Properties in δP_S-Open Sets in Topological Spaces

In this section, some properties of δP_S-open sets using subspaces are studied.

Proposition 2.3.1. Let (Y, τ_Y)be a subspace of a space (X, τ). If $A \in \delta P_S O(X, \tau)$ and $A \subseteq Y$, such that $Y \in \delta PO(X)$ then $A \in \delta P_S O(Y, \tau_Y)$.

Proof. Let $A \in \delta P_S O(X, \tau)$, then $A \in \delta PO(X, \tau)$ and for each x ∈ A, there exists F ∈ SC(X, τ) such that $x \in F \subseteq A$. Since $A \in \delta PO(X, \tau)$ and $A \subseteq Y$, such that $Y \in \delta PO(X)$

Then by Theorem 1.1.20, $A \in \delta PO(Y, \tau_Y)$. Since $F \in SC(X, \tau)$ and $F \subseteq Y$. Then by Theorem 1.1.15(b), F ∈ SC (Y, τ_Y). Hence $A \in \delta P_S O(Y, \tau_Y)$.

Proposition 2.3.2. Let (Y, τ_Y) be a subspace of a space (X, τ). If $A \in \delta P_S O(Y, \tau_Y)$ and $Y \in RO(X, \tau)$, then $A \in \delta P_S O(X, \tau)$.

Proof. Let $A \in \delta P_S O(Y, \tau_Y)$, then $A \in \delta PO(Y, \tau_Y)$ and for each x ∈ A, there exists $F \in SC(Y, \tau_Y)$ such that $x \in F \subseteq A$. Since $Y \in RO(X, \tau)$, then $Y \in \delta O(X, \tau)$ and since $A \in \delta PO(Y, \tau_Y)$, then by Theorem 1.1.20, $A \in \delta PO(X, \tau)$.

Again since $Y \in RO(X, \tau)$, then $Y \in SC(X, \tau)$ and since $F \in SC(Y, \tau_Y)$, by Theorem 1.1.15(c), $F \in SC(X, \tau)$. Hence, $A \in \delta P_S O(X, \tau)$.

Corollary 2.3.3. Let Y be a regular open subspace of a space X and let A be a subset of Y. Then A ∈ $\delta P_S O(Y)$ if and only if A ∈ $\delta P_S O(X)$.

Proof. Follows directly from Proposition 2.3.1 and Proposition 2.3.2.

Proposition 2.3.4. Let A and B be any subsets of a space X. If A ∈ $\delta P_S O(X)$ and B ∈ RO(X), then A∩B ∈ $\delta P_S O(B)$.

Proof: Let A ∈ $\delta P_S O(X)$ and then A ∈ δPO(X) and A = ∪Fα where F_α ∈ SC(X) for each α, by Proposition 2.2.2, Then A ∩ B = (∪Fα) ∩ B = ∪ (Fα ∩ B). Since B ∈ RO(X), B is δ-open and since $A \in \delta PO(X)$ by Theorem 1.1.16, A ∩ B ∈ δPO(X)⇒ $A \cap B \in \delta PO(B)$. Again since B ∈ RO(X), B is open and Hence by Lemma 1.1.17, Fα ∩ B ∈ SC(B) for each α. Thus A ∩ B ∈ $\delta P_S O(B)$.

Proposition 2.3.5. Let A and B be any subsets of a space X. If A ∈ $\delta P_S O(X)$ and B ∈ RSO(X), then A ∩ B ∈ $\delta P_S O(B)$.

Proof. Let A ∈ $\delta P_S O(X)$, then A ∈ δPO(X) and A = ∪Fα where Fα ∈ SC(X) for each α by Proposition 2.2.2. Then A ∩ B = (∪Fα) ∩ B = ∪ (Fα ∩ B). Since B ∈ RSO(X), then B ∈

$\delta SO(X)$ and by Lemma 1.1.29, $A \cap B \in \delta PO(B)$. Again since $B \in RSO(X)$, then $B \in SC(X)$ and hence $F\alpha \cap B \in SC(X)$ for each α. Since $F\alpha \cap B \subseteq B$ and $F\alpha \cap B \in SC(X)$ for each α. Then by Theorem 1.1.15(b), $F\alpha \cap B \in SC(B)$. Therefore, by Proposition 2.2.2, $A \cap B \in \delta P_S O(B)$.

Proposition 2.3.6. If $A \in \delta P_S O(X)$ and B is an δ-open subspace of a space X, then $A \cap B \in \delta P_S O(B)$.

Proof. Let $A \in \delta P_S O(X)$, then $A \in \delta PO(X)$ and $A = \cup F\alpha$ where $F\alpha \in SC(X)$ for each α by Proposition 2.2.2. Then $A \cap B = \cup F\alpha \cap B = \cup (F\alpha \cap B)$. Since B is an δ-open subspace of X, and by Theorem 1.1.16, $A \cap B \in \delta PO(X)$. Again, since B is δ-open then B is open. Then by Lemma 1.1.17, $F\alpha \cap B \in SC(B)$ for each α. Then by Proposition 2.2.2, $A \cap B \in \delta P_S O(B)$.

Corollary 2.3.7. If either $B \in RSO(X)$ or B is an δ-open subspace of a space X and $A \in \delta P_S O(X)$, then $A \cap B \in \delta P_S O(B)$.

Proof. Follows directly from Proposition 2.3.5 and Proposition 2.3.6.

Note 2.3.8. In the case of P_S-open sets, the above Proposition is true when B is open in X. But with δP_S-open sets, it must be modified to that B is a δ-open set.

Proposition 2.3.9. Let A and B be any subsets of a space X. If $A \in \delta P_S O(X)$ and $B \in RO(X)$ then $A \cap B \in \delta P_S O(X)$.

Proof: Let $A \in \delta P_S O(X)$ then $A \in \delta PO(X)$ and $A=\cup F_\alpha$ where $F_\alpha \in SC(X)$, for all α. Let $B \in RO(X)$ and $A \cap B = (\cup F_\alpha) \cap B = \cup (F_\alpha \cap B)$. Since, $B \in RO(X)$, then B is δ-open. Then by Lemma 1.1.16(a), $A \cap B \in \delta PO(X)$. Again, since $B \in RO(X)$ then $B \in SC(X)$ and hence $F_\alpha \cap B \in SC(X)$ for each α. Hence by Proposition 2.2.2, $A \cap B \in \delta P_S O(X)$.

2.4. δP_S-CLOSED SETS

In this section, the concept of δP_S-closed sets is introduced in topological spaces and some of its properties are studied.

Definition 2.4.1: A subset B of a space X is called $\boldsymbol{\delta P_S}$**-closed** if X\B is δP_S-open set. The family of all δP_S-closed subsets of a topological space (X,τ) is denoted by $\delta P_S C(X,\tau)$ or $\delta P_S C(X)$.

Proposition 2.4.2: A δ-preclosed subset B of a space X is called δP_S-closed if and only if B is an intersection of semi-open sets.

Proof: Let $B = \cap B_\lambda$ where $B'_\lambda s$ are semi-open sets.

Then $B^C = (\cap B_\lambda)^C = \cup \left(B_\lambda^C\right)$ = Union of semi-closed sets.

Moreover B^C is δ-preopen.

∴ By Proposition 2.2.2, B^C is δP_S-open

$\Leftrightarrow B$ is δP_S-closed set.

Proposition 2.4.3. Let X be a space and A be subset of X. If A ∈ RSO(X) and A ∈ PR(X), then A ∈ δP_SO(X) and A ∈ δP_SC(X).

Proof. Suppose that A ∈ RSO(X) and A∈ PR(X), then A ∈ SC(X) and A∈ PO(X) which implies A ∈ δPO(X). Hence A ∈ δP_SO(X). Again, since A∈ RSO(X) and A ∈ PR(X), then A∈ SO(X) and A ∈ PC(X) which implies A ∈ δPC(X) Hence, A ∈ δP_SC(X).

Definition 2.4.4: A point $x \epsilon X$ is said to be ***δP_S*- interior point** of A if there exists a δP_S-open set U containing x such that $U \subseteq A$. The set of all δP_S-interior points of A is said to be δP_S-interior of A and is denoted by δP_S-IntA.

Proposition 2.4.5: Let A be any subset a space X. If a point $x \epsilon\ \delta P_S Int\ A$, then there exists $F \in SC(X)$, containing x such that $F \subseteq A$.

Proof: Suppose that $x \epsilon\ \delta P_S Int\ A$, then there exists a δP_S-open set $U\ of\ X$ containing x such that $U \subseteq A$. Since $U \in \delta P_S O(X)$, so there exists $F \in SC(X)$ containing x such that $F \subseteq U \subseteq A$. Hence $x \in F \subseteq A$.

Theorem 2.4.6: For any subsets A of a space X. The following statements are true:

a) The δP_S-Interior of A is the union of all δP_S-open sets which are contained in A.
b) δP_S-IntA is δP_S-open set in X contained in A
c) δP_S-Int A is the largest δP_S-open set contained in A

Proof: Proof follows Directly

Definition 2.4.7: Let A be a set in a space X. A point $x \in X$ is in the ***δP_S*-Closure** of A if and only if $A \cap U \neq \emptyset$ for every $U \in \delta P_S O(X)$ containing x and is denoted by δP_S-ClA.

Definition 2.4.8. The δP_S- interior of A (briefly $\delta P_S\ Int(A)$) in a topological space (X, τ) is defined to be the union of all δP_S-open sets contained in A.

Proposition 2.4.9. If a subset A of (X, τ) is δP_S-open then $\delta P_S Int(A) = A$ but not conversely.

Proof. Let A be a δP_S-open set in X. By Definition 2.4.8, $\delta P_S Int(A) = \cup \{G \subseteq X | G \subseteq A, G \in \delta P_S O(X)\}$. Since A is δP_S-open, it is the largest δP_S-open set contained in itself is A. Therefore $\delta P_S int(A) = A$.

Example 2.4.10. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ Then $\delta P_S O(X, \tau)a = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\}$. Take $A = \{c, d\}$ then $\delta P_S Int(A) = \{c\} \cup \{d\} = \{c, d\} = A$. But A is not δP_S-open in (X, τ).

Proposition 2.4.11: For any subset A in X. If $A \cap F \neq \emptyset$ for every F ∈ SC(X) containing x, then $x \in \delta P_S ClA$.

Proof. Let $U \in \delta P_S O(X)$ containing x. Since $U \in \delta P_S O(X)$, so there exists F ∈ SC(X) containing x such that $F \subseteq U$. By hypothesis, we have $A \cap F \neq \emptyset$, then $A \cap U \neq \emptyset$. Hence $x \in \delta P_S ClA$.

Theorem 2.4.12: For any subset A of a space X. The following statements are true.

a) The δP_S-closure of A is the intersection of all δP_S-closed sets containing A.

b) δP_SClA is δP_S-closed set in X containing A.

c) δP_SClA is the smallest δP_S-closed set containing A.

d) A ∈ δP_SC(X) if and only if A = δP_SClA.

Proof. Obvious.

Example 2.4.13: Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$. Then $\delta P_S C(X, \tau) = \{X, \emptyset, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Take $A = \{a\}$ then $\delta P_S cl(A) = \{a, c\} \cap \{a, d\} \cap \{a, b, c\} \cap \{a, b, d\} \cap \{a, c, d\} = \{a\} = A$ but A is not δP_S-closed in X.

Theorem 2.4.14: For any subsets A and B of a space X. The following statements are true.

a) $Int_\delta A \subseteq \delta P_S IntA \subseteq pIntA \subseteq A \subseteq pClA \subseteq \delta P_S ClA \subseteq Cl_\delta A$.

b) δP_SIntA=X\ δP_SCl(X\A) and δP_SClA= X\ δP_SInt(X\A).

c) δP_SClA ∪ δP_SClB ⊆ δP_SCl(A ∪ B)

d) δP_SCl(A ∩ B) ⊆ δP_SClA ∩ δP_SClB.

Proof. Obvious.

The inclusions in (c), (d), cannot be replaced by equality in general, as it is shown in the following two examples.

Example 2.4.15. Let X = {a, b, c, d} and τ = {∅, X, {a}, {b},{a,b}, {a,b,c}, {a,b,d}}; A={a,b}, B={c} then δP_SO(X) = {∅, X, {d}, {a, b, c}}. Then δP_SInt{a, b} ∪ δP_SInt{c} = δP_SInt{a, b, c}={a, b, c}. But δP_SInt{a, b} ∪ δP_SInt{c}={a,b}∪∅. Therefore $\delta P_S Int(A \cup B) \neq \delta P_S IntA \cup \delta P_S IntB$. Similarly, $C = \{a, c\}, D = \{b, c\}$ then $\delta P_S\, Cl\{C \cap D\} = \delta P_S\, Cl\{c\} = \{c\}$. But $\delta P_S\, Cl\{a, c\} \cap \delta P_S\, Cl\{b, c\} = \{a, c, d\} \cap \{b, c, d\} = \{c, d\}$.

$\therefore \delta P_S\, Cl(C \cap D) \neq \delta P_S Cl(C) \cap \delta P_S Cl(D)$.

Example 2.4.16: In Example 2.2.8. It is clear that A, B ∈ δP_SO(X). Therefore δP_SIntA ∩ δP_SIntB = A ∩ B = {1/2}. But δP_SInt (A ∩ B) = δP_SInt{1/2} = ∅. Therefore, δP_SInt(A∩B) ≠ δP_SIntA ∩ δP_SIntB.

Proposition 2.4.17: For each A ∈ SO(X), Cl_δA = δP_SClA.

Proof. Let x ∉ δP_SClA, then there exists a δP_S-open set G containing x such that G ∩ A = ∅ implies that IntClG ∩ ClIntA = ∅. Since A ∈ SO(X), then A ⊆ ClIntA. So IntClG ∩ A = ∅. Since IntClG ∈ RO(X) containing x. Then x ∉ Cl_δA. Hence Cl_δA ⊆ δP_SClA. But δP_SClA ⊆ Cl_δA in general. Therefore, Cl_δA= δP_SClA.

Proposition 2.4.18: For each A ∈ SO(X), Cl_δA = ClA = δP_SClA = pClA = αClA.

Proof. Let A ∈ SO(X), then A ∈ βO(X). By Theorem 1.1.19(b), Cl_δA = ClA and by Theorem 1.1.19(c), αClA = ClA. Since A ∈ SO(X), then by Proposition 2.4.17, Cl_δA = δP_SClA and by Theorem 1.1.19(a), ClA = pClA. It follows that Cl_δA = ClA = δP_SClA = pClA= αClA.

Proposition 2.4.19: For each A ∈ PO(X), δP_SClA ⊆ ClA.

Proof. Suppose that A ∈ PO(X), then A ∈ βO(X), by Theorem 1.1.38, ClIntClA = ClA. Since A ∈ PO(X), then A ⊆ IntClA and hence δP_SClA ⊆ δP_SClIntClA. Since IntClA ∈ RO(X) and hence IntClA∈ SO(X). By Proposition 2.4.17, $\delta P_S(Cl(Int(Cl(A)))) = Cl\left(Int(Cl(A))\right) = Cl(A)$. Therefore, we have $\delta P_S Cl(A) \subseteq Cl(A)$.

Proposition 2.4.20: For each A ∈ PO(X), δP_SClA ⊆ αClA.

Proof. Follows from Theorem 1.1.19(c) and Proposition 2.4.17.

Proposition 2.4.21. For any subset A of a space X. Then A ∈ SO(X) if and only if δP_SClA = δP_SClIntA.

Proof. Necessity. If A ∈ SO(X), then by Theorem 1.1.40, ClA = ClIntA. Since IntA ∈ SO(X). Then by Proposition 2.4.18, we have δP_SClA = δP_SClIntA.

Sufficiency. Let δP_SClA = δP_SClIntA, then A ⊆ δP_SClA = δP_SClIntA. Since IntA ∈ SO(X). Then by Proposition 2.4.18, we have δP_SClIntA = ClIntA. Therefore, A ⊆ ClIntA. Hence A ∈ SO(X).

Theorem 2.4.22. Let A and Y be subsets of a topological space (X, τ) such that A ⊆ Y ⊆ X. Let $\delta P_S Cl_Y$A denotes the δP_S-closure of A in the subspace Y. Then:

a) If either Y ∈ RSO(X) or Y ∈ τ, then $\delta P_S Cl_Y$A ⊆ δP_SClA.
b) If Y ∈ RO(X), then δP_SClA = $\delta P_S Cl_Y$A

Proof. a) Let x ∉ δP_SClA, then there exists a δP_S-open set G in X containing x such that G ∩ A = ∅. Since A ⊂ Y, then G ∩ Y ∩ A = ∅. Put U = G ∩ Y. Since G ∈ δP_SO(X) and either Y

$\in$ RSO(X) or Y $\in$ τ. Then by Corollary 2.3.5, we have U = G ∩ Y $\in \delta P_S$O(Y) and hence U ∩ A = ∅. Thus x $\notin \delta P_S$Cl$_Y$A. Therefore, δP_SCl$_Y$A ⊆ δP_SClA.

b) Let x $\notin \delta P_S$Cl$_Y$A, then there exists a δP_S-open set U in Y containing x such that U ∩ A = ∅. Since U $\in \delta P_S$O(Y) and Y $\in$ RO(X), so by Proposition 2.3.5, U $\in \delta P_S$O(X). Thus x $\notin$ δP_SClA. Hence δP_SClA ⊆ δP_SCl$_Y$A. Also, since either Y $\in$ RO(X) ⊆ RSO(X) or Y $\in$ RO(X) ⊆ τ, then by (a), we have δP_SCl$_Y$A ⊆ δP_SClA. Therefore, δP_SClA = δP_SCl$_Y$A.

Proposition 2.4.23. Let (X,τ) be a topological space and A,Y subsets of X such that A ⊆ Y ⊆ X and Y $\in$ RO(X). Then δP_SCl$_Y$A = δP_SClA ∩ Y.

Proof. Follows directly from Theorem 2.4.22 (b).

From Proposition 2.4.23, we have the following result.

Corollary 2.4.24. If Y is regular open set in X, then δP_SO(Y) = δP_SO(X) ∩ Y

2.5. $\boldsymbol{\delta P_S}$ -closure and $\boldsymbol{\delta P_S}$-interior

In this section, the notion of δP_S-closure and δP_S-interior of a set is introduced and some of its properties are studied.

Definition 2.5.1. The **$\boldsymbol{\delta P_S}$ -closure of A** (briefly $\delta P_S Cl(A))^c$ in a topological space (X,τ) is defined to be the intersection of all δP_S-closed sets containing A.

Proposition 2.5.2. If a subset A of (X,τ) is δP_S-closed in X then $\delta P_S Cl(A) = A$ but not conversely.

Proof. Let A be a δP_S -closed set in X. By Definition 2.5.1, $\delta P_S Cl(A) = \cap \{F \subseteq X | A \subseteq F, F \in \delta P_S C(X,\tau)\}$. Since A is δP_S-closed, the smallest set containing A is A and hence $\delta P_S cl(A) = A$.

Example 2.5.3. Let $\mathrm{X} = \{\mathrm{a,b,c,d}\}, \tau = \{\mathrm{X}, \emptyset, \{\mathrm{a}\}, \{\mathrm{b}\}, \{\mathrm{a,b}\}, \{\mathrm{a,b,c}\}, \{\mathrm{a,b,d}\}$. Then $\delta P_S C(X,\tau) = \{X, \emptyset, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$. Take $A = \{\mathrm{a}\}$ then $\delta P_S cl(A) = \{a,c\} \cap \{a,d\} \cap \{a,b,c\} \cap \{a,b,d\} \cap \{a,c,d\} = \{a\} = A$ but A is not δP_S-closed in X.

Theorem 2.5.4. For a topological space X with subsets A and B, the following conditions are valid:

a) $\delta P_S Cl(\emptyset) = \emptyset$ and $\delta P_S Cl(X) = X$.

b) If $A \subseteq B$, then $\delta P_S Cl(A) \subseteq \delta P_S Cl(B)$.

c) $A \subseteq \delta P_S Cl(A)$.

d) $\delta P_S Cl(A \cup B) = \delta P_S Cl(A) \cup \delta P_S Cl(B)$.

e) $\delta P_S Cl(A \cap B) \subseteq \delta P_S Cl(A) \cap \delta P_S Cl(B)$.

f) $\delta P_S Cl(\delta P_S Cl(A)) = \delta P_S Cl(A)$.

g) For $A \subseteq X, \delta P_S Cl(A) \subseteq Cl_\delta(A)$.

Proof. (a), (b), (c) and (f) follow from Definition 2.5.1

(d) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (c) we have $\delta P_S Cl(A) \subseteq \delta P_S Cl(A \cup B)$ and $\delta P_S Cl(B) \subseteq \delta P_S Cl(A \cup B)$. Hence $\delta P_S Cl(A) \cup \delta P_S Cl(B) \subseteq \delta P_S Cl(A \cup B)$.

(e) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (b) we have $\delta P_S Cl(A \cap B) \subseteq \delta P_S Cl(A)$ and $\delta P_S Cl(A \cap B) \subseteq \delta P_S Cl(B)$. Hence $\delta P_S Cl(A \cap B) \subseteq \delta P_S Cl(A) \cap \delta P_S Cl(B)$.

(g) is obvious from Proposition-2.2.16.

The reverse inequality does not hold good as observed from the following Example.

Example 2.5.5. a) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ then $\delta P_S C(X, \tau) = \{X, \emptyset, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let A = {a}, B = {b} and A ∩ B =$\emptyset$ which implies δP_SCl(A ∩ B) = $\emptyset, \delta P_S$Cl(A) = $\{a, d\}$ and δP_SCl(B) = $\{b, d\}$. Then $\delta P_S Cl(A) \cap \delta P_S Cl(B) = \{d\}$ but $\delta P_S Cl(A \cap B) = \emptyset$. Hence $\delta P_S Cl(A) \cap \delta P_S Cl(B) \nsubseteq \delta P_S Cl(A \cap B)$.

b) Also, by Theorem 2.4.14 (c), δP_SCl(F ∪ E) = X\ δP_SInt(X\(F ∪ E)) = X\ δP_SInt(X\F ∩ X\E) where F = X\A and E = X\B. So δP_SCl(F ∪ E) = X\ δP_SInt(X\F ∩ X\E) = X\ δP_SInt(A ∩ B). Since δP_SInt(A ∩ B) = δP_SInt{1/2} = ∅. So δP_SCl(F ∪ E) = X\∅ = X. But δP_SClF ∪ δP_SClE = δP_SCl(X\A) ∪ δP_SCl(X\B), by Theorem 2.4.14(c), δP_SCl(X\A) ∪ δP_SCl(X\B) = X\ δP_SIntA ∪ X\ δP_SIntB = X\(δP_SIntA ∩ δP_SIntB). Since δP_SIntA ∩ δP_SIntB = A ∩ B = {1/2}. Then δP_SClF ∪ δP_SClE = X\(δP_SIntA ∩ δP_SIntB) = X\(A ∩ B) = X\{1/2}. Therefore, δP_SClF ∪ δP_SClE ≠ δP_SCl(F ∪ E).

Remark 2.5.6. Theorem 2.5.4 reveals that δP_S-closure is a Closure operator as well as a Kuratowski closure operator.

Definition 2.5.7. A point $x \in X$ is said to be ***δP_S*- interior point** of A if there exists a δP_S-open set U containing x such that $U \subseteq A$. The set of all δP_S-interior points of A is said to be δP_S-interior of A and is denoted by δP_S-IntA.

Theorem 2.5.8. For a topological space X with subsets A and B, the following conditions are valid:

a) $\delta P_S Int(\emptyset) = \emptyset$ and $\delta P_S Int(X) = X$.

b) If $A \subseteq B$, then $\delta P_S Int(A) \subseteq \delta P_S Int(B)$.

c) $\delta P_S Int(A) \subseteq A$.

d) $\delta P_S Int(A \cup B) \supseteq \delta P_S Int(A) \cup \delta P_S Int(B)$.

e) $\delta P_S Int(A \cap B) = \delta P_S Int(A) \cap \delta P_S Int(B)$.

f) $\delta P_S Int\big(\delta P_S\ Int(A)\big) = \delta P_S Int(A)$.

g) For $A \subseteq X, \delta P_S Int(A) \subseteq Int_\delta(A)$.

Proof. Easy verification is omitted.

Remark 2.5.9. Reverse inequality of (e) in Theorem 2.5.8 need not be true as seen from the following example.

Example 2.5.10. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}$. Take $A = \{a, d\}$ and $B = \{b, c\}$ then $A \cup B = X$. We have, $\delta P_S O(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Now, $\delta P_S Int(A) = \{a\}, \delta P_S Int(B) = \{b, c\}$ and $\delta P_S Int(A \cup B) = X$. Therefore $\delta P_S Int(A \cup B) \not\supseteq \delta P_S Int(A) \cup \delta P_S Int(B)$.

Proposition 2.5.11: For any two subsets A and B of (X, τ), then, if B is any δP_S-open set contained in A, then $B \subseteq \delta P_S Int(A)$.

Proof: Let B be any δP_S-open set such that B ⊆ A. Let x ∈ B. Since B is a δP_S-open set contained in A, x is a δP_S-interior point of A. That is $x \in \delta P_S Int(A)$. Hence B ⊆ $\delta P_S Int(A)$.

Remark 2.5.12. Theorem 2.5.8 reveals that δP_S-interior is a interior operator as well as a Kuratowski interior operator.