# *INTRODUCTION*

The necessity to handle uncertainty in real-time problems has been an unending research challenge that has created different methodologies and theories. One of the foundational contributions in this area was the introduction of the fuzzy set (FS) theory by Zadeh L. A. (1965) [96], which marked a significant departure from classical binary logic by allowing partial membership. Extensions of FS such as Intuitionistic Fuzzy Set (IFS), introduced by Atanassov K. T. (1986) [8], type-2 fuzzy set initiated by Karnik N. N (1999) [37], and interval-valued fuzzy set introduced by Liang Q et al. (2000) [51], etc., also enabled one to work in uncertain and ambiguous situations and solve unwell-defined problems as well as problems with incomplete information. Fuzzy logic extends classical logic by assigning a membership function ranging between 0 and 1 to the variables whereas IFS allows both membership and non-membership to define uncertainty.

The neutrosophic set (NS), introduced and explored by Smarandache F. (1998) [74], (1999) [75], extends fuzzy and intuitionistic sets to handle uncertainty, vagueness, and inconsistency, in real situations by incorporating indeterminacy. In a NS, each element is characterized by three independent functions: the truth (T), the indeterminacy (I), and the false (F) membership functions, defined within the universe of discourse X. This structure enables neutrosophic logic to provide an enhanced performance over traditional fuzzy logic.

A NS handles a situation based on two types of definitions, depending on the conditions of the truth T, the indeterminate I, and the false F membership functions. In type 1, the functions are completely independent (2005) [76] and in type 2, the truth membership and indeterminate membership functions are independent, while the false membership function is dependent on the truth membership function (2016) [77].

Pythagorean fuzzy set (PFS) introduced by Yager. R.R, (2013) [92] allows for a greater degree of flexibility by expanding the membership and non-membership degrees. In a PFS, each element is associated with a membership degree $\mu_A(x)$, a non-membership degree $\nu_A(x)$ and these values satisfy the condition $\mu_A(x)^2 + \nu_A(x)^2 \leq 1$. The PFS is capable of representing situations in which both acceptance and rejection degrees are relatively high, offering greater flexibility than traditional FS and IFS.

Spherical Fuzzy Set (SFS), introduced by Kutlu Gundogdu. F., (2019) [48], is an enhanced form of traditional fuzzy set theory developed to manage uncertainty and vagueness in a data more effectively than PFS. It characterizes the degree of uncertainity using three parameters, such as membership, non-membership, and hesitation each positioned on the surface of a sphere graphically and hence the term spherical. The spherical design allows modeling situations where both acceptance and rejection can be relatively high, with meaningful hesitation. This reflects real human assessments more accurately, making SFS a powerful tool for complex decision-making problems such as in the fields of engineering [26], medical diagnosis [53], risk assessment [24], sales personnel selection [13], and pattern recognition [87].

Cubic set, introduced by Jun Y.B et.al. (2012) [32], as an extension of FS, and IVFS, provides a more extensive framework for handling uncertainty by incorporating both fuzzy and interval-valued fuzzy components. A cubic set consists of two parts: a fuzzy set component that assigns a precise membership degree to each element and an interval-valued fuzzy set component that represents uncertainty within a range. The dual representation of cubic sets and its extensions, by combining precise and interval-based membership, effectively model complex real-world problems involving mixed uncertainties. They are valuable in decision-making [80], pattern recognition [2], and other situations where both exact and imprecise data need to be represented simultaneously.

Temporal parameters refer to the measurements related to time within a specific context. A temporal fuzzy set extends the idea of a fuzzy set by allowing the degree of membership to vary with time. Instead of a fixed membership value, a temporal fuzzy set has a membership function that maps each element and time point to a degree of membership within the interval. Allen J. F. (1981) [3] introduced the temporal fuzzy set to address uncertainties involving the time component. The temporal intuitionistic fuzzy set was later introduced by Atanassov K. T. (1991) [9]. TFS, assigns time-dependent membership degree over time, while TIFS enrich this by assigning both membership and non-membership degrees over time, capturing more nuanced uncertainty dynamics. These frameworks are widely used in forecasting, decision-making, and temporal knowledge representation where the nature of fuzziness itself varies with time.

On the other hand rough set theory, introduced by Pawlak. Z. (1982) [59], is a formal approximation of crisp set theory in terms of a pair of sets that provide the upper and lower approximations of the original set. To address incomplete information, rough sets offer an approximate representation. Alongside fuzzy sets, rough set theory is one of the most successful approaches and has a wide variety of applications [66].

Topology is the mathematical study of the properties of spaces that are preserved under continuous deformations. Fuzzy topology extends the concepts of classical topology by incorporating the idea of fuzziness, where the boundaries of set are not sharply defined. The idea of fuzzy topology was first introduced by Chang, C. L. (1968) [18], who defined fuzzy open sets as fuzzy subsets satisfying axioms similar to those of classical open sets. In Chang's definition constant functions between fuzzy topological spaces are not necessarily continuous. To address this, Lowen R. (1976) [52] introduced fuzzy topology to clarify the connection between fuzzy topological spaces and classical topological spaces.

Subsequently, Sostak A. (1985) [82], (1996) [83] introduced fuzzy topology using the notion of fuzzy neighborhood systems in L-topological spaces, offering a unified, general, and categorical framework for fuzzy topological spaces. Unlike Chang and Lowen, who focused on fuzzy open sets, Šostak's approach supported defining fuzzy topology through neighborhoods just like in classical topology. This allowed a better notion of fuzzy continuity and convergence, making fuzzy topology more general and flexible. Also Chattopadhyay K. C. et al. (1992) [19], (1993) [20] further explored gradations of openness, fuzzy closure operators, compactness, and connectedness.

Coker D., et al. (1996) [22] developed intuitionistic fuzzy topology in Sostak's L-topology framework using the lattice structure of intuitionistic fuzzy sets. In 1997 [23]**,** Coker provided an axiomatic foundation extending Lowen's fuzzy topology to intuitionistic fuzzy sets, with definitions of continuity and basic properties. These studies contribute to the development of intutionstic fuzzy topology in dynamic environments. Kutlu. F, Bilgin. T (2015) [47], studied Temporal intuitionistic fuzzy topology in Sostak's sense, while Fatih Kutlu (2019) [46] explored them using Chang's approach.

Decision-making is the process of selecting the best option among several alternatives to achieve a specific goal or solution to a problem. Fuzzy and neutrosophic methods are used in decision-making problems because they can model and manage different types of uncertainty, vagueness, and indeterminacy more effectively than classical approaches.

In Multi-Criteria Decision Making (MCDM) problems, various methods are employed to evaluate alternatives based on multiple criteria, including the Weighted Sum Model (WSM) and Weighted Product Model (WPM) for direct aggregation, TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) [12, 16, 48, 98, 100] for ranking alternatives based on proximity to ideal solutions, and CODAS (Combinative Distance-based Assessment) [ 26, 60, 65, 94], which assesses alternatives by their distance from ideal solutions while considering both qualitative and quantitative criteria. The choice of method depends on the specific requirements of the decision-making problem and the nature of the data. Although much research is still ongoing, a few notable examples are listed here: Li, Y. et al. (2025) [50] established distance measures on Fermatean fuzzy sets, Mengyao Zhan et. al. (2024) [98] proposed Fermatean fuzzy TOPSIS method incorporating prospect theory, Amiri, M. et al. (2023) [4] proposed fuzzy extension of simplified Best-Worst Method, and Peng. et al. (2019) [60] enhanced CODAS with a new score function, improving multi-criteria analysis.

Many theories have been developed to address and handle situations when and where uncertainty exists. However, still, there exist limitations in all those theories. Fermatean Neutrosophic Set (FNS) is a flexible framework and generalized theory introduced by Antony Crispin Sweety et al. (2021) [6] that includes fuzzy, intuitionistic fuzzy, Pythagorean fuzzy, spherical fuzzy, Fermatean fuzzy, and neutrosophic theory.

Though the Fermatean neutrosophic set is a recent research topic, their integration with a temporal component and corresponding topological properties remains unexplored. The incorporation of approximation spaces and cubic sets within this framework has yet to be addressed. Additionally, comprehensive distance measures and decision-making methodologies for Fermatean neutrosophic sets are still underdeveloped. Addressing these gaps is crucial for applying Fermatean neutrosophic set theory to real-time problems involving high indeterminacy.

This thesis introduces various neutrosophic set variants, defines their corresponding topological properties, and develops frameworks for Fermatean neutrosophic rough set, Fermatean neutrosophic cubic set, and Fermatean temporal neutrosophic set. Furthermore, it develops comprehensive distance measures and decision-making methodologies for neutrosophic variants to enhance their applicability to problems involving high uncertainty and indeterminacy.

The thesis is organized as follows:

Chapter I has got an overture to our study which includes the requirements of versatile sets which are of great use in the sub sequel of the thesis.

Exploring neutrosophic variants is essential for handling more complex and high uncertain situations. To address high uncertainties, Chapter II introduces neutrosophic variants as the extension of neutrosophic set theory. This chapter focuses on developing new types of neutrosophic variants and is organized into three sections.

The first section defines neutrosophic variants in two different ways from the definition of a dependency-based neutrosophic set. This section redefines the structure of the neutrosophic set and proposes the new notion of extended Pythagorean neutrosophic set, neutrosophic spherical set, and examines their fundamental properties and compares with Fermatean neutrosophic set.

The second section defines the concept of the Fermatean neutrosophic cubic set and their properties. In the real world, most data is based on intervals because measurements, estimates, and judgments are not exact. Fermatean neutrosophic cubic sets are one of the essential sets for managing such data because they combine both exact and non-exact values that can be represented through intervals to handle different types of uncertainty at the same time. The third section, introduces the notion of the Fermatean temporal neutrosophic set, and explores some propositions.

Chapter III defines and analyzes the topological structure of the established neutrosophic set variants. It develops the concept of topology for neutrosophic variants in the framework of experts such as Chang, Lowen, and Sostak. The chapter is divided into three sections. The first section defines the topological structures of Pythagorean, spherical, and Fermatean neutrosophic sets in the sense of Chang and Lowen, along with an exploration of their properties.

Furthermore, the concept of Fermatean neutrosophic gradation of openness is introduced in second section. Within the framework of Fermatean neutrosophic sets, this gradation assigns a specific degree of openness to each set, quantifying the extent to which elements belong. This measure effectively captures the openness of the set. In this section, Fermatean neutrosophic gradation of openness, subspaces, gradation-preserving maps, bases and subbases, product Fermatean neutrosophic topological spaces, Fermatean neutrosophic compactness, and Tychonoff's theorem are defined and explored. The third section introduces a new class of topology called Fermatean neutrosophic temporal topology. This topology is defined within the frameworks established by Chang, Lowen, and Sostak, accompanied by an exploration of its properties.

Chapter IV constructs the approximation spaces for the defined Neutrosophic variants and develops the concept of rough topology. Approximating a space enable extracting meaningful information from imperfect data, making them valuable in various real-life applications.

Chapter IV is structured into three sections. The first section introduces the Fermatean neutrosophic rough set, defines level cuts, and presents relevant propositions and theorems. The second section defines Fermatean temporal neutrosophic rough sets and develops Fermatean temporal neutrosophic rough topology, exploring their topological properties. The third section establishes a working rule for identifying core attributes by integrating the Fermatean temporal neutrosophic set from Chapter II, the concepts of Fermatean temporal neutrosophic topology from Chapter III, and the Fermatean temporal neutrosophic rough topology introduced in this chapter. Additionally, an attribute reduction technique is provided along with an example.

Chapter V develops aggregation operators to formulate decision-making methods. Aggregation operators integrate multiple decision-maker inputs into a single outcome. In this chapter, the CODAS method is formulated using newly defined aggregation operators, along with a score function and an accuracy function. This chapter is divided into two sections. The first section formulates aggregation operators, including arithmetic and geometric operators, for the neutrosophic spherical set and

Fermatean neutrosophic set. These operators are classified based on the interconnections among criteria, and their properties are analysed. The second section applies aggregation operators to the CODAS technique, introducing the Fermatean/Spherical Neutrosophic CODAS method for scenarios characterized by high uncertainty, vagueness, or conflicting opinions. An illustrative example is also provided to validate the proposed technique.

Chapter VI introduces a distance measure for neutrosophic variants. A distance measure is a mathematical function that quantifies the closeness or separation between two objects in a given space. This chapter is structured into three sections. The first section formulates distance measures for neutrosophic variants, such as the Tangent Metric Neutrosophic Spherical Distance Measure (TMNSDM) and the Tangent Metric Fermatean Neutrosophic Distance Measure (TMFNDM). Additionally, relevant propositions and theorems are discussed. The second section compares the proposed distance measures with existing distance measures such as Euclidean Distance, Normalized Euclidean Distance, Hamming Distance, Normalized Hamming Distance Sine Metric Single-Valued Neutrosophic Distance Measure, supported by an illustrative example. The third section applies the defined distance measures and aggregation operators to the TOPSIS method, enhancing decision-making in high-uncertainty and vagueness scenarios. An example of the proposed TOPSIS method is also provided.

Over all this research make a significant contributions to neutrosophic set theory by introducing new variants and their corresponding mathematical frameworks. By integrating topology, rough set theory, and decision-making methodologies, it provides a robust approach to handle high uncertainty in various applications. A new class of neutrosophic variant aggregation operators has been introduced and employed in the CODAS evaluation method. In addition, a novel distance measure has been formulated and applied within the TOPSIS method. Furthermore, the study will open new directions for future research, particularly in applying neutrosophic variants to real-time problems.