A comparison of Retrial $M^X/G/1$ queueing system under multiple adapted vacation policy with classical vacation policies

MANJU S

(16PMA011)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

April, 2018

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9.4.18.

Signature of the Head of the Department

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Signature of the Supervisor

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CHAPTER- I

INTRODUCTION

Queueing theory is concerned with the development of Mathematical models to analyze the behavior of a system that provides service for randomly arising demands. Since demands for service are governed by some probability law, the theory of queues has been developed within the framework of the theory of stochastic processes. Queueing theory has wide range of applications of queueing models such as assembly lines in production system, manufacturing industries, modeling of digital communication messages, flows through complex networks, time sharing and system design.

The pioneer investigator of queueing theory was the Danish Mathematician A.K.Erlang. He investigated the theory of probabilities and telephone conversations in 1909. Many researchers are motivated by his work and developed the queueing theory for practical applications.

1.1 Characteristics of Queueing Theory

In designing a good queueing system, it is necessary to have good information about the model. The basic characteristics that describe a queueing system are the following:

1.1.1 The input Process or arrival pattern of customers

The arrival pattern describes the manner in which the arrivals occur. It is specified by the inter-arrival time between any two consecutive arrivals or by the mean arrival rate. The input pattern also indicates whether the arrivals occur single or in batches of fixed or random size. The inter-arrival time may be deterministic or stochastic. When it is stochastic the probability distribution associated with it is required. In case of bulk arrivals, not only the time between successive arrivals may be probabilistic but also the number of customers in the batch.

1.1.2 The service pattern

Service pattern can be measured by the number of customers served per some unit of time or the time taken to complete a service. The service time may also be constant (deterministic) or stochastic. If it is stochastic, the probability distribution associated with it will be required. The service can be provided in single or batch.

1.1.3 Queue discipline

It refers to the manner by which customers are selected for service when a queue has formed. The most common discipline that can be observed in everyday life is first come, first served (FCFS) or it is sometime called first in, first out (FIFO). Another important discipline, which is very common in the inventory system, is the last in, last out (LILO). Besides these two, there are other queueing disciplines such as random selection of service (RSS) and a variety of priority schemes (very common in hospital causality), where customers are selected for service on the basis of their priorities.

1.1.4 System capacity

The number of customers in the queue and in service together is called the system capacity. The system may have a queue of finite capacity or effectively infinite capacity.

1.1.5 The number of service channels

A system may have a single server or a number of parallel or series of channels. In parallel channels, each and every channel provides an identical service facility, so that several customers may be served simultaneously. Also a queueing system may have only a single stage of service or it may have several stages operated by a single server.

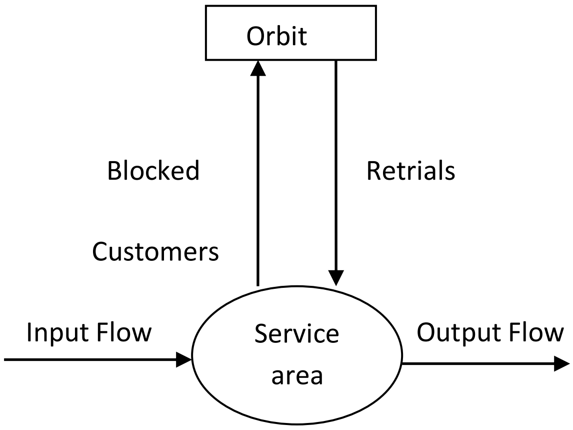
1.1.6 Kendall's Notation

Any queueing system is represented by the notation introduced by Kendall (1951): A/B/C/Y/Z, where A represents the inter arrival time distribution of the customer, B denotes the service time distribution, C is the number of parallel servers, Y represents the capacity of the system and Z denotes the queue discipline. For example, the notation M/G/1 denotes a queueing system with Poisson (Markovian) input, generally distributed service time with single server following FCFS queue discipline and the system capacity is infinite.

1.1.7 Retrial Queueing Models

In classical queueing theory it is assumed that an arriving customer who can not get service immediately either joins the waiting line or leaves the system. This leads to loss of customers. However, such customers after some random period of time return to the system and tire to get service again. Retrial queues have been introduced to solve this deficiency.

The retrial queueing systems are characterized by the feature that an arriving customer who cannot receive service (due to finite capacity of the system, balking, impatience, etc.,) leaves the service area but after some random delay, returns to the system again and requests for service. An arriving customer, encountering the server busy, joins a virtual queue called **orbit**.



1.1 General structure of a Retrial Queue

The classical retrial policy assumes that the probability of a repeated attempt depends on the number of orbiting customers. If the probability of a repeated attempt is independent of the number of orbiting customers, then the retrial policy is called as constant retrial policy. The general structure of retrial queue is shown in Figure 1.1.

The following are some of the special characteristics of queueing systems considered in the thesis.

1.1.8 Bulk or batch arrival queueing models

A queueing system where arrivals or service or both takes place in batches of fixed or random sizes is called a bulk queueing system. Batch arrival queueing models can be used in many practical situations such as the analysis of message packetization in data communication systems.

The basis queueing system considered in the present work is $M^X/G/1$. In this queueing system, the customers arrive in batches and are individually by a single server. The batches arrive according to a time homogeneous Poisson process with parameter λ. The number of customers batches are independent identically distributed positive random variables. The number of customers in the batches denoted by X and the probability distribution of X is given by $Pr(X=k)=g_k$, k=1, 2, 3… that is, the probability that a batch of size k units arrives in an infinitesimal interval (t,t+h) is $\lambda g_k h+O(h)$. The probability generating function g_k is $X(z)=\sum_{k=1}^{\infty} g_k z^k$, with mean $E(X) = X'(1)$. The arrival process is called compound Poison process (Medhi, 2003).

1.1.9 Queue with server vacation

In a classical queueing models, servers are always available in the service facility. However in many practical situations servers may become unavailable for a period of time due to a variety of reasons. This period of server absence is called server vacation i.e., vacation in queueing models represents the period of temporary server absence.

A wide class of policies for governing the vacation mechanism has been discussed in the literature. There are two major types, namely exhaustive and non-exhaustive services with a exhaustive service, the server cannot take a vacation until the system becomes empty. On the other hand, in non-exhaustive system, the server can take a vacation between two services, during busy period. In either case, the rules for resuming services at vacation completion instant and numerous. Based on these rules the two main vacation policies, framed under the exhaustive service discipline are single and multiple vacation policies.

(i) Multiple vacation policy

In multiple vacation policy whenever the system becomes empty (or at the end of each busy period) the server leaves the system for vacation. On returning from the vacation, if the server finds less than the required number of customers then, he may immediately take another vacation. He will continue in this manner, until he finds, upon returning from the vacation, the required number of waiting customers.

(ii) Single vacation policy

In case of single vacation policy when the server returns from the first vacation even if the server finds less than the minimum number of customers required for service, he joins the system and stays idle in the system awaiting the queue length to reach the minimum number for starting his next service.

(iii) Randomized J-vacation or < p, J > -Vacation policy

According to **<p, J>**-vacation policy, if all the customers are served in the queue exhaustively, the server then immediately takes a vacation. Upon returning from a vacation if the queue does not contain required number of customers, then the server either joins the system and remains idle with probability **p** or leaves for another vacation with probability **q(p + q = 1)**. This pattern continues until the number of vacations reaches J. At the end of the J^{th} vacation the server necessarily joins the service facility.

The classical single and multiple vacation policies are the two extreme cases of J-vacation policy (respectively with **J = 1** and J tends to infinity), with **p=1**.

(iv) Multiple adapted vacation policy (MAV)

At the end of each busy period the server either remains idle with probability β_0 or takes a vacation with probability $(\mathbf{1-\beta_0})$. If, at the end of the first vacation no customer has arrived then the server, independently of everything else, takes a new vacation with probabilityβ_1, or remains idle and available to serve the first customer that arrives, with probability $(\mathbf{1-\beta_1})$. This pattern continues until the server on returning from a vacation finds at least one customer in the orbit. That is, the policy is determined by the sequence of probabilities $\{\beta_k\}$, k=0, 1, 2,… . Multiple adapted vacation policy reduces to the randomized J-vacation policy with the following selection of $\beta_k\colon \beta_0 = 1; \beta_k = p \text{ for } 1 \le k \le J-1, \beta_k = 0\ \forall\ k \ge J$. The suitable selection of β_j's (shown in Chapter IV) will reduce MAV policy to Single vacation, multiple vacation policies, and J-vacation policy.

(v) Bernoulli Schedule (Non-exhaustive service) Vacation Policy

In all the vacation policies mentioned above, servers take vacations only when the system becomes empty. But in some situations, especially when the service is done in two or more phases, the maintenance of the system may be required at the completion of each service. In such cases, the service may be stopped for maintenance and overhauling, or continued, if there is no fault in the system. The overhauling may be utilized as a vacation time. The vacation scheme with Bernoulli service discipline originated by Keilson and Servi (1986) is characterized as, after each service, the server may take a vacation with probability **p** or may continued to serve next unit if any, with probability **(1-p)**. The motivation for these types of models comes from computer networks and telecommunication systems were messages are processed in two stages by a single server.

1.1.10 Queue systems with server's Breakdown (or) service interruptions

In queueing situations, servers may breakdown while providing service and the service of the customer being served is then interrupted and cannot resume service until the server is repaired. The period during which the system is in breakdown state is termed as breakdown period. The server breakdown can be classified into operation and time dependent failures. The operation dependent server breakdown can occur only when the server is in operation. The time dependent server breakdown can occur at any time, independent of whether the server is rendering service or not. In queueing models with server breakdown, once failure occurs, the repair process starts immediately and after completion of the repair, service is continued according to the repeat or resume rule.

1.2 Review of Literature

1.2.1 Classical queues

The pioneer investigator of queueing theory is the Danish Mathematician Erlang (1909) who published "The theory of probabilities and telephone conversations" and modeled the telephone traffic systems. Due to its wide applications in many areas, queueing theory has been one of the most active research topics in Operation Research and Management Science for the past several years. Kendall (1951, 1957) was the pioneer who viewed and developed queueing theory from the perspective of stochastic process. Some excellent books on classical queueing theory have been published; they include Takacs (1962), Cooper (1981), Cohen (1982), Gross and Haris (1985), Satty (1983), Wolff (1989), Prabhu (1997), etc.

1.2.2 Retrial queues

The first mathematical results about retrial queues were published in 1950's. Since then numerous papers have been published. Falin& Templeton (1997) stressed the fact that the standard queueing models do not take the retrial phenomenon into account and therefore, cannot be applied in solving a number of practically important problems. To emphasize this idea, they referred to the book by Kosten (1973) and Artalejo (1999). As a consequence of all these efforts, at present, the theory of retrial queues is recognized as an important part of queueing theory.

Cohen (1957) obtained steady state analytical results for the M/M/1 retrial queue. The first result on M/G/1 retrial queues is due to Kielson et al (1968), who used the method of supplementary variables.

Queueing systems with batch arrivals are common in many practical situations. In digital communication systems, messages which are transmitted could consist of a random number of packets. Falin (1976) introduced the batch arrival retrial queueing model. He used the embedded Markov chain technique to derive the joint distribution of the channel state and queue length. Another approach to the problem was proposed by Yang & Templeton (1987).

Queueing systems with vacation time have been found to be useful in modeling those systems in which the server has additional tasks. Many real world systems can be modeled as queues with different vacation policies. A comprehensive and excellent study on the vacation models can be found in Takagi (1988). A comprehensive survey on the recent results for a variety of vacation models can be found in the Doshi (1986, 1990), Krishna Reddy & Anitha (1999). In their research work, the server operates under any one of the vacation policies: single vacation, multiple vacations, gated vacation and so on.

The most classical case in a queue assumes a reliable machine or server. However, in practice, we often meet cases where the servers may fail and can be repaired. Kulkarni & Choi (1990) introduced the retrial queues, considering these server failures and repairs. Queueing systems with repairable service station have been studied by many authors Avi – Itzhak &Naor (1963), Krishnakumar et al (2002), and Artalejo (1994). In the recent years there has been a fast development in the literature on retrial queues. However, very few works take into account both the retrial phenomenon and unreliability of the server. For related literature, one can find the main results and methods about unreliable retrial queues in Atencia et al (2006) and Wang et al (2008).

In retrial queues, the inter-retrial times are modeled according to different disciplines depending on each particular application. In classical retrial policy, the intervals between successive repeated attempts are exponential distributed with a rate $j\gamma$ when the orbit size is j, as discussed by many authors. However, recent application to communication protocols and local area networks show that there are queueing situations in which the retrial rate is independent of the number of customers in the orbit. This constant retrial policy was introduced by Fayolle (1986), who modeled a telephone exchange system.

The first work on the M/G/1 retrial queue with general retrial times is due to Kapyrin (1997) who assumed that each customer in orbit generates a stream of repeated attempts that are independent of the customers in orbit and the server state. Recently there have been some contributions considering queueing systems with general retrial times. Krishna Kumar and Arivudainambi (2002), Atencia and Morena (2005), Li and Wang (2006), Mokaaidis et al. (2007), Wang and Li (2008), Senthil Kumar and Arumuganathan (2010) and Falin (2010) analysed retrial queueing models with general retrial time.

1.3 Thesis Organization

In Retrial queueing systems, a primary customer who does not get service immediately leaves the service area and come back to the system after some period of time to request its demand (service). Between trials the blocked customers join a pool of unsatisfied customers called **orbit.**

Retrial queues play an important role in telephone systems, communication and networking, health care, transportation, manufacturing etc. As far as the analytical results are concerned, it is difficult to obtain them in retrial queues, but there are great number of numerical and approximation methods. In the present work the author makes an attempt to analyse batch arrival single server retrial queueing systems ($M^X\backslash G\backslash 1$) under server breakdowns and server vacations. Chapter II deals with reliable sever and in chapter III unpredictable breakdown during busy period is considered.

The batch arrival retrial queues considered in the present work assume the following operating rule:

"If the sever is busy at the arrival epoch, then the whole arrival batch joins the retrial group , where as if the server is free, then one of the arriving units starts its service and the rest joins the retrial group". The arrivals occur according to the compound Poisson process with group arrival rate $\boldsymbol{\lambda}$. The orbit retreats according to FIFO queue discipline and the retrial time follows a general distribution with finite moments.

In many practical situations, server may become unavailable for a period of time due to various reasons. The period of server absence in some cases is called vacation. The two major types of vacation mechanisms analysed in queueing literature are vacations with exhaustive and non-exhaustive service. In the case of exhaustive service the server can take vacations only when the system becomes empty. In contrast to this, the retrial queues provide an interesting example of vacation models where, after each service completion epoch, the server may take a vacation.

In chapter II, the author considers the batch arrival retrial queue in which the server takes a single vacation between two consecutive services if necessary and takes multiple vacations following Multiple Adapted Vacation policy (MAV) when the system empties. According to this policy whenever the system becomes empty the server may take a vacation (called first vacation) with probability $\boldsymbol{\beta_0}$ or continue to stay in the system to attend the arriving customers with probability $\mathbf{(1-\ \beta_0)}$. After completing the first vacation , if the system is still found empty, the server may take the second vacation with probability β_1 or stays in the system with probability **(1- β_1)**. This pattern continues until the server on returning from a vacation finds at least one customer in the orbit. This policy is referred as Multiple Adapted Vacation policy which generalizes the other vacation policies such as, single multiple and J-vacation policies with the suitable selection of the parameters **βj's**.

Chapter III deals with the Unreliable (M^X\G\1) retrial queueing system by considering the exhaustive service multiple adapted vacation policy. In queueing systems whether the server is a human or a machine, is subjected to unpredictable breakdowns while providing services and the service of the customer being served is then interrupted and cannot resume service until the server is fixed. Thus in practical situations, there is a need to analyze retrial queueing models in which the server is subjected to breakdowns and the customer being served will stay in the service facility to complete the remaining service, as soon as the server is fixed.

Thus the objective of the present work is to develop analytical treatment for two retrial queueing models and obtain some important performance measures for the models. The model descriptions for both the models are presented and the steady state queue size probabilities are derived. Partial probability generating functions are introduced and the total generating function is obtained. The performance measures such as the queue size probabilities when the system in different states including the mean queue length are calculated.

Chapter IV deals with the numerical computations to justify the formulae obtained in chapters II and III and make some effective conclusions. The results of single multiple and J-vacation policies are deduced as special cases.

1.4 Methodology

Techniques for solving problems of Queueing models

Queueing models are classified as Markovian queueing models and non-Markovian queueing models. The techniques generally adopted to solve these types of queueing models are explained below.

Markovian Queueing Models:

Queueing models with exponential inter arrival time and exponential service time are called Markovian queueing models. Some of the techniques used to solve Markovian queueing models are:

- Difference- differential equation method
- Neuts matrix-geometric algorithm
- Continued fraction method

Some queueing systems are studied analytically by deriving the corresponding Difference – differential equations and solving them by applying Rouche's theorem through suitable generating functions. The first method is discussed elaborately by Gross and Saaty (1961). Neuts (1981) developed the matrix-geometric algorithmic approach to study the steady state queueing models. This method involves real arithmetic and avoids the calculation of complex roots based on Rouche's theorem.

Non- Markovian Queueing Models:

The exponential assumption on queueing models, although very convenient, is not always realistic. There is a practical need for models that do not depend on strict Markov assumptions. Queueing models having the inter arrival times and/or service times which are not exponentially distributed are known as non-Markovian queueing models.

The techniques generally used to study non-Markovian queueing models are: Embedded Markov Chain Technique and Supplementary Variable Technique. The author in the present work has analysed the models using supplementary variable technique.

Supplementary variable technique (SVT) (Alfa and Srinivasa Rao (2000))

The first step in analyzing a queueing system is to set it up as a Markov process. In most practical queueing system, supplementary variables are usually needed to achieve this. The alternative to that is an embedded Markov chain method. In the queueing literature, we have two kinds of supplementary variables, in general. They are elapsed time and the remaining time of random variables. For both cases, the approaches of deriving the queueing characteristics are different the main reason for adding supplementary variables to a stochastic process variable is to make a system Markovian. The use of supplementary variable technique (SVT) in queueing dates back to 1942 when it was introduced by Kosten (1973). Later, the technique becomes popular for most stochastic models. This was the result of the article by Cox (1955), in which he used the supplementary variable by considering elapsed service system to study the M/G/1 queueing system. One can assert that the technique of remaining time as the supplementary variable is simple and elegant. The supplementary variable technique analyses for the queueing problem by considering the remaining time as supplementary variables involves, probability density function and its LST and partial differential equations. In the present work, the author used the remaining times as supplementary variables.

1.5 Preliminary Results and the Identities

1.5.1 Transient and steady state solution

Let N(t) denote the number of customer in the system at time t and it is probability distribution denoted by $P_n(t) = Pr(N(t)) = n/N(0) = n)$. For a complete description of the queueing process, the transient or time-dependent solutions are necessary. But it is often difficult to obtain such solutions. Further in many practical situations, we need to know that the behavior in steady state, i.e., when the system reaches an equilibrium state, after being in operation for a pretty long time. The time dependent solutions are called transient solutions. And the solutions obtained as $t \rightarrow \infty$ are called steady state solutions.

There are several methods of solving the difference equations and presenting the probability distributions of the system size and hence calculate important performance measures of the model. Very often the transforms such as the probability generating function $P(z, t) = \sum_{n=0}^{\infty} P_n(t)Z^n$ and the Laplace transforms of the function $L(P_n(t)) = \int_0^{\infty} e^{-\theta t}P_n(t)\,dt$ are used to solve the difference equations.

1.5.2 Probability Generating Function (PGF) of the Random Variable X

In probability theory, the probability generating function of a discrete random variable is a power series representation of the probability mass function of the random variable. The probability generating functions are often employed for their succinct description of the sequence of probabilities $Pr(X = i)$, and to make available the well developed theory of power series with non-negative coefficients.

1.5.3 Laplace Transform

Laplace transform serve as very powerful tools in many situations and an effective means for the solution of many problems arising in queueing theory. The transforms are very effective for solving linear differential equations and reduce a linear differential equation to an algebraic equation. In the study of some probability distributions, this method could be used to find the Laplace transform of a probability distribution rather than the distribution itself.

1.5.4 Laplace Stieltjes Transform (LST)

The Laplace-Stieltjes transform of a non-negative random variable X with distribution function $F(\theta)$, is defined as $F^*(\theta) = \int_{x=0}^{\infty} e^{-\theta t} f(x)dx, \theta \geq 0$. When the random variable X as has a density $f(\theta)$, then the transform simplifies to $F^*(\theta) = \int_{x=0}^{\infty} e^{-\theta t} f(x)dx, \theta \geq 0$. Note that $|F^*(\theta)| \leq 1$ for all $\theta \geq 0$. Further F(0)=1,F`(0) = E(X), $F^{(k)}(0) = (-1)^k E(X^k)$. For numerical study, the algorithms were implemented in computer programs written in C++, using objective oriented tools.

1.5.5 Identities:

1. $$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} QI_{n-k}^*(\theta) g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{k=1}^{\infty} QI_n^*(\theta) z^n \right) = X(z) QI^*(z,\theta)$$

2. $$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{k=1}^{\infty} P_n^*(\theta) z^n \right) = X(z) P^*(z,\theta)$$

1.5.6 Results using L'hospital rule:

If $f(1) = g(1)$, then, $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)_{z=1} = \frac{g'(1)f''(1) - f'(1)g''(1)}{2\left(g'(1)\right)^2}$,

where the dashes represent the derivatives of the functions.

Let $w_x(z) = \lambda\left(1 - X(z)\right)$, $g_\alpha\left(w_x(z)\right) = \alpha + w_x(z)$

and $h_\alpha\,(w_x(z)) = g_\alpha\left(w_x(z)\right) - \alpha\, R^*\left(w_x(z)\right)$. Then

1. $\lim_{z \to 1} \left[\frac{1 - R^*\left(w_x(z)\right)}{w_x(z)} \right] = E(R)$

2. $\lim_{z \to 1} \left[\frac{1 - VI^*\left(w_x(z)\right)}{w_x(z)} \right] = E(VI)$

3. $\lim_{z\to 1}\left[\frac{S^*(h_\alpha(w_x(z)))}{h_\alpha(w_x(z))}\right] = E(S)$

4. $\lim_{z\to 1} S^*(h_\alpha(w_x(z))) = 1$

5. $\frac{d}{dz}\left[\frac{1 - R^*\left(w_x(z)\right)}{w_x(z)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(R^2)}{2}$

6. $\frac{d}{dz}\left[\frac{1 - VI^*\left(w_x(z)\right)}{w_x(z)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(VI^2)}{2}$

7. $\frac{d^2}{dz^2}[S^*(w_x(z))]_{z=1} = (\lambda\, E(X))^2\, E(S^2) + \lambda\, E(X(X-1))\, E(S)$

8. $\frac{d}{dz}\left[S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]_{z=1} = \lambda\, E(X)\, E(S)[1 + \alpha\, E(R)]$

CHAPTER II

A RELIABLE RETRIAL $M^X/G/1$ QUEUEING SYSTEM UNDER TWO TYPES OF SERVER VACATIONS

Introduction

In the present work the author analyses batch arrival retrial queueing model $M^X/G/1$ under server vacations.

In classical queueing theory, it is assumed that any customer, who cannot get service immediately upon arrival, either joins a waiting line or leaves the system forever. But there are real situations where the blocked customers leave the service area temporarily but returns to repeat their demand after some random time. This queueing behavior is referred as retrial queues.

Retrial queueing systems are characterized by the fact that a primary request, finding all servers and waiting positions busy upon arrival, leaves the service area but after some random time, repeats it's demand. Between retrials, the customer is said to be in "orbit". So the repeated attempts for service from the group of blocked customers are super imposed on the normal stream of arrivals of primary requests. Thus the retrial queues can be considered as alternative to queues with losses that do not take repeated attempts into account. Retrial queues have wide applications in telephone systems, computer and communication networks and daily life situation. For a review of main results and methods, the reader is referred to the specific monographs by Failn and Templeton (1997).

In classical queueing models, servers are always available in the service facility. However in many practical situations servers may become unavailable for a period of time due to a variety of reasons. This period of server absences is called server vacation i.e., vacation in queueing models represents the period of temporary server absence. A wide class of policies for governing the vacation mechanism has been discussed in the literature. There are two major types, namely exhaustive and non-exhaustive services. With an exhaustive service, the server cannot take a vacation until the system becomes empty. On the other hand, in non-exhaustive case, the server can take a vacation between two services, during busy period.

In the present chapter, both exhaustive and non-exhaustive service vacation policies are considered. As far as the exhaustive service vacation models are concerned, many of the early work deals with single vacation and repeated vacation policies. Later some of the authors discussed queueing systems under J-vacation policy which generalizes the single and repeated vacation. Very few authors considered the multiple adapted vacation policy for their queueing systems. As far as the retrial queueing systems are concerned the existing literature does not consider multiple adapted vacation policy.

Chapter-II deals with reliable $M^X/G/1$ queueing system in which the arriving customers, who find the server busy, join the retrial queue to try again for their request at random intervals. For the retrial system it is necessary to fix the mechanism of retrial. Most queueing systems with repeated attempts assume that each customer in orbit seeks service independently of each other after a random time exponentially distributed. In the present work, it is assumed that the retrial time follows general distribution. It is also assumed that the server is reliable and whenever the system becomes empty the server takes vacation according to multiple adapted vacation policy which generalizes the other vacation policies. In addition it is also assumed, that the server may take single vacation after completing each service to a customer if necessary. In chapter-III it is considered that the server takes multiple adapted vacations only when the system becomes empty but the server is subjected to unpredictable breakdowns while he is in service.

These retrial models are analysed under steady state and the probability generating functions of queue size probabilities at arbitrary epoch are calculated. The important measures such as system size probabilities and mean queue length are also derived.

2.1 Mathematical Analysis of the System

2.1.1 Model Description

Arrival Pattern

Customers arrive in batches at the system according to a time homogeneous Poisson process with group arrival rate $\boldsymbol{\lambda}$. The batch size **X** is a random variable with probability distribution $\Pr(X = k) = g_k, k = 1,2,\ldots$ and $\sum_{k=1}^{\infty} g_k = 1$. There is no waiting space in front of the server, therefore, if an arriving batch finds the server is idle then one of the arriving customers begins to receive his service immediately and the others leave the service area and enter into the orbit according to **FCFS** queue discipline. The customer at the head of the retrial queue competes with potential primary customers, to decide which customer will enter the next service. If a batch of primary customers arrives first, the retrial customer may cancel its attempt for service and either returns to its position in the retrial queue with probability **q** or quits the system with probability **(1-q)**.

The retrial time of the customer in the retrial queue is generally distributed with distribution function **A(t)**, density function **a(t)** and LST **A*(θ)**. Further it is assumed that, the retrial times begin only when the server is freely available in the system (i.e., either at the completion instants of services or vacation completion instants). When the server is idle, arriving (new arrival) customers must turn on the server immediately. If the server is found busy or on vacation, the arriving batch joins the retrial queue according to **FCFS** discipline.

Multiple Adapted Vacation Policy (MAV)

A cycle starts whenever the system becomes empty and the deactivated server either remains idle in the system with probability $\mathbf{1 - \beta_0}$ or takes a vacation with probability $\boldsymbol{\beta_0}$. At the end of each vacation $\mathbf{j}$, $(\mathbf{j} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \infty)$ if no customers arrive, then the server takes a new vacation with probability $\boldsymbol{\beta_j}$, or remains idle in the system with probability $\mathbf{1 - \beta_j}$. This process continues until a customer arrives and then the server starts a new busy cycle. Vacation times are independent identically distributed random variables with the common distribution function **VI(t)** and corresponding density function **vI(t)**. The time during which the server is either on vacation or idle in the system is called idle period.

Busy Period

Busy period starts at the end of each idle period. During busy period, the server provides service to the arriving customers one at a time. The service time of the arriving customers are assumed to be independent identically distributed random variables with common distribution **S(t)**, density function **s(t)**. The **FCFS** queue discipline is followed, during busy period.

Bernoulli Schedule Vacation

After completing each service, the server is allowed to take a Bernoulli Schedule Vacation. i.e., At the end of each service, the customer leaves the system and the server may take a single vacation with probability **p** or may remain in the system to serve the next customer with probability **(1-p)**. The vacation time **VB** follows a general distribution **VB(t)**. When the system become empty, the server either takes vacation of duration **VI** or waits for the next customer to arrive or retry according to Multiple Adapted Vacation policy. Thus each cycle consists of Idle period, vacation period during idle time, busy period and vacation period between services.

The following notations are introduced to analyses the model. The system is analysed using SVT technique by introducing the remaining time of random variables and SVT variables.

Notations:

λ : Group arrival rate

X : Group size random variable

g_k : $Pr(X = k), k = 1, 2, 3, \dots$

X(z) : Probability generating function of X.

N(t) : The system size at time t

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplaces-Stieltjes Transform (LST) and the k^{th} moments of the random variables are listed below:

	RV	CDF	PDF	LST	K^{th} moments
Retrial time	A	A(w)	a(w)	$A^*(\theta)$	$E(A^k)$
Idle Vacation Time	VI	VI(x)	vI(x)	$VI^*(\theta)$	$E(VI^k)$
Vacation time during busy period	VB	VB(x)	$v_B(x)$	$VB^*(\theta)$	$E(VB^k)$
Service time	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$

Let $A^0(t)$, $VI^0(t)$, $VB^0(t)$ and $S^0(t)$ denote respectively the remaining times of the random variables: namely retrial time, idle vacation time, vacation time during busy period and the service time at time t. Further different states of the server at time **t** are denoted by Y(t) = {0,1,2,3} which respectively denotes idle state, vacation state during idle period, vacation state during busy period and busy state. The supplementary variables are introduced in order to obtain a bivariate Markov Process {N(t), δ(t)} where N(t) denotes the system size random variable and $\delta(t) = (A^0(t), V^0(t), VB^0(t), S^0(t))$ according as Y(t) = (0,1,2,3) respectively.

Let

$PI_0(t) = \Pr\{N(t) = 0, Y(t) = 0\}$ $\quad n = 0$

$PI_n(x, t)\, dt = \Pr\{N(t) = n, w \leq A°(t) \leq w + dt, Y(t) = 0\}$ $\quad n \geq 1$

$QI_{n,j}(x, t)\, dt = \Pr\{N(t) = n, x \leq VI°(t) \leq x + dt, Y(t) = 1\}$ $\quad n \geq 0$

$QB_n (x, t)\, dt = \Pr\{N(t) = n, S_0{}^0(t) = x, y \leq VB°(t) \leq y + dt, Y(t)=2\}$ $\quad n \geq 1$

$P_n(x, t)\, dt = \Pr\{N(t) = n, x \leq S°(t) \leq x + dt, Y(t) = 3\}$ $\quad n \geq 1$

$PI_n(w,t)dt$ is the joint probability that at time **t**, there are **n** customers in the retrial orbit, the server is idle and the remaining retrial time of the server is between **w** and **w + dt**, where **$n \geq 1$** and **$PI_0(t)$** is the probability that the server is idle at time **t**, and there is no customer in the retrial orbit.

$QI_{n,j}(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the retrial orbit, the server is in J^{th} vacation and the remaining vacation time of the server is between **x** and **x + dt**, where **$n \geq 0$**.

$QB_n(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the system, and the remaining vacation time of the server is between **x** and **x + dt** during busy period, where **$n \geq 1$**.

$P_n(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the retrial orbit, the server is busy, and a customer is being served in service and the remaining service time lies between **x** and **x + dt**, **$n \geq 1$**.

The System Size Distribution

Assuming the steady state probabilities exist as t → ∞ and independent of time t, the following steady state equations are obtained for the queueing system using supplementary variables technique:

The steady state probabilities are given by

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_n(x,t) = \frac{d}{dx} P_n(x) \;; \qquad \lim_{t\to\infty} \frac{\partial}{\partial x} QI_{n,j}(x,t) = \frac{d}{dx} QI_{n,j}(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x} QB_n(x,t) = \frac{d}{dx} QB_n(x)$$

$$\lim_{t\to\infty} \left(\frac{\partial}{\partial x} P_n(x,t) = \frac{\partial}{\partial x} QI_{n,j}(x,t) = \frac{\partial}{\partial x} QB_n(x,t) = \right) = 0$$

At steady state $\mathbf{P_n(0)}, \mathbf{QI_{n,j}(0)}, \mathrm{Q}\mathbf{B_n(0)}$ denotes the probability that there are **n** customer in the orbit at the termination of service period, vacation (during idle) period and busy vacation period respectively.

2.2 The Steady State Queue Size Equations

Idle State

$$\lambda \mathrm{PI}_0 = \sum_{j=1}^{\infty}\left(1-\beta_j\right) \mathrm{QI}_{0,j}(0) + \mathrm{P}_0(0)(1-\beta_0) \tag{2.1}$$

$$= \sum_{j=1}^{\infty} \mathrm{QI}_{0,j(0)} + \mathrm{P}_0(0)(1-\beta_o) - \sum_{j=1}^{\infty} \beta_j\, \mathrm{QI}_{0,j(0)}(0) \tag{2.2}$$

$$\begin{aligned} \frac{-d}{dx}\mathrm{PI}_n(w) &= -\lambda \mathrm{PI}_n(w) + (1-p)\mathrm{P}_n(0)a(w) \\ &+ \sum_{j=1}^{\infty} \mathrm{QI}_{n,j}(0)a(w) + \mathrm{QB}_n(0)a(w), \qquad n \geq 1 \end{aligned} \tag{2.3}$$

Idle Vacation

$$-\frac{d}{dx}\mathrm{QI}_{0,1}(x) = -\lambda \mathrm{QI}_{0,1}(x) + \beta_0 \mathrm{P}_0(0) v\mathrm{I}(x) \tag{2.4}$$

$$-\frac{d}{dx}\mathrm{QI}_{0,j}(x) = -\lambda \mathrm{QI}_{0,j}(x) + \mathrm{QI}_{0,j-1}(0)\beta_{j-1} v\mathrm{I}(x) \tag{2.5}$$

$$-\frac{d}{dx}\mathrm{QI}_{n,j}(x) = -\lambda \mathrm{QI}_{n,j}(x) + \sum_{k=1}^{n} \mathrm{QI}_{n-k}(x)\, g_k \tag{2.6}$$

Busy Vacation

$$-\frac{d}{dx}\mathrm{QB}_n(x) = -\lambda \mathrm{QB}_n(x) + p\mathrm{P}_n(0) v\mathrm{B}(x) + \lambda \sum_{k=1}^{n-1} \mathrm{Q}_{n-k}\,(x) g_k \qquad n \geq 1 \tag{2.7}$$

$$-\frac{d}{dx}\mathrm{QB}_0(x) = -\lambda \mathrm{QB}_0(x) + p\mathrm{P}_0(0) v\mathrm{B}(x) \tag{2.8}$$

Busy State

$$\begin{aligned} -\frac{d}{dx}\mathrm{P}_0(x) &= -\lambda \mathrm{P}_0(x) + \mathrm{PI}_1(0)s(x) + (1-q)\lambda s(x)\int_0^{\infty} \mathrm{PI}_1(w)\, dw g_1 \\ &+ \lambda \mathrm{PI}_0 g_1 s(x) \end{aligned} \tag{2.9}$$

$$-\frac{d}{dx}P_n(x) \quad = -\lambda P_n(x) + PI_{n+1}(0)s(x) + \lambda PI_0 g_{n+1} s(x) + \lambda\sum_{k=1}^{n} P_{n-k}(x)g_k$$

$$+ q\lambda s(x)\int_0^\infty \sum_{k=1}^{n} PI_{n-k+1}\,(w)dwg_k$$

$$+(1-q)\lambda(w)\int_0^\infty \sum_{k=1}^{n+1} PI_{n-k+2}(w)g_k, \qquad n \geq 1 \qquad (2.10)$$

Thus the L.S.T of the above equations with respect to x lead to

$$\lambda PI_0 = \sum_{j=1}^{\infty}\left(1-\beta_j\right)QI_{0,j}(0) + P_0(0)\left(1-\beta_0\right) \qquad (2.11)$$

$$\theta PI_n^*(\theta) - PI_n(0) \; = \lambda PI_n^*(\theta) - (1-p)P_n(0)A^*(\theta)$$

$$-\sum_{j=1}^{\infty} QI_{n,j}\,(0)A^*(0) - QB_n(0)A^*(\theta), \qquad n \geq 1 \qquad (2.12)$$

$$\theta QI_{0,1}^*(\theta) - QI_{0,1}(0) = \lambda QI_{0,1}^*(\theta) - P_1(0)\beta_0 VI^*(\theta) \qquad (2.13)$$

$$\theta QI_{0,j}^*(\theta) - QI_{0,j}(0) \quad = \lambda QI_{0,j}^*(\theta) - QI_{0,j-1}(0)\beta_{j-1}VI^*(\theta) \qquad (2.14)$$

$$\theta QI_{n,j}^*(\theta) - QI_{n,j}(0) = \lambda QI_{n,j}^*(\theta) - \lambda\sum_{k=1}^{n} QI_{n-k,j}^*(\theta)\,g_k\,, \qquad n \geq 1 \text{ and } j \geq 1 \quad (2.15)$$

$$\theta QB_n^*(\theta) - QB_n(0) \; = \lambda QB_n^*(\theta) - pP_n(0)VB^*(\theta) - \lambda\sum_{k=1}^{n-1} Q_{n-k}^*(\theta)g_k \qquad (2.16)$$

$$\theta P_0^*(\theta) - P_0(0)$$

$$= (-\lambda)P_0^*(\theta) + PI_1(0)S^*(\theta) - \sum_{j=1}^{\infty} QI_{1,j}(0)S^*(\theta) - \lambda PIg_1 S^*(\theta) \; (2.17)$$

$$
\begin{aligned}
\theta P_n^*(\theta) - P_n(0) &= \lambda P_n^*(\theta) - PI_{n+1}(0)S^*(\theta) - \lambda PI_0 g_{n+1} S^*(\theta) - \lambda \sum_{k=1}^{\infty} P_{n-k}^*(\theta) g_k \\
&- q\lambda S^*(\theta) \int_0^{\infty} \sum_{k=1}^{\infty} PI_{n-k+1}(w) dw g_k \\
&- (1-q)\lambda S^*(\theta) \int_0^{\infty} \sum_{k=1}^{\infty} PI_{n-k+2}(w) dw g_k \qquad n \geq 1 \qquad (2.18)
\end{aligned}
$$

The following PGFs are defined to solve the equations.

2.3 The Probability Generating Functions

$$PI^*(z,\theta) = \sum_{n=1}^{\infty} PI_n^*(\theta) z^n \qquad PI(z,0) = \sum_{n=1}^{\infty} PI_n(0) z^n$$

$$QI_j{}^*(z,\theta) = \sum_{n=0}^{\infty} QI_{n,j}^*(\theta) z^n \qquad QI_j(z,0) = \sum_{n=0}^{\infty} QI_{n,j}(0) z^n$$

$$QB^*(z,\theta) = \sum_{n=1}^{\infty} QB_n^*(\theta) z^n \qquad QB(z,0) = \sum_{n=1}^{\infty} QB_n(0) z^n$$

$$P^*(z,\theta) = \sum_{n=0}^{\infty} P_n{}^*(\theta) z^n \qquad P(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

Multiplying the equations (2.13) and (2.15) at j=1 by appropriate powers of z^n and adding over n=0 to ∞

$$\theta QI_1^*(z,\theta) - QI_1(z,0) = \lambda QI_1^*(z,\theta) - \lambda QI_1^*(z,\theta)X(z) - P_0(0)\beta_0 VI^*(\theta) \quad n \geq 1 \quad (2.19)$$

$$(\theta - w_x(z))QI_1^*(z,\theta) \quad = QI_1(z,0) - P_0(0)\beta_0 VI^*(\theta) \qquad (2.20)$$

where

$w_x(z) = \lambda(1 - x(z))$

At $\theta = w_x(z)$,

$$QI_1(z,0) \; = P_0(0)\beta_0 VI^*(w_x(z)) \tag{2.21}$$

substituting the equation (2.21) in the equation (2.20)

$$(\theta - w_x(z))QI_1^*(z,\theta) = P_0(0)\beta_0 VI^*\big(w_x(z)\big) - P_0(0)\beta_0 VI^*(\theta) \tag{2.22}$$

If $\theta = 0$

$$QI_1^*(z,0) \; = P_0(0)\beta_0 \frac{1 - VI^*(w_x(z))}{w_x(z)} \tag{2.23}$$

Multiplying the equations (2.14) and (2.15) by appropriate powers of z^n and adding over n=0 to ∞

$$\begin{aligned}\theta QI_j^*(z,\theta) - QI_j(z,0) = \lambda QI_j^*(z,\theta) - QI_{j-1}(z,0)\beta_{j-1}VI^*(\theta) \\ -\lambda QI_j^*(z,\theta)X(z), \qquad j \geq 2\end{aligned} \tag{2.24}$$

$$(\theta - w_x(z))QI_j^*(z,\theta) \quad = QI_j(z,0) - QI_{j-1}(z,0)\beta_{j-1}VI^*(\theta), \;\; j \geq 2 \tag{2.25}$$

substituting $\theta = w_x(z)$ in the equation (2.25)

$$QI_j(z,0) \; = QI_{j-1}(z,0)\beta_{j-1}VI^*(w_x(z)) \tag{2.26}$$

Using the equation (2.26) in the equation (2.25)

$$(\theta - w_x(z))QI_j^*(z,\theta) = QI_{j-1}(z,0)\beta_{j-1}VI^*\big(w_x(z)\big) - QI_{j-1}(z,0)\beta_{j-1}VI^*(\theta) \tag{2.27}$$

$$QI_j^*(z,0) \; = QI_{j-1}(z,0)\beta_{j-1}\frac{1 - VI^*(w_x(z))}{w_x(z)} \forall \; j \geq 2 \tag{2.28}$$

If αI_n denotes the probability that n customers arrive during a vacation time

Then, $$VI^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha I_n \, z^n$$

where, $$\alpha I_n \; = \int_0^{\infty} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i g^i}{\lfloor i} \, dv(t)$$

Thus the equations (2.21) and (2.26) respectively imply,

$$\sum_{n=0}^{\infty} QI_{n,1}(0)\, z^n = P_0(0)\, \beta_0 \sum_{n=0}^{\infty} \alpha I_n\, z^n \quad \text{and} \tag{2.29}$$

$$\sum_{n=0}^{\infty} QI_{n,j}(0)\, z^n = QI_{0,j-1}(0)\beta_{j-1} \sum_{n=0}^{\infty} \alpha I_n\, z^n \tag{2.30}$$

The coefficients of z^n of the equations (2.29) and (2.30) give,

$$QI_{n,1}(0) = P_0(0)\beta_0 \alpha I_n, \quad n \geq 0 \tag{2.31}$$

$$QI_{n,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_n, \qquad n \geq 0 \text{and } j \geq 2 \tag{2.32}$$

substituting n=0 in the equations (2.31) and (2.32), we have

$$QI_{0,1}(0) = P_0(0)\beta_0 \alpha I_0 \tag{2.33}$$

$$QI_{0,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_0 \tag{2.34}$$

By recursion

$$QI_{0,j}(0) = QI_{0,j-2}(0)\beta_{j-2} \tag{2.35}$$

$$QI_{0,j}(0) = QI_{0,1}(0) \prod_{i=1}^{j-1} \beta_i (\alpha I_0)^j, \quad j \geq 1 \tag{2.36}$$

$$QI_{0,j}(z,0) = P_0(0) \prod_{i=0}^{j} \beta_i (\alpha I_0)^j \tag{2.37}$$

Then the equation (2.21) gives,

$$QI_1(z,0) = P_0(0)\beta_0 VI^*(w_x(z)) \tag{2.38}$$

Using the equtaion (2.38) in the equation (2.26), we get

$$QI_j(z,0) = P_0(0) \prod_{i=0}^{j-1} \beta_i\, (\alpha I_0)^{j-1} VI^*(w_x(z)) \tag{2.39}$$

Then $QI(z,0) = \sum_{j=0}^{\infty} QI_j(z,0)$ implies (2.40)

$$QI(z,0) = P_0(0)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i\,(\alpha I_0)^j VI^*(w_x(z)) \tag{2.41}$$

Similarly,

$$QI_j^*(z,0) = P_0(0)\prod_{i=0}^{j}\beta_i(\alpha I_0)^j\left(\frac{1-VI^*\big(w_x(z)\big)}{w_x(z)}\right) \tag{2.42}$$

Hence $QI^*(z,0) = \sum_{j=0}^{\infty} QI_j^*(z,0)$ implies (2.43)

$$QI^*(z,0) = P_0(0)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i\left(\frac{1-VI^*\big(w_x(z)\big)}{w_x(z)}\right) \tag{2.44}$$

Thus the partial generating function when the server is in idle vacation state is calculated.

Next we shall calculated the generating function corresponding to the vacation during busy period.

Multiplying the equation (2.16) by appropriate powers of z^n and adding over n=1 to ∞

$$\theta QB^*(z,0) = QB(z,0) + \lambda QB^*(z,\theta) - [P(z,0) - P_0(0)] - \lambda X(z)QB^*(z,\theta) \tag{2.45}$$

$$\big(\theta - w_x(z)\big)QB^*(z,0) = QB(z,0) - p[P(z,0) - P_0(0)]VB^*(\theta) \tag{2.46}$$

At $\theta = w_x(z)$

$$QB(z,0) = p[P(z,0) - P_0(0)]VB^*\big(w_x(z)\big) \tag{2.47}$$

substituting the equation (2.47) in the equation (2.46)

$$\big(\theta - w_x(z)\big)QB^*(z,0) = p[P(z,0) - P_0(0)]\Big(VB^*\big(w_x(z)\big) - VB^*(\theta)\Big) \tag{2.48}$$

At $\theta = 0$ the equation (2.48)leads to

$$\mathrm{QB}^*(z,0) = \frac{p\left(1 - \mathrm{VB}^*\left(w_x(z)\right)\right)}{w_x(z)}\ \left(P(z,0) - P_0(0)\right) \qquad (2.49)$$

The probability that the server is idle can be obtained using equation (2.36) in (2.11)

Substituting the equation (2.36) in the equation (2.11) gives

i. e., $$\lambda \mathrm{PI}_0 = \sum_{j=1}^{\infty}\left(1 - \beta_j\right)\prod_{i=0}^{j-1}\beta_i\,(\alpha I_0)^j P_0(0) + P_0(0)(1 - \beta_0) \qquad (2.50)$$

$$\lambda \mathrm{PI}_0 = (\sum_{j=1}^{\infty}\left(1 - \beta_j\right)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1 - \beta_0))P_0(0) \qquad (2.51)$$

$$\lambda \mathrm{PI}_0 = \phi P_0(0) \qquad (2.52)$$

where

$$\phi = \sum_{j=1}^{\infty}\left(1 - \beta_j\right)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1 - \beta_0) \qquad (2.53)$$

Multiplying the equation (2.12) by appropriate powers of z^n and adding n=1 to ∞ and using equation (2.11)

$$(\theta - \lambda)\mathrm{PI}^*(z,\theta) = \mathrm{PI}(z,0) - A^*(\theta)(\left[(1 - p)\left(P(z,0) - P_0(0)\right)\right]$$
$$+ \sum_{j=1}^{\infty} \mathrm{QI}_j\,(z,0) - \mathrm{QI}_{0,j}\,(0) + \mathrm{QB}(z,0)) \qquad (2.54)$$

Adding $\lambda \mathrm{PI}_0 A^*(\theta)$ on both sides,

$$(\theta - \lambda)\mathrm{PI}^*(z,\theta) = \mathrm{PI}(z,0) - A^*(\theta)(\left[(1 - p)\left(P(z,0) - P_0(0)\right)\right]$$
$$+ \sum_{j=1}^{\infty} Q_j\,(z,0) - \mathrm{QI}_{0,j}\,(0) + \mathrm{QB}(z,0)) + \lambda \mathrm{PI}_0 A^*(\theta)$$
$$- A^*(\theta)(\sum_{j=1}^{\infty} \mathrm{QI}_{0,j}(0) + P_0\,(0)(1 - \beta_0) - \sum_{j=1}^{\infty}\beta_j\,\mathrm{QI}_{0,j}(0)) \qquad (2.55)$$

$$\text{i.e.}, (\theta-\lambda)\mathrm{PI}^*(z,\theta) = \mathrm{PI}(z,0) - A^*(\theta)([(1-p)P(z,0)] + P_0(0)(p-\beta_0) - \lambda \mathrm{PI}_0 + \sum_{j=1}^{\infty} Q_j\,(z,0) + QB(z,0) - \sum_{j=1}^{\infty} \beta_j\, QI_{0,j}(0)) \tag{2.56}$$

Hence

$$(\theta-\lambda)\mathrm{PI}^*(z,\theta) = \mathrm{PI}(z,0) - A^*(\theta)Y_1(z) \tag{2.57}$$

where

$$Y_1(z) = ([(1-p)P(z,0)] + P_0(0)(p-\beta_0) - \lambda \mathrm{PI}_0 + \sum_{j=1}^{\infty} Q_j\,(z,0) + QB(z,0) - \sum_{j=1}^{\infty} \beta_j\, QI_{0,j}(0)) \tag{2.57.1}$$

Substituting for $QI_{0,j}(0), Q_j\,(z,0), QB(z,0)$ from equations $\big((2.37),(2.39),(2.47)\big)$

$Y_1(z)$ is simplified as,

$$Y_1(z) = P(z,0)\Big((1-p) + pVB^*\big(w_x(z)\big)\Big) - \lambda \mathrm{PI}_0 - P_0(0)((pVB^*\big(w_x(z)\big) - 1) + \sum_{j=1}^{\infty} {\alpha_0}^j \prod_{i=0}^{j} \beta_i \Big(1 - VI^*\big(w_x(z)\big)\Big)) \tag{2.58}$$

Then the equation (2.57) at $\theta = \lambda$ gives,

$$\mathrm{PI}(z,0) = A^*(\lambda)Y_1(z) \tag{2.59}$$

substituting the equation (2.59) in the equation (2.57)

$$\mathrm{PI}^*(z,0) = \left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_1(z) \tag{2.60}$$

Now to find the PGF of the system size probabilities in the busy state, the equations (2.17) and (2.18) are used. These equations lead to

$$\theta P^{*}(z,\theta) = P(z,0) + \lambda P^{*}(z,\theta) - \frac{S^{*}(\theta)}{z} PI(z,0) - \frac{\lambda PI_{0}}{z} X(z) S^{*}(\theta)$$

$$-(1-q)\frac{\lambda S^{*}(\theta)}{z^{2}} X(z) PI^{*}(z,0) - \frac{q\lambda S^{*}(\theta)}{z} X(z) PI^{*}(z,0)$$

$$+\lambda X(z) P^{*}(z,\theta) + \lambda P^{*}(z,\theta) \qquad (2.61)$$

since,

$$\sum_{n=0}^{\infty} z^{n} \left(\int_{0}^{\infty} \sum_{k=1}^{n+1} PI_{n-k+2}\,(w)\,dw\, g_{k} \right) = PI^{*}(z,0) \frac{X(z)}{z^{2}}$$

$$\sum_{n=0}^{\infty} z^{n} \left(\int_{0}^{\infty} \sum_{k=1}^{n+1} PI_{n-k+1}\,(w)\,dw\, g_{k} \right) = PI^{*}(z,0) \frac{X(z)}{z}$$

$$\big(\theta - w_{x}(z)\big) P^{*}(z,\theta) = P(z,0) - \frac{S^{*}(\theta)}{z} (PI(z,0) + \lambda X(z) PI_{0}$$

$$+ \frac{\lambda X(z)}{z} (1-q+qz) PI^{*}(z,0)) \qquad (2.62)$$

For further simplification, consider

$$PI(z,0) + \lambda X(z) PI_{0} + \frac{\lambda X(z)}{z} (1-q+qz) PI^{*}(z,0)$$

$$= A^{*}(\lambda) Y_{1}(z) + \lambda X(z) PI_{0}$$

$$+ \left(\frac{1 - A^{*}(\lambda)}{\lambda} \right) Y_{1}(z)\ \frac{\lambda X(z)}{z} (1-q+qz) \qquad (2.63)$$

$$= (A(\lambda) + (1 - A^{*}(\lambda)) \frac{X(z)}{z} (1-q+qz)) Y_{1}(z) + \lambda X(z) PI_{0} (64)$$

$$= M_{1}(z) Y_{1}(z) + \lambda X(z) PI_{0} \qquad (2.64)$$

where

$$M_{1}(z) = A^{*}(\lambda) + \big(1 - A^{*}(\lambda)\big) \frac{X(z)}{z} \qquad (2.65)$$

The equation (2.62) then becomes

$$\big(\theta - w_x(z)\big)P^*(z,\theta) = P(z,0) - \frac{S^*(\theta)}{z}\left[M_1(z)Y_1(z) + \lambda X(z)PI_0\right] \tag{2.66}$$

At $\theta = w_x(z)$ equation (2.66) implies

$$P(z,0) = \frac{S^*\big(w_x(z)\big)}{z}\left[M_1(z)Y_1(z) + \lambda X(z)PI_0\right] \tag{2.68}$$

After substituting for $Y_1(Z)$ and on further simplification, we get

$$\begin{aligned} P(z,0) = \frac{S^*\big(w_x(z)\big)}{z - H^*\big(w_x(z)\big)} (\phi\big(X(z) - M_1(z)\big) - M_1(z)\left(\big(pVB^*\big(w_x(z)\big) - 1\big)\right) \\ + \sum_{j=1}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \left(1 - VI^*\big(w_x(z)\big)\right))P_0(0) \end{aligned} \tag{2.69}$$

where

$$H^*\big(w_x(z)\big) = S^*\big(w_x(z)\big)M_1(z)\left((1-p) + pVB^*\big(w_x(z)\big)\right) \tag{2.70}$$

$$\begin{aligned} \left(z - H^*\big(w_x(z)\big)\right)P(z,0) = S^*\big(w_x(z)\big)(\lambda PI_0\big(X(z) - M_1(z)\big) - M_1(z)P_0(0) \\ (\big(pVB^*\big(w_x(z)\big) - 1\big) + \sum_{j=1}^{\infty} \alpha I_0{}^j \prod_{i=0}^{j} \beta_i \left(1 - VI^*\big(w_x(z)\big)\right) \end{aligned} \tag{2.71}$$

Hence

$$P(z,0) = S^*\big(w_x(z)\big)P_0(0)Q(z) \tag{2.72}$$

where

$$Q(z) = \frac{\begin{aligned} \phi\big(X(z) - M_1(z)\big) - M_1(z)[p\big(VB^*\big(w_x(z)\big) - 1\big) \\ + \textstyle\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=1}^{j} \beta_i \left(1 - VI^*\big(w_x(z)\big)\right)] \end{aligned}}{z - H^*\big(w_x(z)\big)} \tag{2.73}$$

$Y_1(z)$ of equation (2.58) can be further simplified as,

$$Y_1(z) = S^*\big(w_x(z)\big)P_0(0)Q(z)(1 - p + pVB^*\big(w_x(z)\big) - \lambda PI_0$$

$$-P_0(0)[p\big(VB^*\big(w_x(z)\big) - 1\big) - \sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \left(1 - VI^*\big(w_x(z)\big)\right)] \qquad (2.74)$$

(or) equivalently $Y_1(z) =$

$P_0(z)\,Y(z)$ (2.75)

where

$$Y(z) = S^*\big(w_x(z)\big)Q(z)(1 - p + pVB^*\big(w_x(z)\big) - \phi + p\left(1 - VB^*\big(w_x(z)\big)\right)$$

$$-\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \left(1 - VI^*\big(w_x(z)\big)\right) \qquad (2.76)$$

Substituting for P(z,0) and $Y_1(z)$ in equation (2.67), it leads to

$$\big(\theta - w_x(z)\big)P^*(z,\theta) = \frac{P_0(0)Q(z)}{z}(zS^*\big(w_x(z)\big) - S^*(\theta)H^*(w_x(z) - zS^*(\theta)$$

$$+S^*(\theta)H^*(w_x(z)) \qquad (2.77)$$

$$\big(\theta - w_x(z)\big)P^*(z,\theta) \ = \frac{zP_0(0)Q(z)}{z}\left(S^*\big(w_x(z)\big) - S^*(\theta)\right) \qquad (2.78)$$

$$\text{i.e.,}\ \big(\theta - w_x(z)\big)P^*(z,\theta) = zP_0(0)Q(z)\left(S^*\big(w_x(z)\big) - S^*(\theta)\right) \qquad (2.79)$$

At $\theta = 0$ the above equation implies,

$$P^*(z,0) = \frac{\left(1 - S^*\big(w_x(z)\big)\right)P_0(0)}{w_x(z)}Q(z) \qquad (2.80)$$

Thus the partial probability generating functions of the queue size probabilities at arbitrary epoch when the server is in different states are obtained in terms of $P_0(0)$ and are listed below:

$$QI^*(z,0) = P_0(0)\left(\frac{1 - VI^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \qquad (2.80.1)$$

$$PI^*(z,0) = \left(\frac{1-A^*(\lambda)}{\lambda}\right) Y(z)P_0(0) \tag{2.80.2}$$

$$QB^*(z,0) = \frac{p\left(1-VB^*\left(w_x(z)\right)\right)}{w_x(z)} \left(S^*(w_x(z))Q(z)-1\right)P_0(0) \tag{2.80.3}$$

$$P^*(z,0) = \frac{\left(1-S^*\left(w_x(z)\right)\right)P_0(0)}{w_x(z)}Q(z) \tag{2.80.4}$$

$P_0(0)$ can be calculated using the normalizing condition.

2.4 Performance Measures

In this section some useful performance measures such as queue size probabilities and mean queue size of the proposed model are presented. The following results are used to derive these measures.

Treating the equation (2.65) and it's derivatives at z = 1 we have,

$$M_1(1) = A^*(\lambda) + 1 - A^*(\lambda) = 1 \tag{2.81}$$

$$\begin{aligned} M_1'(1) &= \left(1-A^*(\lambda)\right)[-1+E(X)+q] \\ &= \left(1-A^*(\lambda)\right)[E(X)+q-1] \end{aligned} \tag{2.81.1}$$

$$M_1''(1) = \left(1-A^*(\lambda)\right)\left[E\left(X(X-1)\right)+2(1-q)(E(X)+1)\right] \tag{2.81.2}$$

Equation (2.73) can be written as $Q(z) = \dfrac{N_r(z)}{D_r(z)}$ (2.81.3)

where

$$\begin{aligned} N_r(z) = {} & \phi\left(X(z)-M_1(z)\right) - M_1(z)[p\left(VB^*\left(w_x(z)\right)-1\right) \\ & + \sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=1}^{j}\beta_i\left(1-VI^*\left(w_x(z)\right)\right) \end{aligned} \tag{2.81.4}$$

$$D_r(z) = z - H^*\left(w_x(z)\right) \tag{2.81.5}$$

Since $N_r(1) = D_r(1) = 0$

$$Q(1) = \frac{N_r{}'(1)}{D_r{}'(1)} \text{ and} \tag{2.81.6}$$

$$Q'(1) = \frac{D_r'(1)N_r''(1) - D_r''(1)N_r'(1)}{2(D_r{}'(1))^2} \tag{2.81.7}$$

It is calculated that,

$$N_r'(1) = \phi\big(E(X) - M_1{}'(1)\big) + \lambda E(X)\left(E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=1}^{j} \beta_i - pVB \right) \tag{2.81.8}$$

$$\begin{aligned} N_r{}''(1) \; &= \phi\big(EX(X-1) - M_1{}''(1)\big) + \sum_{j=0}^{\infty} \alpha_0{}^j \prod_{i=1}^{j} \beta_i \,(2M'(1)\lambda E(X)E(VI) \\ &\quad + \lambda E\big(X(X-1)\big)E(VI) + \big(\lambda E(X)\big)^2 E(VI^2)) \\ &\quad - p\left(2\lambda E(X)M'(1)E(VB) + \lambda E\big(X(X-1)\big)E(VB) + \big(\lambda E(X)\big)^2 E(VB)\right) \end{aligned}$$

$$D_r{}'\,(1) = 1 - \rho - M_1{}'(1)\,, \quad \text{where } \rho = \lambda E(X)(E(S) + pE(VB)) \tag{2.81.9}$$

$$\begin{aligned} D_r{}''(1) \; &= M_1{}''(1) + 2\,\lambda E(X)[E(S) + pE(VB)]M_1{}'(1) + \big(\lambda E(X)\big)^2 (E(S^2) \\ &\quad + 2E(S)pE(VB) + pE(VB^2)) + \lambda E(X(X-1))[E(S) + pE(VB)] \end{aligned}$$

The equation (2.76) and its derivative at z =1 implies,

$$Y(1) \; = Q(1) - \phi \tag{2.81.10}$$

$$Y'(1) \; = Q'(1) + Q(1)\rho + \lambda E(X)[E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=1}^{j} \beta_i - pE(VB)] \tag{2.81.11}$$

2.4.1 The Steady State Queue Size Probabilities:

Let P_{VI} , P_{VB}, PI and P_{busy} denote the probability that the server is on idle vacation, idle busy vacation, idle and busy state respectively. Then the corresponding probabilities are obtained, by considering the equations in (2.80.1 to 2.80.4) at z=1 and the using the results (2.81 to 2.81.11). Thus

$$\text{(i)}\quad P_{VI} = \lim_{z\to 1} QI^*(z,0)$$

$$= P_0 E(VI) \sum_{j=0}^{\infty} P_0(0) \prod_{i=0}^{j} \beta_i$$

$$\text{(ii)}\ P_{busy} = \lim_{z\to 1} P^*(z,0)$$

$$= E(S)Q(1)P_0(0)$$

$$\text{(iii)}\ P_{VB} = \lim_{z\to 1} QB^*(z,0)$$

$$= p\, E(VB)P_0(0)(Q(1)-1)$$

$$\text{(iv)}\quad PI = \lim_{z\to 1} PI^*(z,0)$$

$$= \left(\frac{1-A^*(\lambda)}{\lambda}\right) Y(1)P_0(0)$$

$$\text{(v)}\ \lambda PI_0 = \phi P_0(0)$$

$$PI_0 = \frac{\phi P_0(0)}{\lambda}$$

Thus the system size probability are expressed terms of $P_0(0)$. $P_0(0)$ can be evaluated using the normal condition,

$$P_I + P_{VI} + P_{VB} + P_{busy} + PI_0 = 1$$

$$\text{i.e.,}\, 1 = \left(\frac{1-A^*(\lambda)}{\lambda} Y(1)P_0(0)\right) + P_0(0)E(VI) \sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \beta_i$$

$$+ (p\, E(VB)P_0(0)(Q(1)-1)) + (E(S)Q(1)P_0(0)) + \left(\frac{\phi P_0(0)}{\lambda}\right)$$

$$1 = P_0(0)(\frac{1-A^*(\lambda)}{\lambda} Y(1) + E(VI) \sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \beta_i + p\, E(VB)(Q(1)-1)$$

$$+ E(S)Q(1) + \frac{\phi}{\lambda})$$

$$\left(P_0(0)\right)^{-1} = \frac{1-A^*(\lambda)}{\lambda}Y(1) + E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i + p\,E(VB)(Q_1(1)-1)$$

$$+ E(S)Q_1(1) + \frac{\phi}{\lambda} \qquad (2.82)$$

2.4.2 Mean System Size:

In this section the average number of customers waiting in the queue, when the server is in different states are calculated.

Let L_I, L_{VI}, L_{busy} and L_{VB} denote the expected system size when the server is in idle state, idle vacation state, busy state and busy vacation state respectively. Then the derivatives of equations in (2.80.1 to 2.80.4) at z=1 give the required measures. The results given in equations (2.81 to 2.81.11) are used for further simplification.

Thus the mean system size corresponding to different states are given by,

(i) $$L_{VI} = P_0(0)\lambda E(X)\frac{E(VI^2)}{2}\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i$$

(ii) $$L_{busy} = \frac{d}{dz}P^*(z,0)\bigg|_{z=1}$$

$$= P_0(0)\left(E(S)Q'(1) + \lambda E(X)\frac{E(S^2)}{2}Q(1)\right)$$

(iii) $$L_{VB} = \frac{d}{dz}QB^*(z,0)\bigg|_{z=1}$$

$$= P_0(0)\left[\frac{\lambda E(X)E(VB^2)}{2}(Q(1)-1) + E(VB)(\lambda E(X)E(S)Q(1) + Q'(1))\right]$$

(iv) $$L_I = \frac{d}{dz}PI^*(z,0)\bigg|_{z=1}$$

$$= \frac{1-A^*(\lambda)}{\lambda}Y'(1)P_0(0)$$

The total expected system size for the model is given by,

$$L = L_{VI} + L_I + L_{VB} + L_{busy}$$

$$L = P_0(0)\Bigg[\lambda E(X)\frac{E(VI^2)}{2}\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i + \frac{1-A^*(\lambda)}{\lambda}Y'(1) + \frac{\lambda E(X)E(VB^2)}{2}(Q(1)-1) + E(VB)\big(\lambda E(X)E(S)Q(1) + Q'(1)\big) + \left(E(S)Q'(1) + \lambda E(X)\frac{E(S^2)}{2}Q(1)\right)\Bigg] \tag{2.83}$$

Thus the system performance measures for the reliable Retrial queuing system under multiple adapted vacation policy are derived.

CHAPTER- III

A REPAIRABLE RETRIAL $M^X/G/1$ QUEUEING SYSTEM UNDER MULTIPLE ADAPTED VACATION POLICY

In chapter II reliable retrial batch arrival queueing system in which the server follows multiple adapted vacation policy during the idle time is analysed. But it is of basic importance to study the retrial queue with server breakdowns and repairs because of limited ability of repairs and heavy influence of breakdowns on the performance measures of the system. In the present chapter a repairable, batch arrival single server retrial queueing model is analyzed at steady-state. By using SVT, a detailed analysis for system performance measures including the queue size probabilities and average queue lengths are obtained.

3.1 Mathematical Analysis of the System

3.1.1 Model Description

Arrival Pattern

Customers arrive in batches at the system according to a time homogeneous Poisson process with group arrival rates $\boldsymbol{\lambda}$. The batch size X is a random variable with probability distribution $\Pr(X = k) = g_k, k = 1,2, \ldots$ and $\sum_{k=1}^{\infty} g_k = 1$. There is no waiting space in front of the server, therefore, if an arriving batch finds the server idle, then one of the arriving customers begins to receive his service immediately and the others leave the service area and enter into "orbit" according to **FCFS** queue discipline. The customer at the head of the retrial queue competes with potential primary customers, to decide which customer will enter the next service. If a batch of primary customers arrives first, the retrial customers may cancel its attempt for service and either returns to its position in the retrial queue with probability $\mathbf{q}$ or quits the system with probability **(1-q)**.

The retrial time of the customer in the retrial queue is generally distributed with distribution function **A(t)**, density function **a(t)** and LST **A*(θ)**. Further it is assumed that, the retrial times begin only when the server is freely available in the system (i.e., either at the completion instants of services or vacation completion instants). When the server is idle, arriving (new arrival) customers must turn on the server immediately. If the server is found busy or on vacation or in breakdown state, the arriving batch joins the retrial queue according to **FCFS** discipline.

Multiple Adapted Vacation Policy (MAV)

A cycle starts whenever the system becomes empty and the server is deactivated. The deactivated server either leaves the system for a vacation (first vacation) of random length **(VI)** with probability $\boldsymbol{\beta_0}$ or remains idle in the system with probability $\mathbf{(1-\beta_0)}$. Upon returning from the vacation, if the server finds at least one customer waiting in the orbit, then he immediately joins the system and waits for the customers retrial. Otherwise, if there are no customers found waiting in the queue then the server either joins the system with the probability $\mathbf{(1-\beta_1)}$ or takes a new vacation (second vacation) with probability $\boldsymbol{\beta_1}$. This pattern continues until at least one customer is found in the system. The vacation policy is determined by the sequence of probabilities $\{\boldsymbol{\beta_i}\}$, i=0,1,2,3… The vacations have independent duration with common distribution function **VI(t)** and density function **vI(t)** of finite moments.

Busy Period and Breakdown Period

During busy period, the server provides service. The service times follow general (arbitrary) distributions with distribution functions **S(t)** and the density functions **s(t)** of finite moments $\mathbf{E(S^k)}$, k=1,2.

The server may breakdown at any time, while working and the service channels will not function for a short interval of time. The breakdowns occur according to the Poisson process with $\boldsymbol{\alpha}$.

As soon as the server fails, the server is sent for repair immediately and the service is stopped for the customer until the channel is repaired. The customer whose service is interrupted waits in the server facility for the server to return from the repair facility to complete the remaining service. The repair times (denoted by R) of the server is assumed to be arbitrarily distributed with distribution function **R(t)**, density function **r(t)** for $\mathbf{t \geq 0}$.

The customers continue to arrive according to the Compound Poisson process, independent of the state of the system. The service completion period of a customer consists of the service time of the server and the repair time. Thus a cycle is made up of idle vacation period, completion period.

We denote the model by **M^X/G/1/MAV/breakdown**, where **MAV** denotes the Multiple Adapted Vacation policy. Various stochastic processes involved in the queueing system are assumed to be independent to each other. Using supplementary variable technique the steady state system equations under the steady state condition are analyzed and the **PGF** of the system size is obtained so that various performance measures of the model can be derived from it.

Notations:

λ : Group arrival rate

X : Group size random variable

g_k : Pr(X = k), k = 1, 2, 3, …

X(z) : Probability generating function of X.

N(t) : The system size at time **t**

The notations of Random Variables **(RV)**, Cumulative Distribution Functions **(CDF)**, Probability Density Function **(PDF)**, Laplace-Stieltjes Transform **(LST)** and the **k^{th}** moments of the random variables are listed below:

	RV	**CDF**	**PDF**	**LST**	**K^{th} moments**
Retrial time	A	A(w)	a(w)	$A^*(\theta)$	$E(A^k)$
Idle Vacation Time	VI	VI(x)	vI(x)	$VI^*(\theta)$	$E(VI^k)$
Repair time	R	R(x)	r(x)	$R^*(\theta)$	$E(R^k)$
Service time	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$

If f(x) is the density function of the probability distribution F(x) then

$$F^{*}(\theta) = \int_{0}^{\infty} e^{-\theta x} f(x)dx = \int_{0}^{\infty} e^{-\theta x} d(F(x))$$

Let $VI^0(t)$, $A^0(t)$, $S^0(t)$ and $R^0(t)$ denote respectively the remaining times of the random variables: namely vacation time, retrial time, the service time and repair time at time t. Further the different states of the server at time t are denoted by Y(t) = {0,1,2,3} according as the server is in idle state when there is no customers, idle state when there is n customers, vacation state, busy state and repair state respectively. The supplementary variables are introduced in order to obtain a Markov Process $\{N(t),\delta(t)\}$ where N(t) denotes the system size random variable and $\delta(t) = (A^0(t), VI^0(t), S^0(t), R^0(t))$ according as Y(t) = (0,1,2,3) respectively.

Let

$PI_0(t) = \Pr\{N(t) = 0, Y(t) = 0\}$ $\quad n = 0$

$PI_n(t) = \Pr\{N(t) = n, Y(t) = 0\}$ $\quad n \geq 0$

$QI_{n,j}(x,t)dt = \Pr\{N(t) = n, x \leq VI^{\circ}(t) \leq x + dt, Y(t) = 1\}$ $\quad n \geq 0$

$P_n(x,t)dt = \Pr\{N(t) = n, w \leq S^{\circ}(t) \leq x + dt, Y(t) = 2\}$ $\quad n \geq 1$

$B_n(x,y,t)dt = \Pr\{N(t) = n, x < R^{\circ}(t) \leq x + dt, Y(t) = 3\}$ $\quad n \geq 1$

$PI_n(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the retrial orbit, and the server is idle and the remaining retrial time of the server is between **w** and **w + dt**, where **$n \geq 1$** and **PI_0** is the probability that the server is idle at time **t**, and there is no customer in the retrial orbit.

$QI_{n,j}(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the system, the srever is in J^{th} vacation and the remaining vacation time of the server is between **x** and **x + dt**, where **$n \geq 0$**.

$P_n(x,t)dt$ is the joint probability that at time **t**, there are **n** customers in the retrial orbit, the server is busy and a customer is being served in the service channel with remaining service time lies between **x** and **x + dt**, where **$n \geq 1$**.

$B_n(x,y,t)dt$ denotes the probability that there are **n** customers in the system at time **t**, the server is under repair and the remaining service time for the customer is equal to **x**, and the server is being repaired with the remaining repair time between **y** and **y +dt.**

PI(t) denotes that the server is idle in the empty system at time **t**.

Assuming that at steady-state, the probabilities are independent of time **t** we have

$$\lim_{t\to\infty}\frac{\partial}{\partial x}P_n(x,t)=\frac{d}{dx}P_n(x)\ ;\qquad \lim_{t\to\infty}\frac{\partial}{\partial x}QI_{n,j}(x,t)=\frac{d}{dx}QI_{n,j}(x)$$

$$\lim_{t\to\infty}\frac{\partial}{\partial x}B_n(x,y,t)=\frac{d}{dx}B_n(x,y)$$

$$\lim_{t\to\infty}\left(\frac{\partial}{\partial x}P_n(x,t)=\frac{\partial}{\partial x}QI_{n,j}(x,t)=\frac{\partial}{\partial x}B_n(x,y,t)\right)=0$$

At steady state $\mathbf{P_n(0)}, \mathbf{QI_{n,j}(0)}, \mathbf{B_n(0)}$ denote the probability that there are **n** customer in the system at the termination of service period, vacation (during idle) period and repair period respectively.

3.2 The Steady State System Size Equations

Idle state

$$\lambda PI_0=\sum_{j=1}^{\infty}(1-\beta_j)\,QI_{0,j}(0)+P_0(0)(1-\beta_0) \tag{3.1}$$

$$-\frac{d}{dw}PI_n(w)=-\lambda\,PI_n(w)+P_n(0)a(w)+\sum_{j=1}^{\infty}QI_{n,j}(0)\,a(w),\quad n\geq 1 \tag{3.2}$$

Vacation during idle state

$$-\frac{d}{dx}QI_{0,1}(x)\ =-\lambda QI_{0,1}(x)+P_0(0)\beta_0 vI(x) \tag{3.3}$$

$$-\frac{d}{dx}QI_{0,j}(x)\ =-\lambda QI_{0,j}(x)+QI_{0,j-1}(0)\beta_{j-1}vI(x),\quad j\geq 2 \tag{3.4}$$

$$-\frac{d}{dx}QI_{n,j}(x) = -\lambda QI_{n,j}(x) + \sum_{k=1}^{n} QI_{n-k,j}(x)\, g_k, \quad n \geq 1, j \geq 1 \tag{3.5}$$

Busy state

$$\begin{aligned}-\frac{d}{dx}P_0(x) &= -(\lambda+\alpha)P_0(x) + B_0(x,0) + PI_0(0)s(x)\\ &+ \lambda \int_0^\infty P_1(w)d(w)g_1(1-q)s(x) + \lambda PI_0(x)\, g_1\, s(x)\end{aligned} \tag{3.6}$$

$$\begin{aligned}-\frac{d}{dx}P_n(x) &= -(\lambda+\alpha)P_n(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x)g_k + PI_{n+1}(0)s(x) + B_n(x,0)\\ &+ \lambda PI_0(x)g_{n+1}s(x) + q\lambda s(x)\left(\int_0^\infty \sum_{k=1}^{n} PI_{n-k+1}(w)dwg_k\right)\\ &+ (1-q)\lambda s(x)\left(\int_0^\infty \sum_{k=1}^{n+1} PI_{n-k+2}(w)dwg_k\right), \quad n \geq 1\end{aligned} \tag{3.7}$$

Breakdown state

$$-\frac{\partial}{\partial y}B_0(x,y) = -\lambda B_0(x,y) + \alpha P_0(x)r(y) \tag{3.8}$$

$$-\frac{\partial}{\partial y}B_n(x,y) = -\lambda B_n(x,y) + \lambda \sum_{k=1}^{n} B_{n-k}(x,y)g_k + \alpha P_n(x)r(y), \quad n \geq 1 \tag{3.9}$$

The L.S.T of the equations with respect to x and y are given by

$$\lambda PI_0 = \sum_{j=1}^{\infty}(1-\beta_j)Q_{0,j}(0) + P_0(0)(1-\beta_0) \tag{3.10}$$

$$\theta PI_n^*(\theta) - PI_n(0) = \lambda PI_n^*(\theta) - P_n(0)A^*(\theta) - \sum_{j=1}^{\infty} QI_{n,j}(0)A^*(\theta), \quad n \geq 1 \tag{3.11}$$

$$\theta QI_{0,1}^*(\theta) - QI_{0,1}(0) = \lambda QI_{0,1}^*(\theta) - P_0(0)\beta_0 VI^*(\theta) \tag{3.12}$$

$$\theta QI_{0,j}^*(\theta) - QI_{0,j}(0) = \lambda QI_{0,j}^*(\theta) - QI_{0,j-1}(0)\beta_{j-1}VI^*(\theta), \quad j \geq 1 \tag{3.13}$$

$$\theta QI_{n,j}^{*}(\theta) - QI_{n,j}(0) \;= \lambda QI_{n,j}^{*}(\theta) - \lambda \sum_{k=1}^{n} QI_{n-k}^{*}(\theta)\, g_k, \quad j \geq 1, n \geq 1 \tag{3.14}$$

$$\begin{aligned} \theta P_0^{*}(\theta) - P_0(0) \;&= (\lambda + \alpha)P_0^{*}(\theta) - B_0^{*}(\theta, 0) - PI_1(0)S^{*}(\theta) - \lambda PI_0 g_1\, S^{*}(\theta) \\ &\quad -(1-q)\lambda\, S^{*}(\theta)\left(\int_0^{\infty} PI_1(w)dwg_1\right) \end{aligned} \tag{3.15}$$

$$\begin{aligned} \theta P_n^{*}(\theta) - P_n(0) \;&= (\lambda + \alpha)P_n^{*}(\theta) - PI_{n+1}(0)S^{*}(\theta) - B_n^{*}(\theta, 0) - \lambda PI_0 g_{n+1}\, S^{*}(\theta) \\ &\quad - \lambda \sum_{k=1}^{n} P_{n-k}^{*}(\theta)\, g_k \; - q\lambda S^{*}(\theta)\left(\int_0^{\infty} \sum_{k=1}^{n} PI_{n-k+1}(w)dwg_k\right) \\ &\quad -(1-q)\lambda S^{*}(\theta)\left(\int_0^{\infty} \sum_{k=1}^{n+1} PI_{n-k+2}(w)dwg_k\right) \quad n \geq 1 \end{aligned} \tag{3.16}$$

$$-\frac{\partial}{\partial y} B_0^{*}(\theta, y) \;= -\lambda B_0(\theta, y) + \alpha P_0^{*}(\theta)r(y) \tag{3.17}$$

$$-\frac{\partial}{\partial y} B_n^{*}(\theta, y) \;= -\lambda B_n^{*}(\theta, y) + \alpha P_n^{*}(\theta)r(y) + \lambda \sum_{k=1}^{n} B_{n-k}^{*}(\theta, y)\, g_k \tag{3.18}$$

Taking LST with respect to y for breakdown equations

$$\theta_1 B_0^{**1}(\theta, \theta_1) - B_0^{*}(\theta, 0) \;= \lambda B_0^{**1}(\theta, \theta_1) - \alpha P_n^{*}(\theta)R^{*1}\,(\theta_1) \tag{3.19}$$

$$\begin{aligned} \theta_1 B_n^{**1}(\theta, \theta_1) - B_n^{*}(\theta, 0) \;&= \lambda B_n^{**1}(\theta, \theta_1) - \alpha P_n^{*}(\theta)R^{*1}(\theta_1) \\ &\quad -\lambda \sum_{k=1}^{n} B_{n-k}^{**1}(\theta, \theta_1)\, g_k \quad n \geq 0 \end{aligned} \tag{3.20}$$

The following PGFs are defined to solve the equations.

3.3 Probability Generating Functions

Now to obtain the partial PGFs of the number of customers in the queue, the following partial PGFs are defined

$$PI^*(z,\theta)=\sum_{n=1}^{\infty} PI_n^*(\theta)z^n \qquad PI(z,0)=\sum_{n=1}^{\infty} PI_n(0)z^n$$

$$QI_j{}^*(z,\theta)=\sum_{n=0}^{\infty} QI_{n,j}^*(\theta)z^n \qquad QI(z,0)=\sum_{n=0}^{\infty} QI_{n,j}(0)z^n$$

$$P^*(z,\theta)=\sum_{n=0}^{\infty} P_n^*(\theta)z^n \qquad P(z,0)=\sum_{n=0}^{\infty} P_n(0)z^n$$

$$B^{**1}(z,\theta,\theta_1)=\sum_{n=0}^{\infty} B_n^{**1}(\theta,\theta_1)z^n \qquad B^*(z,\theta,0)=\sum_{n=0}^{\infty} B_n^*(\theta,0)z^n$$

Multiplying the equations (3.19) and (3.20) by z^n adding over n=0 to ∞ calculate the partial probability generating function of breakdown equation

$$\theta B^{**1}(z,\theta,\theta_1)-B^*(z,\theta,0) = \lambda B^{**1}(z,\theta,\theta_1)-\alpha P^*(z,\theta)R^{*1}(\theta_1)-\lambda X(z)B^{**1}(z,\theta,\theta_1) \qquad (3.21)$$

$$(\theta_1-w_x(z))B^{**1}(z,\theta,\theta_1)=B^*(z,\theta,0)-\alpha P^*(z,\theta)R^{*1}(\theta_1) \qquad (3.22)$$

substituting $\theta_1=w_x(z)$ in the equation (3.22) $B^*(z,\theta,0)=\alpha P^*(z,\theta)R^{*1}\big(w_x(z)\big)$ (3.23)

substituting the equation (3.23) in the equation (3.22), we get

$$(\theta_1-w_x(z))B^{**1}(z,\theta,\theta_1)=\alpha P^*(z,\theta)\big[R^{*1}\big(w_x(z)\big)-R^{*1}(\theta_1)\big] \qquad (3.24)$$

$$B^{**1}(z,\theta,\theta_1)=\frac{\alpha P^*(z,\theta)\big[R^{*1}\big(w_x(z)\big)-R^{*1}(\theta_1)\big]}{\theta_1-w_x(z)} \qquad (3.25)$$

At $\theta_1=0$,

$$B^{**1}(z,\theta,0)=\frac{\alpha P^*(z,\theta)\big[1-R^{*1}\big(w_x(z)\big)\big]}{w_x(z)} \qquad (3.26)$$

Next to obtain the partial probability generating function of idle vacation period.

Multiplying the equations (3.12) and (3.14) by appropriate powers of z^n and adding over n=0 to ∞

$$\theta QI_1^*(z,\theta) - QI_1(z,0) = \lambda QI_1^*(z,\theta) - \lambda QI_1^*(z,\theta)X(z) - P_0(0)\beta_0 VI^*(\theta) \qquad (3.27)$$

$$[\theta - w_x(z)]QI_1^*(z,\theta) = QI_1(z,0) \ - P_0(0)\beta_0 VI^*(\theta) \qquad (3.28)$$

where $w_x(z) = \lambda - \lambda X(z)$

At $\theta = w_x(z)$,

$$QI_1(z,0) = P_0(0)\beta_0 VI^*(w_x(z)) \qquad (3.29)$$

substituting the equation (3.29) in (3.28)

$$[\theta - w_x(z)]QI_1^*(z,\theta) = P_0(0)\beta_0\left[VI^*\left(w_x(z)\right) - VI^*(\theta)\right] \qquad (3.30)$$

At $\theta = 0$,

$$QI_1^*(z,0) = P_0(0)\beta_0 \frac{\left(1 - VI^*(w_x(z)\right)}{w_x(z)} \qquad (3.31)$$

Similarly,

Multiplying the equations (3.13) and (3.14) appropriate powers of z^n and adding over n=0 to ∞

$$\theta QI_j^*(z,\theta) - QI_j(z,0) = \lambda QI_j^*(z,\theta) - QI_{j-1}(z,0)\beta_{j-1}VI^*(\theta) - \lambda QI_j^*(z,\theta)X(z) \quad (3.32)$$

$$[\theta - w_x(z)]QI_j^*(z,\theta) = QI_j(z,0) - QI_{j-1}(z,0)\beta_{j-1}VI^*(\theta) \qquad (3.33)$$

At $\theta = w_x(z)$

$$QI_j(z,0) = QI_{j-1}(z,0)\beta_{j-1}VI^*(w_x(z)) \qquad (3.34)$$

substituting the equation (3.34) in (3.33)

$$[\theta - w_x(z)]QI_j^*(z,\theta) = QI_{j-1}(z,0)\beta_{j-1}\left(VI^*\left(w_x(z)\right) - VI^*(\theta)\right) \qquad (3.35)$$

substituting $\theta = 0$ in the equation (3.35)

$$QI_j^*(z,0) = QI_{j-1}(0)\beta_{j-1}\frac{1 - VI^*(w_x(z))}{w_x(z)} \quad \forall\, j \geq 2 \qquad (3.36)$$

If αI_n denotes the probability that n-customers arrive during a vacation time

Then, $$VI^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha I_n \, z^n$$

where, $$\alpha I_n = \int_0^{\infty} e^{-\lambda t} \sum_{i=0}^{n} \frac{(\lambda t)^i g^i}{\lfloor i} \, dv(t)$$

Thus the equations (3.29) and (3.34)

$$\sum_{n=0}^{\infty} QI_{n,1}(0)\, z^n = P_0(0)\, \beta_0 \sum_{n=0}^{\infty} \alpha I_n \, z^n \tag{3.37}$$

$$\sum_{n=0}^{\infty} QI_{n,j}(0)\, z^n = QI_{0,j-1}(0)\beta_{j-1} \sum_{n=0}^{\infty} \alpha I_n \, z^n \tag{3.38}$$

Thus the coefficient z^n of equations (3.37) and (3.38) gives,

$$QI_{n,1}(0) = P_0(0)\beta_0 \alpha I_n \quad , n \geq 0 \tag{3.39}$$

$$QI_{n,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_n \quad n \geq 0 \text{ and } j \geq 2 \tag{3.40}$$

substituting n=0 in the equation (3.39) and (3.40)

$$QI_{0,1}(0) = P_0(0)\beta_0 \alpha I_0 \tag{3.41}$$

$$QI_{0,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_0 \tag{3.42}$$

By recursion

$$QI_{0,j}(0) = QI_{0,j-2}(0)\beta_{j-2}\beta_{j-1}\,(\alpha I_0)^2 \tag{3.43}$$

$$QI_{0,j}(0) = P_0(0) \prod_{i=0}^{j-1} \beta_i \alpha I_0^j \;\; \forall\, j \geq 1 \tag{3.44}$$

$$QI_j(z,0) = P_0(0) \prod_{i=0}^{j} \beta_i (\alpha I_0)^j \tag{3.45}$$

$$QI_{j-1}(z,0) = P_0(0)\prod_{i=0}^{j-1}\beta_i\,(\alpha I_0\,)^{j-1} \tag{3.46}$$

Substituting the equation (3.46) in equation (3.34), we get

$$QI_j(z,0) = P_0(0)\prod_{i=0}^{j-1}\beta_i\,(\alpha I_0\,)^{j-1}\beta_{j-1}VI^*(w_x(z)) \tag{3.47}$$

$$\text{Then}, QI(z,0) \;\; = \sum_{j=0}^{\infty} QI_j\,(z,0) \;\text{ implies} \tag{3.48}$$

$$QI(z,0) \;\; = P_0(0)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i\,(\alpha I_0\,)^{j}VI^*(w_x(z)) \tag{3.49}$$

Similarly, equation (3.36) implies

$$QI_j^*(z,0) \;\; = P_0(0)\prod_{i=0}^{j}\beta_i(\alpha I_0\,)^{j}\left(\frac{1 - VI^*\big(w_x(z)\big)}{w_x(z)}\right) \tag{3.50}$$

$$QI^*(z,0) = \sum_{j=0}^{\infty} QI_j^*(z,0)\text{ implies} \tag{3.51}$$

$$QI^*(z,0) = P_0(0)\sum_{j=0}^{\infty}(\alpha I_0\,)^{j}\prod_{i=0}^{j}\beta_i\left(\frac{1 - VI^*\big(w_x(z)\big)}{w_x(z)}\right) \tag{3.52}$$

By substituting the equation (3.44) in the equation (3.1), we get

$$\text{i.e.,} \qquad \lambda PI_0 \; = \sum_{j=1}^{\infty}\big(1-\beta_j\big)\prod_{i=0}^{j-1}\beta_i\,(\alpha I_0)^{j}P_0(0) + P_0(0)(1-\beta_0)$$

$$\lambda PI_0 \;\; = (\sum_{j=1}^{\infty}\big(1-\beta_j\big)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^{j} + (1-\beta_0))P_0(0)$$

$$\lambda PI_0 \; = \phi P_0(0) \tag{3.53}$$

where

$$\phi = \sum_{j=1}^{\infty}\left(1-\beta_j\right)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1-\beta_0) \qquad (3.53.1)$$

Then to calculate partial generating function of idle period.

Multiplying the equations (3.10) and (3.11) by appropriate powers of z^n and adding over n=1 to ∞

$$(\theta-\lambda)PI^*(z,\theta)$$

$$= PI(z,0) - A^*(\theta)(P(z,0) - P_0(0) + QI(z,0) - \sum_{j=1}^{\infty} QI_{0,j}(0)) \quad (3.54)$$

$$(\theta-\lambda)PI^*(z,\theta) = PI(z,0) - A^*(\theta)\big(P(z,0) - P_0(0)\big)$$

$$+A^*(\theta)(\,QI(z,0) - \sum_{j=1}^{\infty} QI_{0,j}(0)) \qquad (3.55)$$

we have

$$\lambda PI_0\,A^*(\theta) = A^*(\theta)(\sum_{j=1}^{\infty} Q_{0,j}(0) - \sum_{j=1}^{\infty}\beta_j\,Q_{0,j}(0) + P_0(0) - \beta_0 P_0(0)) \qquad (3.56)$$

By adding the equations (3.55) and (3.56), we get

$$(\theta-\lambda)PI^*(z,\theta) = PI(z,0) - A^*(\theta)(P(z,0) + QI(z,0)$$

$$-\left(\sum_{j=1}^{\infty}\beta_j\prod_{i=0}^{j-1}\beta_i\,{\alpha_0}^j + \beta_0\right)P_0(0)) - \lambda PI_0\,A^*(\theta) \qquad (3.57)$$

At $\theta = \lambda$

$$PI(z,0) = A^*(\lambda)\left(P(z,0) + P_0(0)\left(VI^*\left(w_x(z)\right) - 1\right)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i\alpha I_0{}^j - \lambda PI_0\right) \qquad (3.58)$$

$$PI(z,0) = A^*(\lambda)[Y_1(z)] \qquad (3.59)$$

where

$$Y_1(z) = P(z,0) + P_0(0)\left(VI^*\left(w_x(z)\right) - 1\right)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i\alpha I_0{}^j - \lambda PI_0 \qquad (3.60)$$

At $\theta = 0$ the equation (3.57) becomes

$$PI^*(z,0) = \left(\frac{1 - A^*(\lambda)}{\lambda}\right)Y_1(z) \qquad (3.61)$$

To obtain the partial probability generating functions of queue size when the server is in busy state, equation (3.15) and (3.16) are used.

Multiplying the equations (3.15) and (3.16) by appropriate powers of z^n and adding over n=0 to ∞

$$\theta P^*(z,0) - P(z,0) = (\lambda + \alpha)P^*(z,\theta) - B^*(z,\theta,0) - \lambda X(z)P^*(z,\theta)$$

$$-\frac{\lambda}{z}PI_0X(z)S^*(\theta) - \frac{S^*(\theta)}{z}PI(z,0) - (1-q)\lambda S^*(\theta)PI^*(z,0)\frac{X(z)}{z^2}$$

$$-\lambda S^*(\theta)PI^*(z,0)\frac{X(z)}{z}q \qquad (3.62)$$

since,

$$\sum_{n=0}^{\infty} z^n\left(\int_0^{\infty}\sum_{k=1}^{n+1}PI_{n-k+2}\,(w)\,dw\,g_k\right) = PI^*(z,0)\frac{X(z)}{z^2}$$

$$\sum_{n=0}^{\infty} z^n\left(\int_0^{\infty}\sum_{k=1}^{n+1}PI_{n-k+1}\,(w)\,dw\,g_k\right) = PI^*(z,0)\frac{X(z)}{z}$$

$$(\theta - g_\alpha(w_x(z)))P^*(z,\theta) = P(z,0) - B^*(z,\theta,0) - \frac{S^*(\theta)}{z}(PI(z,0) + \lambda X(z)PI_0$$

$$+\frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)) \quad (3.63)$$

where $g_\alpha(w_x(z)) = \alpha + w_x(z)$

$$(\theta - g_\alpha(w_x(z)))P^*(z,\theta) = P(z,0) - P^*(z,\theta)\alpha R^{*1}\big(w_x(z)\big) - \frac{S^*(\theta)}{z}(PI(z,0)$$

$$+\lambda X(z)PI_0 + \frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)) \quad (3.64)$$

$$\Big(\theta - h_\alpha\big(w_x(z)\big)\Big)P^*(z,\theta) = P(z,0) - \frac{S^*(\theta)}{z}(PI(z,0) + \lambda X(z)PI_0$$

$$+\frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)) \quad (3.65)$$

where $h_\alpha\big(w_x(z)\big) = g_\alpha(w_x(z)) - \alpha R^{*1}\big(w_x(z)\big)$

At $\theta = h_\alpha(w_x(z))$ in (3.65)

$$P(z,0) = \frac{S^*\big(h_\alpha w_x(z)\big)}{z}(PI(z,0) + \lambda X(z)PI_0 + \frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)) \quad (3.66)$$

Substituting the equation (3.66) in (3.65)

$$P^*(z,\theta) = \frac{S^*\Big(h_\alpha\big(w_x(z)\big)\Big) - S^*(\theta)}{z\Big(\theta - h_\alpha\big(w_x(z)\big)\Big)}\Bigg(PI(z,0) + \lambda X(z)PI_0$$

$$+\frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)\Bigg) \quad (3.67)$$

At $\theta = 0$

$$P^*(z,0) = \frac{1 - S^*\Big(h_\alpha\big(w_x(z)\big)\Big)}{zh_\alpha(w_x(z))}(PI(z,0) + \lambda X(z)PI_0$$

$$+\frac{\lambda X(z)}{z}(1-q+qz)PI^*(z,0)) \quad (3.68)$$

$$P^*(z,0) = \frac{1 - S^*(h_\alpha(w_x(z)))}{zh_\alpha\big(w_x(z)\big)}(A^*(\lambda)Y_1(z) + \lambda X(z)PI_0$$

$$+\frac{\lambda X(z)}{z}(1-q+qz)\left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_1(z)) \qquad (3.69)$$

$$P^*(z,0) = \frac{1-S^*\left(h_\alpha\left(w_x(z)\right)\right)}{zh_\alpha(w_x(z))}(\lambda X(z)PI_0 + (A^*(\lambda)$$

$$+\frac{(1-q+qz)}{z}\left(1-A^*(\lambda)\right)X(z))Y_1(z)) \qquad (3.70)$$

$$P^*(z,0) = \frac{1-S^*(h_\alpha(w_x(z)))}{zh_\alpha(w_x(z))}[\lambda X(z)PI_0 + M_1(z)Y_1(z)] \qquad (3.71)$$

where

$$M_1(z) = A^*(\lambda) + \frac{X(z)}{z}(1-q+qz)\left(1-A^*(\lambda)\right) \qquad (3.72)$$

Using the equation (3.72) in (3.66)

$$P(z,0) = \frac{S^*(h_\alpha(w_x(z)))}{z}[\lambda X(z)PI_0 + M_1(z)Y_1(z)] \qquad (3.73)$$

The equation (3.65) can be written as

$$(\theta - h_\alpha(w_x(z)))P^*(z,0) = P(z,0) - \frac{S^*(\theta)}{z}[M_1(z)Y_1(z) + \lambda X(z)PI_0] \qquad (3.74)$$

Substituting for $Y_1(z)$ from equation (3.60), the right hand side is

$$= P(z,0) - \frac{S^*(\theta)}{z}(M_1(z)(P(z,0) + P_0(0)\left(VI^*\left(w_x(z)\right) - 1\right)$$

$$\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\beta_i - \lambda PI_0) + \lambda X(z)PI_0) \qquad (3.75)$$

$$= \left(1 - \frac{S^*(\theta)}{z}M_1(z)\right)P(z,0) - \frac{S^*(\theta)}{z}(M_1(z)P_0(0)\left(VI^*\left(w_x(z)\right) - 1\right)$$

$$\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\beta_i + \lambda PI_0\left(X(z) - M_1(z)\right)) \qquad (3.76)$$

$Y_1(z)$ from equation (3.60), the right hand side is

$$= \left(\frac{z - S^*(\theta)M_1(z)}{z}\right)P(z,0) - \frac{S^*(\theta)}{z}P_0(0)(M_1(z)\big(VI^*\big(w_x(z)\big) - 1\big)$$

$$\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i + \varphi\big(X(z) - M_1(z)\big))$$

$$= \left(\frac{z - S^*(\theta)M_1(z)}{z}\right)P(z,0) - \frac{S^*(\theta)}{z}P_0(0)Q_1(z) \tag{3.77}$$

where

$$Q_1(z) = M_1(z)\big(VI^*\big(w_x(z)\big) - 1\big)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i + \varphi\big(X(z) - M_1(z)\big) \tag{3.78}$$

At $\theta = h_\alpha\big(w_x(z)\big)$ the equation (3.77) gives

$$\big(z - S^*(h_\alpha(w_x(z)))M_1(z)\big)P(z,0) = S^*\left(h_\alpha\big(w_x(z)\big)\right)P_0(0)Q_1(z) \tag{3.79}$$

$$P(z,0) = \frac{S^*(h_\alpha(w_x(z)))P_0(0)Q_1(z)}{z - S^*(h_\alpha(w_x(z)))M_1(z)} \tag{3.80}$$

$$= S^*\left(h_\alpha\big(w_x(z)\big)\right)P_0(0)Q_R(z) \tag{3.81}$$

where

$$Q_R(z) = \frac{Q_1(z)}{z - S^*(h_\alpha(w_x(z)))M_1(z)} \tag{3.82}$$

Now the equation (3.60) becomes

$$Y_1(z) = S^*\left(h_\alpha\big(w_x(z)\big)\right)P_0(0)Q_R(z)$$

$$+\ P_0(0)\big(VI^*\big(w_x(z)\big) - 1\big)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i(\alpha I_0)^j - \lambda PI_0$$

$$= Y_R(z)P_0(0) \tag{3.82.1}$$

where

$$Y_R(z) = S^*\left(h_\alpha\left(w_x(z)\right)\right)Q_R(z) + \left(VI^*\left(w_x(z)\right) - 1\right)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i(\alpha I_0)^j - \phi \quad (3.82.2)$$

Substituting the equation (3.81) in the equation (3.77)

$$\left(\theta - h_\alpha\left(w_x(z)\right)\right)P^*(z,\theta) = \left(\frac{z - S^*(\theta)M_1(z)}{z}\right)S^*\left(h_\alpha\left(w_x(z)\right)\right)Q_R(z)P_0(0)$$

$$-\frac{S^*(\theta)}{z}P_0(0)Q_1(z) \quad (3.83)$$

$$= \frac{P_0(0)Q_1(z)}{z}\left(\frac{z - S^*(\theta)M_1(z)S^*\left(h_\alpha\left(w_x(z)\right)\right)}{z - S^*\left(h_\alpha\left(w_x(z)\right)\right)M_1(z)} - S^*(\theta)\right) \quad (3.84)$$

$$= \frac{P_0(0)Q_1(z)}{\left(z - H^*\left(w_x(z)\right)\right)z}(z\,S^*\left(h_\alpha\left(w_x(z)\right)\right)$$

$$- S^*(\theta)M_1(z)S^*\left(h_\alpha\left(w_x(z)\right)\right) + S^*(\theta)S^*\left(h_\alpha\left(w_x(z)\right)\right)M_1(z)$$

$$- z\,S^*(\theta)z\,S^*\left(h_\alpha\left(w_x(z)\right)\right) - S^*(\theta)M_1(z)S^*\left(h_\alpha\left(w_x(z)\right)\right)$$

$$+ S^*(\theta)S^*\left(h_\alpha\left(w_x(z)\right)\right)M_1(z) - z\,S^*(\theta)) \quad (3.85)$$

where $H^*\left(w_x(z)\right) = S^*\left(h_\alpha\left(w_x(z)\right)\right)M_1(z)$

$$\left(\theta - h_\alpha\left(w_x(z)\right)\right)P^*(z,\theta) = \frac{P_0(0)Q_1(z)}{\left(z - H^*\left(w_x(z)\right)\right)}\left[S^*\left(h_\alpha\left(w_x(z)\right)\right) - S^*(\theta)\right] \quad (3.86)$$

$$\left(\theta - h_\alpha\left(w_x(z)\right)\right)P^*(z,\theta) = P_0(0)Q_R(z)\left[S^*\left(h_\alpha\left(w_x(z)\right)\right) - S^*(\theta)\right] \quad (3.87)$$

Now $Q_R(z)$ can be also written as

$$Q_R(z) = \frac{Q_1(z)}{\left(z - H^*\left(w_x(z)\right)\right)} = \frac{N_R(z)}{D_R(z)} \quad (3.88)$$

At $\theta = 0$ the equation (3.87) gives

$$P^*(z,0) = \frac{P_0(0)Q_R(z)}{\left(h_\alpha\left(w_x(z)\right)\right)}\left[1 - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right] \tag{3.89}$$

Thus the partial probability generating functions of the system size probabilities at arbitrary epoch when the server is in different states are obtained and are listed below.

$$PI^*(z,0) = \left(\frac{1 - A^*(\lambda)}{\lambda}\right)Y_1(z) \tag{3.89.1}$$

$$QI^*(z,0) = P_0(0)\left(\frac{1 - VI^*\left(w_x(z)\right)}{\left(w_x(z)\right)}\right)\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i \tag{3.89.2}$$

$$B^{**1}(z,\theta,0) = \frac{\alpha P^*(z,\theta)\left[1 - R^{*1}\left(w_x(z)\right)\right]}{w_x(z)} \tag{3.89.3}$$

$$P^*(z,0) = \frac{P_0(0)Q_R(z)}{\left(h_\alpha\left(w_x(z)\right)\right)}\left[1 - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right] \tag{3.89.4}$$

3.4 Performance Measures

In this section some useful performance measures of the proposed model are presented. The following results are used to derive the system size probability and the queue size.

From equations (3.88) and (3.78) it is found that

Q(1) is of 0/0 form hence it can be evaluated using the L'hospital rule.

At z =1 equation (3.88) implies

$$Q_R(1) = \frac{N_R'(1)}{D_R'(1)} \tag{3.90}$$

$$Q_R'(1) = \frac{D_R'(1)N_R''(1) - D_R''(1)N_R'(1)}{2D_R'^2} \tag{3.90.1}$$

By calculation

$$D_R{}'(1) = 1 - \left(M_1{}'(1) + \lambda E(X)E(S)(1 + \alpha E(R))\right)$$

$$= 1 - M_1{}'(1) - \rho_R \qquad (3.90.2)$$

where,

$$\rho_R = \lambda E(X)E(S)\left(1 + \alpha E(R)\right) \qquad (3.90.3)$$

$$M_1(1) = A(\lambda) + 1 - A(\lambda) = 1 \qquad (3.90.4)$$

$$M_1{}'(1) = \left(1 - A^*(\lambda)\right)(E(X) - 1 + q) \qquad (3.90.5)$$

$$M_1''(1) \;= \left(1 - A^*(\lambda)\right)[E\left(X(X-1)\right) - 2(1-q)(E(X) + 1) \qquad (3.90.6)$$

Similarly $N_R{}'(1)$ and $Q_R(1)$ are simplified as,

$$N_R{}'(1) = \phi\left(E(X) - M_1{}'(1)\right) + \lambda E(X)E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \qquad (3.90.7)$$

Thus,

$$Q_R(1) = \frac{\phi\left(E(X)\right) - M_1{}'(1) + \lambda E(X)E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i}{1 - M_1{}'(1) - \rho_R} \qquad (3.90.8)$$

Now to calculate $Q_R{}'(1)$, the following results are obtained.

$$\begin{aligned}-D_R''(1) = M_1''(1) &+ 2M_1'(1)\lambda E(X)E(S)\left(1 + \alpha E(R)\right) \\ &+ \lambda E\left(X(X-1)\right)E(S)\left(1 + \alpha E(R)\right) \\ &+ \left(\lambda E(X)\right)^2\left[E(S^2)\left(1 + \alpha E(R)\right)^2 + E(S)\,E(R^2)\right]\end{aligned}$$

$$\begin{aligned}N_R''(1) \;= \phi\;\left(E\left(X(X-1)\right) - M_1{}''(1)\right) &+ \sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i\,(2M_1''(1)\lambda E(X)E(VI) \\ &+ \lambda E\left(X(X-1)\right)E(VI) + \left(\lambda E(X)\right)^2 E(VI^2))\end{aligned}$$

$$Y_R(1) = Q_R(1) - \phi \qquad (3.90.9)$$

$$Y_R'(1) = \lambda E(X)E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i + Q_R(1)\lambda E(X)E(S)\big(1+\alpha E(R)\big) + Q_R{}'(1)$$

$$= \lambda E(X)\left(Q_R(1)E(S)\big(1+\alpha E(R)\big) + E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{\infty}\beta_i\right) + Q_R'(1) \qquad (3.90.10)$$

Using the results (3.90 to 3.90.10), the performance measures are calculated.

3.4.1 The Steady State System Size Probabilities

Let PI_0, PI, P_{VI}, P_{busy}, and P_{br} denote the probability that the server is idle when there is no customer, idle when there are customers in orbit, on vacation, busy state and breakdown state respectively. Then the corresponding probabilities are obtained, by considering the equations in (3.89.1 to 3.89.4) at z=1. Thus

(i) $PI_0 = \dfrac{\phi}{\lambda}P_0(0)$

(ii) $PI = \lim_{z\to 1} PI^*(z,0)$

$$= \left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_R(1)P_0(0)$$

(iiI) $P_{VI} = QI^*(z,0)|_{z=1}$

$$= P_0(0)\sum_{j=0}^{\infty}\prod_{i=0}^{j}\beta_i(\alpha I_0)^j E(VI)$$

(iv) $P_{busy} = \lim_{z\to 1} P^*(z,0)$

$$= E(S)Q_R(1)P_0(0)$$

(v) $P_{br} = \lim_{z\to 1} B^{**1}(z,\theta,0)$

$$= \alpha P_{busy}E(R)$$

Thus the system size probabilities are expressed in terms of $P_0(0)$. $P_0(0)$ can be evaluated using the normalizing condition,

$$PI_0 + PI_n + P_{VI} + P_{busy} + P_{br} = 1$$

$$\text{i.e. } 1 = P_0(0)(E(S)Q_R(1) + E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\beta_i + \left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_R(1)$$

$$+\frac{\phi}{\lambda} + \alpha E(S)E(R)Q_R(1)) \tag{3.91}$$

$$P_0(0)^{-1} = E(S)Q_R(1) + E(VI)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\beta_i + \left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_R(1) + \frac{\phi}{\lambda} + \alpha E(S)E(R)Q_R(1)$$

where $Q_R(1), Y_R(1)$ is given by equations (3.90.8), (3.90.9)

3.4.2 Mean System Size

In this section the average number of customers waiting in the system, when the server is in different states are calculated.

Let L_{PI}, L_{VI}, L_{busy} and L_{br} denote the expected system size when the server is in idle state, vacation state, busy state and breakdown state respectively. Then the derivatives of equations in (3.89.1 to 3.89.4) at z=1 give the required measures.

Thus the mean system size corresponding to different states are obtained by differentiating the equations (3.89.1 to 3.89.4) with respect to z and letting z=1. Then,

(i) $$L_{PI} = \left(\frac{1-A^*(\lambda)}{\lambda}\right)Y_R'(1)P_0(0)$$

(ii) $$L_{VI} = P_0(0)\lambda E(X)\frac{E(VI^2)}{2}\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\beta_i$$

(iii) $$L_{busy} = \left[E(S)Q_R{}'(1) + \lambda E(X)\frac{E(S^2)}{2}\big(1 + \alpha E(R)\big)Q_R(1)\right]P_0(0)$$

(iv) $$L_{br} = \alpha\left(P_{busy}\lambda E(X)\frac{E(R^2)}{2} + L_{busy}E(R)\right)$$

The total expected system size for the model

$$L = L_{PI} + L_{VI} + L_{busy} + L_{br}$$

$$\begin{aligned}
&= P_0(0)[\lambda\, E(X)\frac{E(VI^2)}{2}\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\beta_i + \left(\frac{1 - A^*(\lambda)}{\lambda}\right)Y'_R(1) + E(S)Q_R{}'(1) \\
&\qquad + \lambda\, E(X)\left(\frac{E(S^2)}{2}\right)\big(1 + \alpha E(R)\big)Q_R(1) \\
&\qquad + \alpha\left[P_{busy}\lambda E(X)\frac{E(R^2)}{2} + L_{busy}E(R)\right] \qquad (3.92)
\end{aligned}$$

Where $Q_R(1), Q_R{}'(1), Y_R{}'(1)$ are given by equation (3.90), (3.90.1), (3.90.10).

Conclusion:

The $M^X/G/1$ Retrial queueing system without and with server's breakdown are analyzed in chapters II and III respectively under MAV policy. In the following chapter IV, numerical computations are made to justify the formulae obtained some particular cases are above obtained.

CHAPTER-IV

NUMERICAL ANALYSIS AND PARTICULAR CASES

In section-1 of the Present chapter, numerical values for system measures are calculated for a set of parameters and their effects on performance measures are studied. In section-2, the results of chapters II and III are compared and steady-state results for other vacation policies are also deduced.

4.1 Numerical Analysis

The distribution of each random variable and the common parameters used for the numerical computations of both the models are listed in the following table.

Random variable (Y)	**Distribution**	**Mean E(Y)**	**Second order moments E(Y^2)**
Vacation time (VI)	Erlang 3-type $\eta = 0.5$	$\frac{1}{\eta}$	$\frac{4}{3\eta^2}$
Repair time (R)	Gamma ($5,\beta_1$) $\beta_1 = 5$	$\frac{5}{\beta_1}$	$\frac{30}{{\beta_1}^2}$
Vacation time during busy period (VB)	Gamma 3-type ($3,\eta_b$) $\eta_b = 0.6$	$\frac{3}{\eta_b}$	$\frac{12}{{\eta_b}^2}$
Breakdown (α)	Poisson	$\alpha = 1$	
Batch size (X)	Geometric p=0.5	$\frac{1}{(1-p)}$	$\frac{2p}{(1-p)}$
Service time (S)	2- stage hyper exponential $\alpha = 1 , a = 0.3$ $\mu_1 = 2 , \mu_2 = 6$	$S^*(\alpha)$ $= \frac{a}{\mu_1} + \frac{(1-a)}{\mu_2}$	$S^{*\prime}(\alpha)$ $= 2(\frac{a}{{\mu_1}^2} + \frac{(1-a)}{{\mu_2}^2})$

4.1.1

To justify the results derived in the chapters II and III the following numerical computations are made. A set of sample parameters is used to justify the statements. A similar conclusion can be made for other sets of parameters also. The values given in tables 4.1.2(a) to 4.1.4(a) show that the value for the model of chapter II. The numerical values of the model of chapter III are given in tables 4.1.2(b) to 4.1.4(b).

In tables 4.1.2(a) and (b), the system size probabilities when the system is in different states are calculated by increasing the values of the arrival rate **λ** and utilization factor **ρ**. It is verified for both the models that the total probability is **1**.

λ	**PI**	$\mathbf{P_{Idle}}$	$\mathbf{P_{VI}}$	$\mathbf{P_{VB}}$	$\mathbf{P_{busy}}$	**ρ**
0.0069	0.1380	0.0042	0.0011	0.0282	0.8283	0.8599
0.0070	0.1254	0.0043	0.0010	0.0288	0.8403	0.8724
0.0071	0.1128	0.0044	0.0009	0.0294	0.8523	0.8848
0.0072	0.1002	0.0045	0.0008	0.0300	0.8643	0.8973
0.0073	0.0876	0.0046	0.0007	0.0306	0.8762	0.9098
0.0074	0.0751	0.0047	0.0006	0.0312	0.8882	0.9222
0.0075	0.0625	0.0048	0.0005	0.0319	0.9002	0.9347
0.0076	0.0499	0.0049	0.0004	0.0325	0.9122	0.9472
0.0077	0.0372	0.0050	0.0003	0.0331	0.9241	0.9596
0.0078	0.0246	0.0051	0.0002	0.0338	0.9361	0.9721

4.1.2 (a)

λ	**PI**	$\mathbf{P_{Idle}}$	$\mathbf{P_{VI}}$	$\mathbf{P_{br}}$	$\mathbf{P_{busy}}$	**ρ**
0.6700	0.0316	0.3186	0.0782	0.2539	0.3174	0.6432
0.6800	0.0279	0.3226	0.0703	0.2573	0.3217	0.6528
0.6900	0.0243	0.3266	0.0623	0.2607	0.3259	0.6624
0.7000	0.0208	0.3306	0.0543	0.2641	0.3301	0.6720
0.7100	0.0174	0.3345	0.0462	0.2674	0.3343	0.6816
0.7200	0.0140	0.3385	0.0380	0.2708	0.3385	0.6912
0.7300	0.0108	0.3423	0.0299	0.2741	0.3442	0.7008
0.7400	0.0077	0.3462	0.0216	0.2774	0.3468	0.7104
0.7500	0.0047	0.3500	0.0134	0.2807	0.3509	0.7200
0.7600	0.0017	0.3538	0.0051	0.2841	0.3551	0.7296

4.1.2 (b)

In tables 4.1.3(a) and (b), the number of customers waiting in the system when the server is idle (or) busy (or) in breakdown states are calculated corresponding to increasing the values of the arrival rate **λ** for both the models. It is shown that L increases as **λ** increases.

λ	L_{busy}	L_{VI}	L_{VB}	L_{Idle}	L
0.0069	9.8488	0.00010	0.6894	0.0459	10.5841
0.0070	11.0308	0.00090	0.7840	0.0520	11.8669
0.0071	12.4739	0.00009	0.9002	0.0598	13.4339
0.0072	14.2761	0.00008	1.0462	0.0695	15.3919
0.0073	16.5923	0.00007	1.2349	0.0820	17.9093
0.0074	19.6809	0.00006	1.4878	0.0988	21.2676
0.0075	24.0089	0.00005	1.8438	0.1225	25.9753
0.0076	30.5156	0.00004	2.3809	0.1582	33.0547
0.0077	41.4095	0.00003	3.2828	0.2182	44.9106
0.0078	63.4123	0.00002	5.1086	0.3396	68.8605

4.1.3(a)

λ	L_{busy}	L_{VI}	L_{br}	L_{Idle}	L
0.6700	7.2329	0.1398	5.9564	7.3506	20.6799
0.6800	8.5312	0.1275	7.0002	8.6493	24.3081
0.6900	10.1930	0.1147	8.3343	10.3101	28.9523
0.7000	12.3839	0.1013	10.0920	12.4980	35.0753
0.7100	15.3871	0.0875	12.4996	15.4951	43.4694
0.7200	19.7316	0.0731	15.9802	19.8280	55.6130
0.7300	26.5308	0.0582	21.4248	26.6056	74.6195
0.7400	38.5999	0.0428	31.0853	38.6306	108.3587
0.7500	65.7136	0.0268	52.7815	65.6360	184.1580
0.7600	180.9189	0.0104	144.9510	180.3550	506.2355

4.1.3(b)

In tables 4.1.4(a) & 4.1.4(b) it is shown that for both the models, the system size increases as, mean vacation time increases. The effect of mean repair time on L for the model of chapter III is given by the table 4.1.4 (b).

EVI	**L**	**EVB**	**L**
6.6666	0.0058	62.0833	65.1912
7.1428	0.0063	62.2499	71.7525
7.6923	0.0069	62.2930	73.9736
8.3333	0.0077	62.3016	74.0499
9.0909	0.0087	62.3249	75.1341
10.0000	0.0100	62.4999	84.3493
11.1111	0.0117	62.8333	109.6243
12.5000	0.0140	62.0666	138.1898
14.2857	0.0174	62.1499	152.2419
16.6666	0.0226	62.6883	423.7543

4.1.4 (a)

EVI	**L**	**ER**	**L**
1.0000	5.5597	0.3333	2.2070
1.1111	5.8347	0.3636	2.4001
1.2500	6.1907	0.4000	2.6543
1.4285	6.6640	0.4444	3.0020
1.6666	7.3151	0.5000	3.5022
2.0000	8.2517	0.5714	4.2737
2.5000	9.6872	0.6666	5.5917
3.3333	12.1142	0.8000	8.2517
5.0000	16.9989	1.0000	15.7390
10.0000	31.6536	1.3333	91.9644

4.1.4 (b)

4.2 Particular Cases:

In this section it is verified that when the breakdown parameter **α** is zero and the probability **p** that the server takes a single vacation between services is zero, the probability generating functions of both models coincide.

At α=0 and p=0, equations (2.73) & (2.76) of chapter II and equations (3.88) & (3.82.2) imply,

$$Q(z) = Q_R(z) = \frac{\phi(X(z) - M_1(z) + M_1(z)(VI^*(w_x(z) - 1)(\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i)}{z - M_1(z)S^*(w_x(z)}$$

and

$$Y(z) = Y_R(z) = S^*\big(w_x(z)\big)Q(z) + (VI^*(w_x(z) - 1)(\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i) - \phi$$

The partial generating function given in equations (2.80.1) to (2.80.4) are compared with (3.89.1) to (3.89.4) when $\alpha = p = 0$ and it is found that the corresponding partial probability generating functions are equal and are given by

$$PI^*(z, 0) = \left(\frac{1 - A^*(\lambda)}{\lambda}\right) Y_1(z)$$

$$QI^*(z, 0) = P_0(0)\left(\frac{1 - VI^*\big(w_x(z)\big)}{(w_x(z))}\right)\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i$$

$$P^*(z, 0) = \frac{P_0(0)Q_1(z)}{\Big(h_\alpha\big(w_x(z)\big)\Big)}\left[1 - S^*\Big(h_\alpha\big(w_x(z)\big)\Big)\right]$$

For the results of the reliable $M^X/G/1$ retrial queueing model corresponding to different vacation policies, it is enough to calculate the expressions of

$$MAV = \sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j} \beta_i \text{ and } \phi = \sum_{j=1}^{\infty}\left(1 - \beta_j\right)\prod_{i=0}^{j-1} \beta_i\, \alpha I_0^j + \left(1 - \beta_0\right) \text{ of the models}$$

Case 1: Single vacation model:

When $\beta_0 = 1, \beta_j = 0$ for $j \geq 1$, the results for single vacation model are obtained. ϕ and MAV in this case are,

$$\phi = \alpha_0$$

MAV = 1

Case 2: Multiple vacation model:

With the selection of , $\beta_i = 1$ for every $i \geq 0$ results are obtained for multiple vacation model. It is noted that

$$\phi = 0$$

$$\text{MAV} = \frac{1}{1-\alpha_0}$$

Case 3: J- vacation model:

For this policy $\beta_i{}'$s are selected as, $\beta_0 = 1, \beta_j = p$ for $1 \leq j \leq J-1$ and

$\beta_j = 0$ for $j \geq J$.

Then,

$$\phi = \sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1-\beta_0)$$

$$\text{MAV} = 1 + \frac{\alpha_0(1-(\alpha_0 p)^J)}{1-\alpha I_0 p}$$

Summary and Conclusion:

There are various situations in our everyday life when blocked customers are not willing to work and they temporarily have the service facility for a while but try again after some random time. In call centers, if a customer makes a phone call when all the agents are busy then, the customer will try to make a call again after some random time. In computer networks, if a packet is lost, the packet may be retransmitted at a later time by a retransmission mechanism. The analysis of retrial queue is more complex and challenging than that of standard queues.

In the present work, the author makes an attempt to analyse batch arrival retrial non-Markovian queueing models, in which the retrial time of each customer is generally distributed. Chapter II deals with reliable server who takes a single vacation between two consecutive service and multiple vacations, controlled by a sequence of probabilities $\{\beta_i\}$ i=0,1,2,… during idle time. Chapter III consider the unreliable case, in which the server is subjected to unpredictable breakdowns. The steady-state queue size probabilities when the system is in different states are obtained and average numbers of customers waiting in the queue during idle and busy period are also calculated. The results of both the models are compared and some conclusions regarding the performance measures are made through numerical computations. The results for single, multiple and J-vacation queueing models are deduced as special cases through suitable selection of the vacation controlled parameters.

The analysis of the present work will help the researchers to analyses retrial queueing models under Multiple Adapted Vacation policy so that, the results of other vacation policies can be deduced a special cases.

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