INTRODUCTION

"If people do not believe that mathematics is simple, it is
Only because they do not realize how complicated life is"

- **John Von Neumann**

In 1965, the concept of fuzzy sets was introduced by Zadeh [43]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as semi groups, groups, rings and vector spaces. In 1971, Rosen Feld [36] introduced the notion of fuzzy subgroup of a group. Since then, many scholars have studied the theories of fuzzy subgroups of a group. In 1975, Zadeh [44] introduced the concept of an interval valued fuzzy set which is an extension of the concept of a fuzzy set.

In 1986, the idea of intuitionistic fuzzy set was first introduced by K.T.Atanassav [2], as a generalization of the notion of fuzzy sets. In 2012, based on the (interval-valued) fuzzy sets, Jun et al. [20] introduced the notion of (internal, external) cubic sets, and investigated several properties.

In 1966, Y.Imai and K.Iseki [14, 17] introduced two classes of abstract algebras : BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2007, as a generalization of a BCK/BCI-algebras, Kim and Kim [21] introduced the notion of a BE-algebra which was deeply studied by many mathematicians. In 2010, B.L.Meng [28] introduced the notion of CI-algebras and studied its elementary properties as a generalization of BE-algebras.

This thesis is devoted to the study of Fuzzification of ideals and Filters in CI-algebras. The following articles are chosen for our discussion :

1. "CI-Algebras" by B.L.Meng [3]

2. "Closed filters in CI-algebras" by B.L.Meng [4]

3. "On filters and upper sets in CI-algebras" by Bozena Piekart and Andrzek Walendziak [11]

4. "A note on CI-algebras" by K.H.Kim [24]

5. "Fuzzy ideals in CI-algebras" by Samy M.Mostafa, Mokhtar A.Abdel Naby and Osama R.Elgendy [37]

6. "On $(\in, \in \vee q_k)$-fuzzy filters of CI-algebras" by Ameneh Namdar, A.Borumand Saie and Ghazanfar Jabbari [1]

7. "Anti fuzzy ideals of CI-algebras and its lower level cuts" by T.Priya and T.Ramachandran [31]

8. "Anti fuzzy subalgebras and homomorphism of CI-algebras" by P.M.Sithar Selvam, T.Priya and T.Ramachandran [38]

9. "Intuitionistic (T, S)-fuzzy CI-algebras" by A.Borumand Saeid and A.Rezaei [8]

10. "Cubic subalgebras and filters of CI-algebras" by S.S.Ahn, Y.H.Kim and Jung Mi Ko [40]

This thesis is split into five chapters :

In the first chapter, the preliminary definitions and results of CI-algebras are presented due to B.L.Meng [4, 3], Bozena Piekart et al. [11] and K.H.Kim [24].

The interesting results discussed in this chapter are given as follows :

i. Let X be a self-distributive CI-algebra and I an ideal of X. Then $I_W = \{x \in X / w * x \in I\}$ is an ideal of an CI-algebra X.

ii. Let X be a transitive CI-algebra. Then a non empty subset A of X is an ideal of X if and only if A is a filter of X.

iii. Let X be an CI-algebra. If (X ; $*$, 1) is a self distributive CI-algebra, then $A(x, y) = \{z \in X / z = 1 \text{ (or) } x * (y * z) = 1\}$ is a subalgebra of X.

iv. A filter of a CI-algebra X is closed if and only if it is subalgebra of X.

Second chapter deals with the study of fuzzy ideals and fuzzy filters in CI-algebras. In this chapter, the fuzzification of ideals in CI-algebras and their several properties are discussed. Also the relationship between fuzzy filters and $(\in, \in \vee qk)$-fuzzy filters are investigated.

The important results discussed in this chapter are given as follows :

i. Let μ be a fuzzy set in a CI-algebra X. Then μ is a fuzzy ideal of X iff it satisfies : $U(\mu ; \alpha) \neq \Phi \Rightarrow U(\mu ; \alpha)$ is an ideal of X, $\forall \alpha \in [0, 1]$ where $U(\mu ; \alpha) = \{x \in X / \mu(x) \geq \alpha\}$.

ii. Let X be a transitive CI-algebra. A fuzzy set μ in X is a fuzzy ideal of X. Then it satisfies condition :
   a. $\mu(1) \geq \mu(x), \forall x \in X$ and
   b. $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

iii. Every fuzzy filter is an $(\in, \in \vee qk)$-fuzzy filter.

Chapter three deals with the study of antifuzzy ideals and antifuzzy subalgebras in CI-algebras. In this chapter, few results of antifuzzy ideals of CI-algebras under transitive and self-distributive, few results of antifuzzy ideals of CI-algebras with lower level cuts are discussed. Also some properties of antifuzzy ideals in CI-algebra under homomorphism are discussed.

Interesting results proved in this chapter are given as follows :

i. Any subalgebra of a CI-algebra X can be realized as a level subalgebra of some antifuzzy subalgebra of X.

ii. Let $f : (X ; *, 1) \to (Y ; \Delta, 1')$ be a homomorphism of CI-algebras. If μ is an antifuzzy ideal of Y then μ_f defined by $\mu_f(x) = \mu(f(x))$ is an antifuzzy ideal of X.

iii. Let μ be a fuzzy set of CI-algebra X. If for each $t \in [0, 1]$, the lower level cut $L(\mu ; t) = \{x \in X / \mu(x) \leq t\}$ is an ideal of X, then μ is an antifuzzy ideal of X.

iv. If μ and δ are antifuzzy ideals in a CI-algebra X, then $\mu \times \delta$ is an antifuzzy ideal in $X \times X$, where $(\mu \times \delta)(x, y) = \max\{\mu(x), \delta(y)\}$, for all $x, y \in X$.

Chapter four deals with the study of intuitionistic fuzzy CI-algebras under T-norm and S-norm. In this chapter, intuitionistic (T, S)-fuzzy subalgebras in CI-algebras and their fundamental properties are discussed. Also the relationship between intuitionistic (T, S)-fuzzy subalgebras and intuitionistic (T, S)-fuzzy (closed) filters of CI-algebras are investigated.

Some interesting results discussed in this chapter are given as follows :

i. If $\{A_i\}_{i\in\Lambda}$ is a family of intuitionistic (T, S)-fuzzy subalgebras of a CI-algebra X. Then $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic (T, S)-fuzzy subalgebras of X, where $\bigcap_{i\in\Lambda} A_i = (\wedge \mu_i, \vee \gamma_i)$.

ii. An intuitionistic fuzzy set A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra of a CI-algebra X iff the fuzzy sets μ_A and γ_A^c are T-fuzzy subalgebra of X.

iii. Every intuitionistic (T, S)-fuzzy closed filter is an intuitionistic (T, S)-fuzzy subalgebra.

iv. Let A = (μ_A, γ_A) be an intuitionistic (T, S)-fuzzy subalgebra of X such that the non-empty sets U(μ_A ; α) and L(γ_A ; α) are closed filters of X and $\alpha \in [0, 1]$. Then A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy closed filter of X.

Chapter five deals with the study of cubic subalgebras and filters of CI-algebras. In this chapter, several related properties of cubic subalgebras and cubic filters in CI-algebras are investigated. Also, conditions for a cubic set to be a cubic filter are provided.

The following interesting results are discussed in this chapter.

i. If $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of a CI-algebra X, then the set $\mathscr{S} = \{x \in X \,/\, \tilde{\mu}_A(x) = \tilde{\mu}_A(1),\ \lambda(x) = \lambda(1)\}$ is a subalgebra of X.

ii. For a homomorphism f : X → Y of CI-algebras, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by f, where $\mathcal{C}(X)$ is the family of cubic sets in a set X.

a. If $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ is a cubic subalgebra of X which has the cubic property, then $\mathcal{C}_f(\mathscr{A})$ is a cubic subalgebra of Y.

b. If $\mathscr{B} = (\tilde{\mu}_B, \kappa) \in \mathcal{C}(Y)$ is a cubic subalgebra of Y, then $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subalgebra of X.

iii. Let X be a CI-algebra. If a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ satisfies the conditions :

a. $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$, $\lambda(1) \le \lambda(x)$ $\forall$ x, y $\in$ X and

$$a * (b * x) = 1 \Rightarrow \begin{pmatrix} \tilde{\mu}_A(x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \\ \lambda(x) \le \max\{\lambda(a), \lambda(b)\} \end{pmatrix}$$

(or)

b. $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$, $\lambda(1) \le \lambda(x)$ $\forall$ x, y $\in$ X and

$\tilde{\mu}_A(x * z) \succeq r\min\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$,

$\lambda(x * z) \le \max\{\lambda(x * (y * z)), \lambda(y)\}$

(or)

c. $$\begin{cases} \tilde{\mu}_A((a * (b * x)) * x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \\ \lambda((a * (b * x)) * x) \le \max\{\lambda(a), \lambda(b)\} \text{ and} \end{cases}$$

$\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x), \lambda(y * x) \le \lambda(x)$, $\forall$ x, y $\in$ X

Then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.