Chapter I

CHAPTER I

FUZZY SOFT SETS

Definition: 1.1

Let U be a non-empty set. A **Fuzzy Set** in U is a function with domain U and values in the closed interval I = [0, 1].

Definition: 1.2

Let $\mu, \rho \in F(U)$. We define the following fuzzy sets.

1) $\mu \leq \rho \Rightarrow \mu(x) \leq \rho(x) \ \forall x \in U$.
2) **Intersection of two Fuzzy Sets**:

   $\mu \wedge \rho \in F(U)$ by $(\mu \wedge \rho)(x) = \min\{\mu(x), \rho(x)\} \ \forall x \in U$.
3) **Union of two Fuzzy Sets:**

   $\mu \vee \rho \in F(U)$ by $(\mu \vee \rho)(x) = \max\{\mu(x), \rho(x)\} \ \forall x \in U$.
4) **Complement of Fuzzy Set:**

   $\mu^C \in F(U)$ by $\mu^C(x) = 1 - \mu(x) \ \forall x \in U$. μ^C is called the complement of μ and is denoted as μ' or $(1-\mu)$.

Definition: 1.3

Let U be the initial universe set and let E be the set of parameters. A pair (F, E) is called a **Soft Set** (over U) if and only if F is a mapping of E into the set of all subsets of the set U.

In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(\varepsilon), \varepsilon \in E$, from this family may be considered as the set of ε - elements of the soft set (F, E), or as the set of ε - approximate elements of the soft set.

Example: 1.4

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E = { e_1 (Costly), e_2 (Beautiful), e_3 (Fuel Efficient), e_4 (Luxurious) } be the set of parameters and A = { e_1, e_2, e_3 } $\subseteq$ E.

Then (F, A) = { F(e_1) = { c_1, c_4 }, F(e_2) = { c_1, c_2, c_4 }, F(e_3) = { c_3 } } is the soft set representing the 'attractiveness of the car' which Mr. X going to buy.

Definition: 1.5

Let U be the initial universe set, E be the set of parameters, A $\subseteq$ E and F(U) is the set of all fuzzy subsets of U. A pair (F, A) is called a **Fuzzy Soft Set** over U where $F : A \rightarrow F(U)$ is a mapping from A into F(U).

Example: 1.6

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E = { e_1 (Costly), e_2 (Beautiful), e_3 (Fuel Efficient), e_4 (Luxurious) } be the set of parameters and A = { e_1, e_2, e_3 } $\subseteq$ E.

Then

(F, A) = { F(e_1) = { c_1/0.7, c_2/0.1, c_3/0.2, c_4/0.6 } ,

F(e_2) = { c_1/0.8, c_2/0.6, c_3/0.1, c_4/0.5 } ,

F(e_3) = { c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 } }

is the fuzzy soft set representing the 'attractiveness of the car' which Mr. X going to buy.

Definition: 1.7

Let U be the universe and E a set of attributes. Then the pair (U, E) denotes the collection of all fuzzy soft sets on U with attributes from E and is called **Fuzzy Soft Class**.

Definition: 1.8

For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class (U, E), we say that (F, A) is a **Fuzzy Soft Subset** of (G, B), if

1) $A \subseteq B$,
2) For all $\varepsilon \in A, F(\varepsilon) \leq G(\varepsilon)$ and is written as (F, A) $\tilde{\subseteq}$ (G, B).

Example: 1.9

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E = { e_1 (Costly), e_2 (Beautiful), e_3 (Fuel Efficient), e_4 (Luxurious) } be the set of parameters, A = { e_1, e_2, e_3 } $\subseteq$ E and B = { e_1, e_2, e_3, e_5 } $\subseteq$ E.

Then

(F, A) = { F(e_1) = { c_1/0.7, c_2/0.1, c_3/0.2, c_4/0.6 } ,

F(e_2) = { c_1/0.8, c_2/0.6, c_3/0.1, c_4/0.5 } ,

F(e_3) = { c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 } }

is the fuzzy soft set representing the 'attractiveness of the car' which Mr. X going to buy and

(G, B) = { G(e_1) = { c_1/0.7, c_2/0.2, c_3/0.2, c_4/0.7 } ,

G(e_2) = { c_1/0.9, c_2/0.6, c_3/0.5, c_4/1 } ,

G(e_3) = { c_1/0.3, c_2/0.2, c_3/0.8, c_4/0.3 } ,

G(e_4) = { c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 } }

is the fuzzy soft set representing the 'attractiveness of the car' which Mr. Y going to buy.

Here $A \subseteq B$, and for all $\varepsilon \in A, F(\varepsilon) \leq G(\varepsilon)$.

Thus (F, A) $\tilde{\subseteq}$ (G, B).

Definition: 1.10

(F, A) is said to be a **Fuzzy Soft Super Set** of (G, B), if (G, B) is a fuzzy soft subset of (F, A). We denote it by (F, A) $\tilde{\supseteq}$ (G, B).

Definition: 1.11

A fuzzy soft set (F, A) over U is said to be **Null Fuzzy Soft Set** (with respect to the parameter set A), denoted by $\tilde{\varphi}$ if $\forall \varepsilon \in A$, $F(\varepsilon)$ is the null fuzzy set φ.

Definition: 1.12

A fuzzy soft set (F, A) over U is said to be **Absolute Fuzzy Soft Set** (with respect to the parameter set A), denoted by $\tilde{A}$ if $\forall \varepsilon \in A$, $F(\varepsilon)$ is the absolute fuzzy set U.

Definition: 1.13

The **Complement of a Fuzzy Soft Set** (F, A) is denoted by $(F, A)^C$ and is defined by $(F, A)^C = (F^C, A)$, where $F^C : A \to F(U)$ is a mapping given by $F^C(\sigma) = (F(\sigma))^C$ for all $\sigma \in A$.

Definition: 1.14

If (F, A) and (G, B) be two fuzzy soft sets, then **"(F, A) AND (G, B)"** is a fuzzy soft set denoted by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall \alpha \in A$ and $\forall \beta \in B$, where $\cap$ is the operation fuzzy intersection of two fuzzy sets.

Definition: 1.15

If (F, A) and (G, B) be two fuzzy soft sets, then **"(F, A) OR (G, B)"** is a fuzzy soft set denoted by $(F, A) \vee (G, B) = (O, A \times B)$, where $O(\alpha, \beta) = F(\alpha) \cup G(\beta)$, $\forall \alpha \in A$ and $\forall \beta \in B$, where $\cup$ is the operation fuzzy union of two fuzzy sets.

Definition: 1.16

Union of two Fuzzy Soft Sets (F, A) and (G, B) in a soft class (U, E) is a Fuzzy soft set (H, C) where C = A $\cup$ B and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & if\ x \in A - B \\ G(\varepsilon), & if\ x \in B - A \\ F(\varepsilon) \cup G(\varepsilon), & if\ x \in A \cap B \end{cases}$$

and is written as (F, A) $\tilde{\vee}$ (G, B) = (H, C).

Example: 1.17

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E = { e_1 (Costly), e_2 (Beautiful), e_3 (Fuel Efficient), e_4 (Luxurious) } be the set of parameters, A = { e_1, e_2, e_3 } $\subseteq$ E and B = { e_1, e_2, e_3, e_5 } $\subseteq$ E. We consider the fuzzy soft sets

(F, A) = { F(e_1) = { c_1/0.9, c_2/0.1, c_3/0.4, c_4/0.6 } ,

F(e_2) = { c_1/1, c_2/0, c_3/0.9, c_4/0.5 } ,

F(e_3) = { c_1/0.8, c_2/0.2, c_3/0.7, c_4/0.6 } } and

(G, B) = { G(e_1) = { c_1/0.7, c_2/0.2, c_3/0.2, c_4/0.7 } ,

G(e_2) = { c_1/0.9, c_2/0.6, c_3/0.5, c_4/1 } ,

G(e_3) = { c_1/0.3, c_2/0.2, c_3/0.8, c_4/0.3 } ,

G(e_4) = { c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 } }

Then (F, A) $\tilde{\vee}$ (G, B) = (H, C) ,

where C = A $\cup$ B = { e_1, e_2, e_3, e_5 } and

(H, C) = { H(e_1) = { c_1/0.9, c_2/0.2, c_3/0.4, c_4/0.7 } ,

H(e_2) = { c_1/1, c_2/0.6, c_3/0.9, c_4/1 } ,

$H(e_3) = \{ c_1/0.8, c_2/0.2, c_3/0.8, c_4/0.6 \}$,

$H(e_4) = \{ c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 \} \}$

Definition: 1.18

Intersection of two Fuzzy Soft Sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where C = A $\cap$ B and $\forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \, or \, G(\varepsilon)$ (as both are same fuzzy set) and is written as (F, A) $\tilde{\wedge}$ (G, B) = (H, C).

Generally $F(\varepsilon) \, or \, G(\varepsilon)$ may not be identical. Moreover, A $\cap$ B must be non-empty to avoid the degenerate case. Thus the revised Definition of 1.19 is as follows.

Definition: 1.19

Let (F, A) and (G, B) be two fuzzy soft sets in a soft class (U, E) with A $\cap$ B$\neq\phi$. Then **Intersection of two Fuzzy Soft Sets** (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where C = A $\cap$ B and $\forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. We write (F, A) $\tilde{\wedge}$ (G, B) = (H, C).

Example: 1.20

For the two fuzzy soft sets (F, A) and (G, B) given in Example 1.17,

(F, A) $\tilde{\wedge}$ (G, B) = (H, C), where C = A $\cap$ B = $\{ e_1, e_2, e_3 \}$ and

(H, C) = $\{ H(e_1) = \{ c_1/0.7, c_2/0.1, c_3/0.2, c_4/0.6 \}$,

$H(e_2) = \{ c_1/0.9, c_2/0, c_3/0.5, c_4/0.5 \}$,

$H(e_3) = \{ c_1/0.3, c_2/0.2, c_3/0.7, c_4/0.3 \} \}$

Definition: 1.21

Let (F, A) and (G, B) be two fuzzy soft sets over a common Universal set. Then a **Relation** R of (F, A) on (G, B) may be defined as a mapping

$R:A\times B\rightarrow P(U^2)$ such that for each $e_i\in A$, $e_j\in B$ and for all $u_l\in F(e_i)$, $u_k\in G(e_j)$ the relation R is characterized by the following membership function, $\mu_R(u_l,u_k)=\mu_{F(e_i)}(u_l)\times\mu_{G(e_j)}(u_k)$, where $u_l\in F(e_i), u_k\in G(e_j)$.

Theorem: 1.22 Commutative Property

Let (F_1,A_1) and (F_2,A_2) be two fuzzy soft sets in a soft class (U, E). Then

1) $(F_1,A_1)\,\tilde{\wedge}\,(F_2,A_2)=(F_2,A_2)\,\tilde{\wedge}\,(F_1,A_1)$
2) $(F_1,A_1)\,\tilde{\vee}\,(F_2,A_2)=(F_2,A_2)\,\tilde{\vee}\,(F_1,A_1)$

Proof:

1) Let $(F_1,A_1)\,\tilde{\wedge}\,(F_2,A_2)=(H_1,C_1)$

   Where $C_1=A_1\cap A_2$ and

   $\forall\varepsilon\in C_1, H_1(\varepsilon)=F_1(\varepsilon)\cap F_2(\varepsilon)$

   Again, let $(F_2,A_2)\,\tilde{\wedge}\,(F_1,A_1)=(H_1,C_1)$

   Where $C_2=A_2\cap A_1$ and

   $\forall\varepsilon\in C_2, H_2(\varepsilon)=F_2(\varepsilon)\cap F_1(\varepsilon)$

   Clearly $C_1=C_2$ and $H_1=H_2$.

   Thus $(F_1,A_1)\,\tilde{\wedge}\,(F_2,A_2)=(F_2,A_2)\,\tilde{\wedge}\,(F_1,A_1)$.

2) Let $(F_1,A_1)\,\tilde{\vee}\,(F_2,A_2)=(H_1,C_1)$

   Where $C_1=A_1\cup A_2$ and

$$\forall\varepsilon\in C_1,\ H_1(\varepsilon)=\begin{cases}F_1(\varepsilon), & if\ \varepsilon\in A_1-A_2\\ F_2(\varepsilon), & if\ \varepsilon\in A_2-A_1\\ F_1(\varepsilon)\cup F_2(\varepsilon), & if\ \varepsilon\in A_1\cap A_2\end{cases}$$

Again, let $(F_2,A_2)\,\tilde{\vee}\,(F_1,A_1)=(H_1,C_1)$

Where $C_2=A_2\cup A_1$ and

$$\forall \varepsilon \in C_2,\ H_2(\varepsilon) = \begin{cases} F_2(\varepsilon), & if\ \varepsilon \in A_2 - A_1 \\ F_1(\varepsilon), & if\ \varepsilon \in A_1 - A_2 \\ F_2(\varepsilon) \cup F_1(\varepsilon), & if\ \varepsilon \in A_2 \cap A_1 \end{cases}$$

Thus $C_1 = C_2$ and $H_1 = H_2$

Hence $(F_1, A_1) \tilde{\vee} (F_2, A_2) = (F_2, A_2) \tilde{\vee} (F_1, A_1)$.

Theorem: 1.23 Associative Property

Let $(F_1, A_1), (F_2, A_2)$ and (F_3, A_3) be three fuzzy soft sets in a soft class (U, E). Then

1) $(F_1, A_1) \tilde{\wedge} ((F_2, A_2) \tilde{\wedge} (F_3, A_3)) = ((F_1, A_1) \tilde{\wedge} (F_2, A_2)) \tilde{\wedge} (F_3, A_3)$
2) $(F_1, A_1) \tilde{\vee} ((F_2, A_2) \tilde{\vee} (F_3, A_3)) = ((F_1, A_1) \tilde{\vee} (F_2, A_2)) \tilde{\vee} (F_3, A_3)$

Proof:

1) Let $(F_2, A_2) \tilde{\wedge} (F_3, A_3) = (H_1, C_1)$ and $(F_1, A_1) \tilde{\wedge} (H_1, C_1) = (H_2, C_2)$

   Where $C_1 = A_2 \cap A_3$ and $\forall \varepsilon \in C_1, H_1(\varepsilon) = F_2(\varepsilon) \cap F_3(\varepsilon)$

   And $C_2 = A_1 \cap C_1 = A_1 \cap (A_2 \cap A_3)$,

   $\forall \varepsilon \in C_2, H_2(\varepsilon) = F_1(\varepsilon) \cap H_1(\varepsilon) = F_1(\varepsilon) \cap (F_2(\varepsilon) \cap F_3(\varepsilon))$

   Again, let $(F_1, A_1) \tilde{\wedge} (F_2, A_2) = (H_3, C_3)$ and $(H_3, C_3) \tilde{\wedge} (F_3, A_3) = (H_4, C_4)$

   Where $C_3 = A_1 \cap A_2$ and $\forall \varepsilon \in C_3, H_3(\varepsilon) = F_1(\varepsilon) \cap F_2(\varepsilon)$

   And $C_4 = C_3 \cap A_3 = (A_1 \cap A_2) \cap A_3$,

   $\forall \varepsilon \in C_4, H_4(\varepsilon) = H_3(\varepsilon) \cap F_3(\varepsilon) = (F_1(\varepsilon) \cap F_2(\varepsilon)) \cap F_3(\varepsilon)$

   It is clear that $C_2 = C_4$ and $H_2 = H_4$

   Thus $(F_1, A_1) \tilde{\wedge} ((F_2, A_2) \tilde{\wedge} (F_3, A_3)) = ((F_1, A_1) \tilde{\wedge} (F_2, A_2)) \tilde{\wedge} (F_3, A_3)$

2) Let $(F_2, A_2) \tilde{\vee} (F_3, A_3) = (H_1, C_1)$ and $(F_1, A_1) \vee (H_1, C_1) = (H_2, C_2)$

Where $C_1 = A_2 \cup A_3$ and $C_2 = A_1 \cup C_1 = A_1 \cup (A_2 \cup A_3)$ (1)

Case (i)

When $\varepsilon \in A_1 - C_1 = A_1 - (A_2 \cup A_3)$, $H_2(\varepsilon) = F_1(\varepsilon)$ (2)

Case (ii)

When $\varepsilon \in C_1 - A_1 = (A_2 \cup A_3) - A_1$,

$$H_2(\varepsilon) = H_1(\varepsilon) = \begin{cases} F_2(\varepsilon) & if\ \varepsilon \in A_2 - A_3 \\ F_3(\varepsilon) & if\ \varepsilon \in A_3 - A_2 \\ F_2(\varepsilon) \cup F_3(\varepsilon) & if\ \varepsilon \in A_2 \cap A_3 \end{cases} \quad (3)$$

Case (iii)

When $\varepsilon \in A_1 \cap C_1 = A_1 \cap (A_2 \cup A_3)$, $H_2(\varepsilon) = F_1(\varepsilon) \cup H_1(\varepsilon)$

Now, $A_1 \cap C_1 = A_1 \cap (A_2 \cup A_3)$

$= A_1 \cap ((A_2 - A_3) \cup (A_3 - A_2) \cup (A_2 \cap A_3))$

$= (A_1 \cap (A_2 - A_3)) \cup (A_1 \cap (A_3 - A_2)) \cup (A_1 \cap (A_2 \cap A_3))$

When $\varepsilon \in A_1 \cap (A_2 - A_3)$, $H_1(\varepsilon) = F_2(\varepsilon)$, by (3)

Thus $H_2(\varepsilon) = F_1(\varepsilon) \cup H_1(\varepsilon) = F_1(\varepsilon) \cup F_2(\varepsilon)$ (4)

When $\varepsilon \in A_1 \cap (A_3 - A_2)$, $H_1(\varepsilon) = F_3(\varepsilon)$, by (3)

Thus $H_2(\varepsilon) = F_1(\varepsilon) \cup H_1(\varepsilon) = F_1(\varepsilon) \cup F_3(\varepsilon)$ (5)

When $\varepsilon \in A_1 \cap (A_2 \cap A_3)$, $H_1(\varepsilon) = F_2(\varepsilon) \cup F_3(\varepsilon)$, by (3)

Thus $H_2(\varepsilon) = F_1(\varepsilon) \cup H_1(\varepsilon) = F_1(\varepsilon) \cup (F_2(\varepsilon) \cup F_3(\varepsilon))$ (6)

Again, Let $(F_1, A_1)\ \tilde{\vee}\ (F_2, A_2) = (H_3, C_3)$ and $(H_3, C_3)\tilde{\vee}(F_3, A_3) = (H_4, C_4)$

Where $C_3 = A_1 \cup A_2$ and $C_4 = C_3 \cup A_3 = (A_1 \cup A_2) \cup A_3$ (7)

Case (i)

When $\varepsilon \in C_3 - A_3 = (A_1 \cup A_2) - A_3$,

$$H_4(\varepsilon) = H_3(\varepsilon) = \begin{cases} F_1(\varepsilon) & if\ \varepsilon \in A_1 - A_2 \\ F_2(\varepsilon) & if\ \varepsilon \in A_2 - A_1 \\ F_1(\varepsilon) \cup F_2(\varepsilon) & if\ \varepsilon \in A_1 \cap A_2 \end{cases} \quad (8)$$

Case (ii)

When $\varepsilon \in A_3 - C_3 = A_3 - (A_1 \cup A_2) = (A_3 - A_1) \cap (A_3 - A_2)$, $H_4(\varepsilon) = F_3(\varepsilon)$ (9)

Case (iii)

When $\varepsilon \in C_3 \cap A_3 = (A_1 \cup A_2) \cap A_3$

$$= ((A_1 - A_2) \cup (A_2 - A_1) \cup (A_1 \cap A_2)) \cap A_3$$

$$= ((A_1 - A_2) \cap A_3) \cup ((A_2 - A_1) \cap A_3) \cup ((A_1 \cap A_2) \cap A_3)$$

$H_4(\varepsilon) = H_3(\varepsilon) \cup F_3(\varepsilon)$

When $\varepsilon \in (A_1 - A_2) \cap A_3$, $H_3(\varepsilon) = F_1(\varepsilon)$, by (8)

Thus $H_4(\varepsilon) = H_3(\varepsilon) \cup F_3(\varepsilon)$

i.e. $H_4(\varepsilon) = F_1(\varepsilon) \cup F_3(\varepsilon)$ (10)

When $\varepsilon \in (A_2 - A_1) \cap A_3$, $H_3(\varepsilon) = F_2(\varepsilon)$, by (8)

Thus $H_4(\varepsilon) = H_3(\varepsilon) \cup F_3(\varepsilon)$

i.e. $H_4(\varepsilon) = F_2(\varepsilon) \cup F_3(\varepsilon)$ (11)

When $\varepsilon \in (A_1 \cap A_2) \cap A_3$, $H_3(\varepsilon) = F_1(\varepsilon) \cup F_2(\varepsilon)$, by (8)

Thus $H_4(\varepsilon) = H_3(\varepsilon) \cup F_3(\varepsilon)$

i.e. $H_4(\varepsilon) = (F_1(\varepsilon) \cup F_2(\varepsilon)) \cup F_3(\varepsilon)$ (12)

Now, when $\varepsilon \in A_1 - (A_2 \cup A_3) = (A_1 - A_2) \cap (A_1 - A_3)$

$H_2(\varepsilon) = F_1(\varepsilon)$, by (2) and $H_4(\varepsilon) = F_1(\varepsilon)$, by (8)

When $\varepsilon \in (A_2 \cup A_3) - A_1 = ((A_2 - A_3) \cup (A_3 - A_2) \cup (A_2 \cap A_3)) - A_1$

$$= ((A_2 - A_3) - A_1) \cup ((A_3 - A_2) - A_1) \cup ((A_2 \cap A_3)) - A_1)$$

When $\varepsilon \in (A_2 - A_3) - A_1$

$H_2(\varepsilon) = F_2(\varepsilon)$, by (3) and $H_4(\varepsilon) = F_2(\varepsilon)$, by (8)

$\{ (A_2 - A_3) - A_1 = (A_2 \cap A_3^C) \cap A_1^C = A_2 \cap (A_3^C \cap A_1^C) = A_2 \cap (A_3 \cup A_1)^C = A_2 - (A_3 \cup A_1)$

$= (A_2 - A_3) \cap (A_2 - A_1) \}$

When $\varepsilon \in (A_3 - A_2) - A_1$

$H_2(\varepsilon) = F_3(\varepsilon)$, by (3) and $H_4(\varepsilon) = F_3(\varepsilon)$, by (9)

$\{ (A_3 - A_2) - A_1 = (A_3 \cap A_2^C) \cap A_1^C = A_3 \cap (A_1^C \cap A_2^C) = A_3 \cap (A_1 \cup A_2)^C = A_3 - (A_1 \cup A_2)$

$= A_3 - C_3 \}$

When $\varepsilon \in (A_2 \cap A_3) - A_1$

$H_2(\varepsilon) = F_2(\varepsilon) \cup F_3(\varepsilon)$, by (3) and $H_4(\varepsilon) = F_2(\varepsilon) \cup F_3(\varepsilon)$, by (11)

$\{ (A_2 \cap A_3) - A_1 = (A_2 \cap A_3) \cap A_1^C = (A_2 \cap A_1^C) \cap A_3 = (A_2 - A_1) \cap A_3 \}$

When $\varepsilon \in A_1 \cap (A_2 \cup A_3) = A_1 \cap ((A_2 - A_3) \cup (A_3 - A_2) \cup (A_2 \cap A_3))$

$$= (A_1 \cap (A_2 - A_3)) \cup (A_1 \cap (A_3 - A_2)) \cup (A_1 \cap (A_2 \cap A_3))$$

When $\varepsilon \in A_1 \cap (A_2 - A_3) = A_1 \cap (A_2 \cap A_3^C) = (A_1 \cap A_2) \cap A_3^C = (A_1 \cap A_2) - A_3$

$H_2(\varepsilon) = F_1(\varepsilon) \cup F_2(\varepsilon)$, by (4) and $H_4(\varepsilon) = F_1(\varepsilon) \cup F_2(\varepsilon)$, by (8)

When $\varepsilon \in A_1 \cap (A_3 - A_2) = A_1 \cap (A_3 \cap A_2^C) = A_3 \cap (A_1 \cap A_2^C) = A_3 \cap (A_1 - A_2)$

$H_2(\varepsilon) = F_1(\varepsilon) \cup F_3(\varepsilon)$, by (5) and $H_4(\varepsilon) = F_1(\varepsilon) \cup F_3(\varepsilon)$, by (10)

When $\varepsilon \in A_1 \cap (A_2 \cap A_3)$

$H_2(\varepsilon) = F_1(\varepsilon) \cup (F_2(\varepsilon) \cup F_3(\varepsilon))$, by (6) and $H_4(\varepsilon) = (F_1(\varepsilon) \cup F_2(\varepsilon)) \cup F_3(\varepsilon)$, by (12)

Thus $(F_1, A_1) \tilde{\vee} ((F_2, A_2) \tilde{\vee} (F_3, A_3)) = ((F_1, A_1) \tilde{\vee} (F_2, A_2)) \tilde{\vee} (F_3, A_3)$

Theorem: 1.24 Idempotent Property

Let (F_1, A_1) be the fuzzy soft set in a soft class (U, E). Then

1) $(F_1, A_1) \tilde{\wedge} (F_1, A_1) = (F_1, A_1)$
2) $(F_1, A_1) \tilde{\vee} (F_1, A_1) = (F_1, A_1)$

Proof:

1) Let $(F_1, A_1) \tilde{\wedge} (F_1, A_1) = (H, C)$

   Where $C = A_1 \cap A_1 = A_1$ and $\forall \varepsilon \in C = A_1$,

   $H(\varepsilon) = F_1(\varepsilon) \cap F_1(\varepsilon) = F_1(\varepsilon)$

   Thus $(F_1, A_1) \tilde{\wedge} (F_1, A_1) = (F_1, A_1)$
2) Let $(F_1, A_1) \tilde{\vee} (F_1, A_1) = (H, C)$

   Where $C = A_1 \cup A_1 = A_1$ and $\forall \varepsilon \in C = A_1$,

   $H(\varepsilon) = F_1(\varepsilon) \cup F_1(\varepsilon) = F_1(\varepsilon)$

   Thus $(F_1, A_1) \tilde{\vee} (F_1, A_1) = (F_1, A_1)$

Theorem: 1.25 Absorption Property

Let (F_1, A_1) and (F_2, A_2) be two fuzzy soft sets in a soft class (U, E). Then

1) $(F_1, A_1) \tilde{\wedge} ((F_1, A_1) \tilde{\vee} (F_2, A_2)) = (F_1, A_1)$
2) $(F_1, A_1) \tilde{\vee} ((F_1, A_1) \tilde{\wedge} (F_2, A_2)) = (F_1, A_1)$

Proof:

1) Let $(F_1,A_1)\tilde{\vee}(F_2,A_2)=(H_1,C_1)$ and $(F_1,A_1)\tilde{\wedge}(H_1,C_1)=(H_2,C_2)$

Where $C_1=A_1\cup A_2$ and $\forall\varepsilon\in C_1$, $H_1(\varepsilon)=\begin{cases}F_1(\varepsilon), & \text{if } \varepsilon\in A_1-A_2\\ F_2(\varepsilon), & \text{if } \varepsilon\in A_2-A_1\\ F_1(\varepsilon)\cup F_2(\varepsilon), & \text{if } \varepsilon\in A_1\cap A_2\end{cases}$

Also $C_2=A_1\cap C_1=A_1$ and

$\forall\varepsilon\in C_2$, $H_2(\varepsilon)=F_1(\varepsilon)\cap H_1(\varepsilon)$

Now if $\varepsilon\in A_1-A_2$, $H_2(\varepsilon)=F_1(\varepsilon)\cap F_1(\varepsilon)=F_1(\varepsilon)$

And if $\varepsilon\in A_1\cap A_2$,

$H_2(\varepsilon)=F_1(\varepsilon)\cap H_1(\varepsilon)=F_1(\varepsilon)\cap(F_1(\varepsilon)\cup F_2(\varepsilon))=F_1(\varepsilon)$

Thus $(F_1,A_1)\tilde{\wedge}((F_1,A_1)\tilde{\vee}(F_2,A_2))=(F_1,A_1)$

2) Let $(F_1,A_1)\tilde{\wedge}(F_2,A_2)=(H_1,C_1)$ and $(F_1,A_1)\tilde{\vee}(H_1,C_1)=(H_2,C_2)$

Where $C_1=A_1\cap A_2$ and

$\forall\varepsilon\in C_1$, $H_1(\varepsilon)=F_1(\varepsilon)\cap F_2(\varepsilon)$

Also $C_2=A_1\cup C_1=A_1$ and

$\forall\varepsilon\in C_2$, $H_2(\varepsilon)=F_1(\varepsilon)\cup H_1(\varepsilon)$

$=F_1(\varepsilon)\cup(F_1(\varepsilon)\cap F_2(\varepsilon))=F_1(\varepsilon)$

$=F_1(\varepsilon)$

Thus $(F_1,A_1)\tilde{\vee}((F_1,A_1)\tilde{\wedge}(F_2,A_2))=(F_1,A_1)$

Theorem: 1.26 Distributive Property

Let (F_1,A_1) , (F_2,A_2) and (F_3,A_3) be three fuzzy soft sets in a soft class (U, E). Then

1) $(F_1,A_1)\tilde{\wedge}((F_2,A_2)\tilde{\vee}(F_3,A_3))=((F_1,A_1)\tilde{\wedge}(F_2,A_2))\tilde{\vee}((F_1,A_1)\tilde{\wedge}(F_3,A_3))$

2) $(F_1,A_1)\tilde{\vee}((F_2,A_2)\tilde{\wedge}(F_3,A_3))=((F_1,A_1)\tilde{\vee}(F_2,A_2))\tilde{\wedge}((F_1,A_1)\tilde{\vee}(F_3,A_3))$

Proof:

1) Let $(F_1, A_1)\tilde{\wedge}(F_2, A_2)=(H_1, C_1)$ and $(F_1, A_1)\tilde{\wedge}(F_3, A_3)=(H_2, C_2)$

Where $C_1 = A_1 \cap A_2$, $C_2 = A_1 \cap A_3$

Let $(H_1, C_1)\tilde{\vee}(H_2, C_2)=(H_3, C_3)$

where $C_3 = C_1 \cup C_2 = (A_1 \cap A_2)\cup(A_1 \cap A_3) = A_1 \cap(A_2 \cup A_3)$

Case (i)

When $\varepsilon \in C_1 - C_2 = (A_1 \cap A_2)-(A_1 \cap A_3) = A_1 \cap(A_2 - A_3)$,

$H_3(\varepsilon)=H_1(\varepsilon)=F_1(\varepsilon)\cap F_2(\varepsilon)$

Case (ii)

When $\varepsilon \in C_2 - C_1 = (A_1 \cap A_3)-(A_1 \cap A_2) = A_1 \cap(A_3 - A_2)$,

$H_3(\varepsilon)=H_2(\varepsilon)=F_1(\varepsilon)\cap F_3(\varepsilon)$

Case (iii)

When $\varepsilon \in C_1 \cap C_2 = (A_1 \cap A_2)\cap(A_1 \cap A_3) = A_1 \cap A_2 \cap A_3$,

$$
\begin{aligned}
H_3(\varepsilon) &= H_1(\varepsilon)\cup H_2(\varepsilon) \\
&= (F_1(\varepsilon)\cap F_2(\varepsilon))\cup(F_1(\varepsilon)\cap F_3(\varepsilon)) \\
&= F_1(\varepsilon)\cap(F_2(\varepsilon)\cup F_3(\varepsilon))
\end{aligned}
$$

Let $(F_2, A_2)\tilde{\vee}(F_3, A_3)=(H_4, C_4)$ and $(F_1, A_1)\tilde{\wedge}(H_4, C_4)=(H_5, C_5)$

Where $C_4 = A_2 \cup A_3$,

$$
\begin{aligned}
C_5 = A_1 \cap C_4 = A_1 \cap(A_2 \cup A_3) &= A_1 \cap((A_2 - A_3)\cup(A_3 - A_2)\cup(A_2 \cap A_3)) \\
&= (A_1 \cap(A_2 - A_3))\cup(A_1 \cap(A_3 - A_2))\cup(A_1 \cap(A_2 \cap A_3)) \\
&= C_3
\end{aligned}
$$

When $\varepsilon \in A_2 - A_3$, $H_4(\varepsilon)=F_2(\varepsilon)$

$\varepsilon \in A_3 - A_2$, $H_4(\varepsilon)=F_3(\varepsilon)$

$\varepsilon \in A_2 \cap A_3$, $H_4(\varepsilon)=F_2(\varepsilon)\cup F_3(\varepsilon)$

Case (i)

$\forall \varepsilon \in C_5$, When $\varepsilon \in A_1 \cap (A_2 - A_3)$,

$H_5(\varepsilon) = F_1(\varepsilon) \cap H_4(\varepsilon)$

$= F_1(\varepsilon) \cap F_2(\varepsilon)$ (When $\varepsilon \in A_2 - A_3$, $H_4(\varepsilon) = F_2(\varepsilon)$)

$= H_3(\varepsilon)$

Case (ii)

$\forall \varepsilon \in C_5$, When $\varepsilon \in A_1 \cap (A_3 - A_2)$,

$H_5(\varepsilon) = F_1(\varepsilon) \cap H_4(\varepsilon)$

$= F_1(\varepsilon) \cap F_3(\varepsilon)$ (When $\varepsilon \in A_3 - A_2$, $H_4(\varepsilon) = F_3(\varepsilon)$)

$= H_3(\varepsilon)$

Case (iii)

$\forall \varepsilon \in C_5$, When $\varepsilon \in A_1 \cap (A_2 \cap A_3)$,

$H_5(\varepsilon) = F_1(\varepsilon) \cap H_4(\varepsilon)$

$= F_1(\varepsilon) \cap (F_2(\varepsilon) \cup F_3(\varepsilon))$ (When $\varepsilon \in A_2 \cap A_3$, $H_4(\varepsilon) = F_2(\varepsilon) \cup F_3(\varepsilon)$)

$= H_3(\varepsilon)$

It is clear from the above that

$(F_1, A_1) \tilde{\wedge} ((F_2, A_2) \tilde{\vee} (F_3, A_3)) = ((F_1, A_1) \tilde{\wedge} (F_2, A_2)) \tilde{\vee} ((F_1, A_1) \tilde{\wedge} (F_3, A_3))$

2) This can be proved in a similar way.

Theorem: 1.27

Let (F_1, A_1) and (F_2, A_2) be three fuzzy soft sets in a soft class (U, E).

1) $(F_1, A_1) \tilde{\wedge} (F_2, A_2) \tilde{\subseteq} (F_1, A_1)$, $(F_1, A_1) \tilde{\wedge} (F_2, A_2) \tilde{\subseteq} (F_2, A_2)$

2) $(F_1, A_1) \tilde{\subseteq} (F_1, A_1) \tilde{\vee} (F_2, A_2)$, $(F_2, A_2) \tilde{\subseteq} (F_1, A_1) \tilde{\vee} (F_2, A_2)$

Proof:

1) Let $(F_1, A_1) \tilde{\wedge} (F_2, A_2) = (H, C)$, where $C = A_1 \cap A_2$

   And $\forall \varepsilon \in C$, $H(\varepsilon) = F_1(\varepsilon) \cap F_2(\varepsilon)$

   Now, $C = A_1 \cap A_2 \subseteq A_1$ and

   $\forall \varepsilon \in C = A_1 \cap A_2, H(\varepsilon) = F_1(\varepsilon) \cap F_2(\varepsilon) \leq F_1(\varepsilon)$

   Thus $(F_1, A_1) \tilde{\wedge} (F_2, A_2) \tilde{\subseteq} (F_1, A_1)$.

   The other result can also be proved in a similar way.

2) Let $(F_1, A_1) \tilde{\vee} (F_2, A_2) = (H, C)$, where $C = A_1 \cup A_2$

   And $\forall \varepsilon \in C_1$, $H_2(\varepsilon) = \begin{cases} F_1(\varepsilon), & \text{if } \varepsilon \in A_1 - A_2 \\ F_2(\varepsilon), & \text{if } \varepsilon \in A_2 - A_1 \\ F_1(\varepsilon) \cup F_2(\varepsilon), & \text{if } \varepsilon \in A_1 \cap A_2 \end{cases}$

   Now, $A_1 \subseteq C$ and

   $\forall \varepsilon \in A_1$, $H(\varepsilon) = \begin{cases} F_1(\varepsilon), & \text{if } \varepsilon \in A_1 - A_2 \\ F_1(\varepsilon) \cup F_2(\varepsilon), & \text{if } \varepsilon \in A_1 \cap A_2 \end{cases}$

   Thus $\forall \varepsilon \in A_1$, $H(\varepsilon) \leq F_1(\varepsilon)$

   Thus $(F_1, A_1) \tilde{\subseteq} (F_1, A_1) \tilde{\vee} (F_2, A_2)$.

   The other result can also be proved in a similar way.

Theorem: 1.28

Let (F_1, A_1) and (F_2, A_2) be three fuzzy soft sets in a soft class (U, E).

1) $(F_1, A_1) \tilde{\subseteq} (F_2, A_2) \Rightarrow (F_1, A_1) \tilde{\wedge} (F_2, A_2) = (F_1, A_1)$
2) $(F_1, A_1) \tilde{\subseteq} (F_2, A_2) \Rightarrow (F_1, A_1) \tilde{\vee} (F_2, A_2) = (F_2, A_2)$

Proof:

1) Let $(F_1, A_1) \tilde{\subseteq} (F_2, A_2)$.

   Then $A_1 \subseteq A_2$ and $\forall \varepsilon \in A_1$, $F_1(\varepsilon) \leq F_2(\varepsilon)$

Now, let $(F_1, A_1)\tilde{\wedge}(F_2, A_2) = (H, C)$.

Then $C = A_1 \cap A_2 = A_1$, as $A_1 \subseteq A_2$

And $\forall \varepsilon \in C = A_1$, $H(\varepsilon) = F_1(\varepsilon) \cap F_2(\varepsilon) = F_1(\varepsilon)$ as $\forall \varepsilon \in A_1$, $F_1(\varepsilon) \leq F_2(\varepsilon)$

Thus $(F_1,A_1)\tilde{\subseteq}(F_2,A_2) \Rightarrow (F_1,A_1)\tilde{\wedge}(F_2,A_2) = (F_1,A_1)$

2) Let $(F_1, A_1)\tilde{\subseteq}(F_2, A_2)$.

Then $A_1 \subseteq A_2$ and $\forall \varepsilon \in A_1$, $F_1(\varepsilon) \leq F_2(\varepsilon)$

Now, let $(F_1, A_1)\tilde{\vee}(F_2, A_2) = (H, C)$.

Then $C = A_1 \cup A_2 = A_2$, as $A_1 \subseteq A_2$

And $\forall \varepsilon \in C = A_2$, $H(\varepsilon) = F_1(\varepsilon) \cup F_2(\varepsilon) = F_2(\varepsilon)$ as $\forall \varepsilon \in A_1$, $F_1(\varepsilon) \leq F_2(\varepsilon)$

Thus $(F_1,A_1)\tilde{\subseteq}(F_2,A_2) \Rightarrow (F_1,A_1)\tilde{\vee}(F_2,A_2) = (F_2,A_2)$

Theorem: 1.29 De Morgan Inclusions

For fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class (U, E), one has the following:

1) $(F, A)^C \tilde{\wedge} (G, B)^C \tilde{\supseteq} [\ (F, A) \tilde{\vee} (G, B)\ ]^C$

2) $[\ (F, A) \tilde{\wedge} (G, B)\ ]^C \tilde{\subseteq} (F, A)^C \tilde{\vee} (G, B)^C$

Proof:

1) Let $(F, A) \vee (G, B) = (H, C)$,

where $C = A \cup B$ and

$$\forall \varepsilon \in C,\ H(\varepsilon) = \begin{cases} F(\varepsilon) & if\ \varepsilon \in A - B \\ G(\varepsilon) & if\ \varepsilon \in B - A \\ F(\varepsilon) \cup G(\varepsilon) & if\ \varepsilon \in A \cap B \end{cases}$$

Thus $[\ (F, A) \vee (G, B)\ ]^C = (H, C)^C$

$= (H^C, C)$, where $C = A \cup B$ and

$= \left(H(\varepsilon)\right)^C$

$$\forall \varepsilon \in C,\ H^C(\varepsilon) = \begin{cases} (F(\varepsilon))^C & if\ \varepsilon \in A-B \\ (G(\varepsilon))^C & if\ \varepsilon \in B-A \\ (F(\varepsilon) \cup G(\varepsilon))^C & if\ \varepsilon \in A \cap B \end{cases}$$

$$= \begin{cases} F^C(\varepsilon) & if\ \varepsilon \in A-B \\ G^C(\varepsilon) & if\ \varepsilon \in B-A \\ (F(\varepsilon))^C \cap (G(\varepsilon))^C & if\ \varepsilon \in A \cap B \end{cases}$$

Again, $(F,A)^C \wedge (G,B)^C = (F^C,A) \wedge (G^C,B) = (I,J)$, say, where $J = A \cap B$ and

$\forall \varepsilon \in J,\ I(\varepsilon) = F^C(\varepsilon) \cap G^C(\varepsilon)$

We see that $J \subseteq C$ and $\forall \varepsilon \in J,\ I(\varepsilon) \subseteq H^C(\varepsilon)$

It follows immediately that $[\ (F,A) \tilde{\wedge} (G,B)\ ]^C \tilde{\supseteq} (F,A)^C \tilde{\vee} (G,B)^C$

2) Let $(F,A) \wedge (G,B) = (H,C)$, where $C = A \cap B$ and $\forall \varepsilon \in C,\ H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$

Thus $[\ (F,A) \wedge (G,B)\ ]^C = (H,C)^C$

$= (H^C, C)$, where $C = A \cup B$ and

$$\forall \varepsilon \in C,\ H^C(\varepsilon) = (F(\varepsilon) \cap G(\varepsilon))^C$$

$$= (F(\varepsilon))^C \cup (G(\varepsilon))^C$$

$$= F^C(\varepsilon) \cup G^C(\varepsilon)$$

Again, $(F,A)^C \vee (G,B)^C = (F^C,A) \vee (G^C,B) = (I,J)$, say, where $J = A \cup B$ and

$$\forall \varepsilon \in J,\ I(\varepsilon) = \begin{cases} F^C(\varepsilon) & if\ \varepsilon \in A-B \\ G^C(\varepsilon) & if\ \varepsilon \in B-A \\ (F(\varepsilon))^C \cup (G(\varepsilon))^C & if\ \varepsilon \in A \cap B \end{cases}$$

We see that $C \subseteq J$ and $\forall \varepsilon \in C,\ H^C(\varepsilon) \subseteq I(\varepsilon)$

It follows that $[\ (F,A) \tilde{\wedge} (G,B)\ ]^C \tilde{\subseteq} (F,A)^C \tilde{\vee} (G,B)^C$

Theorem: 1.30 De Morgan Laws

For fuzzy soft sets (F,A) and (G,A) in a fuzzy soft class (U, E), one has the following:

1) $[\ (F,A)\vee(G,A)\ ]^C=(F,A)^C\wedge(G,A)^C$

2) $[\ (F,A)\tilde{\wedge}(G,A)\ ]^C=(F,A)^C\tilde{\vee}(G,A)^C$

Proof:

1) Let $(F,A)\vee(G,A)=(H,A)$, where $\forall\varepsilon\in A,\ H(\varepsilon)=F(\varepsilon)\cup G(\varepsilon)$

   Thus $\left((F,A)\vee(G,A)\right)^C=(H,A)^C=(H^C,A)$,

   Where $\forall\varepsilon\in A,\ H^C(\varepsilon)=\left(H(\varepsilon)\right)^C$

$$=\left(F(\varepsilon)\cup G(\varepsilon)\right)^C$$

$$=\left(F(\varepsilon)\right)^C\cap\left(G(\varepsilon)\right)^C$$

$$=F^C(\varepsilon)\cap G^C(\varepsilon)$$

   Again, $(F,A)^C\wedge(G,A)^C=(F^C,A)\wedge(G^C,A)=(I,A)$, say,

   Where $\forall\varepsilon\in A,\ I(\varepsilon)=F^C(\varepsilon)\cap G^C(\varepsilon)$

   Thus $[\ (F,A)\vee(G,A)\ ]^C=(F,A)^C\wedge(G,A)^C$

2) Let $(F,A)\wedge(G,A)=(H,A)$, where $\forall\varepsilon\in A,\ H(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$

   Thus $\left((F,A)\wedge(G,A)\right)^C=(H,A)^C=(H^C,A)$,

   Where $\forall\varepsilon\in A,\ H^C(\varepsilon)=\left(H(\varepsilon)\right)^C$

$$=\left(F(\varepsilon)\cap G(\varepsilon)\right)^C$$

$$=\left(F(\varepsilon)\right)^C\cup\left(G(\varepsilon)\right)^C$$

$$=F^C(\varepsilon)\cup G^C(\varepsilon)$$

   Again, $(F,A)^C\vee(G,A)^C=(F^C,A)\vee(G^C,A)=(I,A)$, say, Where $\forall \varepsilon \in A,\ I(\varepsilon) = F^C(\varepsilon) \cup G^C(\varepsilon)$

Thus $[\ (F,A) \wedge (G,A)\ ]^C = (F,A)^C \vee (G,A)^C$

Theorem: 1.31

For fuzzy soft sets (F,A) and (G,A) in a fuzzy soft class (U, E), one has the following:

1) $\left((F,A) \wedge (G,B)\right)^C = (F,A)^C \vee (G,B)^C$

2) $\left((F,A) \vee (G,B)\right)^C = (F,A)^C \wedge (G,B)^C$

Proof:

1) Let $(F,A) \wedge (G,B) = (H, A \times B)$, where $H(\alpha,\beta) = F(\alpha) \cap G(\beta),\ \forall \alpha \in A$ and $\forall \beta \in B$, where $\cap$ is the operation 'fuzzy intersection' of two fuzzy sets.

   Thus $\left((F,A) \wedge (G,B)\right)^C = (H, A \times B)^C = (H^C, A \times B)$ where $\forall (\alpha,\beta) \in A \times B$,

   $$H^C(\alpha,\beta) = \left(H(\alpha,\beta)\right)^C$$
   $$= \left(F(\alpha) \cap G(\beta)\right)^C$$
   $$= \left(F(\alpha)\right)^C \cup \left(G(\beta)\right)^C$$
   $$= F^C(\alpha) \cup G^C(\beta)$$

   Let $(F,A)^C \vee (G,B)^C = (F^C, A) \vee (G^C, B) = (O, A \times B)$,

   where $O(\alpha,\beta) = F^C(\alpha) \cup G^C(\beta),\ \forall \alpha \in A$ and $\forall \beta \in B$, where $\cup$ is the operation 'fuzzy union' of two fuzzy sets.

   Thus $\left((F,A) \wedge (G,B)\right)^C = (F,A)^C \vee (G,B)^C$

2) Let $(F,A) \vee (G,B) = (H, A \times B)$, where $H(\alpha,\beta) = F(\alpha) \cup G(\beta),\ \forall \alpha \in A$ and $\forall \beta \in B$, where $\cup$ is the operation 'fuzzy union' of two fuzzy sets.

   Thus $\left((F,A) \vee (G,B)\right)^C = (H, A \times B)^C = (H^C, A \times B)$ where $\forall (\alpha,\beta) \in A \times B$,

$H^C(\alpha,\beta) = (H(\alpha,\beta))^C$

$= (F(\alpha) \cup G(\beta))^C$

$= (F(\alpha))^C \cap (G(\beta))^C$

$= F^C(\alpha) \cap G^C(\beta)$

Let $(F,A)^C \wedge (G,B)^C = (F^C,A) \wedge (G^C,B) = (O, A\times B)$,

where $O(\alpha,\beta) = F^C(\alpha) \cap G^C(\beta)$, $\forall \alpha \in A$ and $\forall \beta \in B$, where $\cap$ is the operation 'fuzzy intersection' of two fuzzy sets.

Thus $((F,A) \vee (G,B))^C = (F,A)^C \wedge (G,B)^C$

Definition: 1.32

Let $\Im = \{(F_i, A_i) / i \in I\}$ be a family of fuzzy soft sets in a fuzzy soft class (U, E). Then the **Union of Fuzzy Soft Sets** in $\Im$ is a fuzzy soft set (H, C), where $C = \underset{i}{\cup} A_i$ and for all $\varepsilon \in C$, $H(\varepsilon) = \underset{i}{\vee} \Delta_i(\varepsilon, A_i)$ where $\Delta_i(\varepsilon, A_i) = \begin{cases} F_i(\varepsilon), & if\ \varepsilon \in A_i \\ \varphi, & if\ \varepsilon \notin A_i \end{cases}$

Example: 1.33

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E ={e_1 (Costly),e_2 (Beautiful),e_3 (Fuel Efficient),e_4 (Luxurious)} be the set of parameters and A_1 = {e_1, e_2, e_3} $\subseteq$ E, A_2 = {e_1, e_4} $\subseteq$ E, A_3 = {e_1, e_2, e_3, e_4} $\subseteq$ E. We consider three fuzzy soft sets (F_1, A_1), (F_2, A_2) and (F_3, A_3) as follows.

(F_1, A_1) = { $F_1(e_1)$= { c_1/0.7, c_2/0.1, c_3/0.2, c_4/0.6 } ,

$F_1(e_2)$= { c_1/0.8, c_2/0.6, c_3/0.1, c_4/0.5 } ,

$F_1(e_3)$= { c_1/0.1, c_2/0.2, c_3/0.7, c_4/0.3 } }

(F_2, A_2) = { $F_2(e_1)$= { $c_1/0.4$, $c_2/0.6$, $c_3/0.1$, $c_4/0.6$ } ,

$F_2(e_4)$= { $c_1/0.8$, $c_2/0.1$, $c_3/0.4$, $c_4/0.5$ } }

(F_3, A_3) = { $F_3(e_1)$= { $c_1/0.2$, $c_2/0.3$, $c_3/0.5$, $c_4/0$ } ,

$F_3(e_2)$= { $c_1/0.3$, $c_2/0.9$, $c_3/0.5$, $c_4/0.5$ } ,

$F_3(e_3)$= { $c_1/0.1$, $c_2/0.2$, $c_3/0.6$, $c_4/0.3$ } ,

$F_4(e_4)$= { $c_1/0.8$, $c_2/0.3$, $c_3/0.5$, $c_4/0.1$ } }

Thus $(F_1, A_1)\tilde{\vee}(F_2, A_2)\tilde{\vee}(F_3, A_3) = (H, C)$,

Where $C = A_1 \cup A_2 \cup A_3$ = { e_1, e_2, e_3, e_4 } and

(H, C) = { $H(e_1)$= { $c_1/0.7$, $c_2/0.6$, $c_3/0.5$, $c_4/0.6$ } ,

$H(e_2)$= { $c_1/0.8$, $c_2/0.9$, $c_3/0.5$, $c_4/0.5$ } ,

$H(e_3)$= { $c_1/0.1$, $c_2/0.2$, $c_3/0.7$, $c_4/0.3$ } ,

$H(e_4)$= { $c_1/0.8$, $c_2/0.3$, $c_3/0.5$, $c_4/0.5$ } }

Definition: 1.34

Let $\Im = \{(F_i, A_i) / i \in I\}$ be a family of fuzzy soft sets in a fuzzy soft class (U, E), with $\bigcap_i A_i \neq \varphi$. Then the **Intersection of Fuzzy Soft Sets** in $\Im$ is a fuzzy soft set (H, C), where $C = \bigcap_i A_i$ and for all $\varepsilon \in C$, $H(\varepsilon) = \wedge_i F_i(\varepsilon)$.

Example: 1.35

Let U = { c_1, c_2, c_3, c_4 } be the set of four cars under consideration and E = { e_1 (Costly), e_2 (Beautiful), e_3 (Fuel Efficient), e_4 (Luxurious) } be the set of parameters and A_1={e_1, e_2, e_3}$\subseteq$E, A_2={e_1, e_4}$\subseteq$E, A_3={e_1, e_2, e_3, e_4}$\subseteq$E. We consider three fuzzy soft sets (F_1, A_1), (F_2, A_2) and (F_3, A_3) as follows.

(F_1, A_1) = { $F_1(e_1)$= { $c_1/0.7$, $c_2/0.1$, $c_3/0.2$, $c_4/0.6$ } ,

$F_1(e_2)$= { $c_1/0.8$, $c_2/0.6$, $c_3/0.1$, $c_4/0.5$ } ,

$F_1(e_3)$= { $c_1/0.1$, $c_2/0.2$, $c_3/0.7$, $c_4/0.3$ } }

(F_2, A_2) = { $F_2(e_1)$= { $c_1/0.4$, $c_2/0.6$, $c_3/0.1$, $c_4/0.6$ } ,

$F_2(e_4)$= { $c_1/0.8$, $c_2/0.1$, $c_3/0.4$, $c_4/0.5$ } }

(F_3, A_3) = { $F_3(e_1)$= { $c_1/0.2$, $c_2/0.3$, $c_3/0.5$, $c_4/0$ } ,

$F_3(e_2)$= { $c_1/0.3$, $c_2/0.9$, $c_3/0.5$, $c_4/0.5$ } ,

$F_3(e_3)$= { $c_1/0.1$, $c_2/0.2$, $c_3/0.6$, $c_4/0.3$ } ,

$F_4(e_4)$= { $c_1/0.8$, $c_2/0.3$, $c_3/0.5$, $c_4/0.1$ } }

Thus $(F_1, A_1) \tilde{\wedge} (F_2, A_2) \tilde{\wedge} (F_3, A_3) = (H, C)$, where $C = A_1 \cap A_2 \cap A_3$ = { e_1 } and

(H, C) = { $H(e_1)$= { $c_1/0.2$, $c_2/0.3$, $c_3/0.1$, $c_4/0$ } }

Theorem: 1.36

Let $\Im = \{(F_i, A_i) / i \in I\}$ be a family of fuzzy soft sets in a fuzzy soft class (U, E). Then $\forall\, i \in I$, $(F_i, A_i) \tilde{\subseteq} \tilde{\vee}_i (F_i, A_i)$

Proof:

Let $\tilde{\vee}_i (F_i, A_i) = (H, C)$, where $C = \cup_i A_i$ and $\forall \alpha \in C$, $i \in I$

$$H(\alpha) = \vee_i \Delta_i(\alpha, A_i) \text{ where } \Delta_i(\alpha, A_i) = \begin{cases} F_i(\alpha), & \text{if } \alpha \in A_i \\ \varphi, & \text{if } \alpha \notin A_i \end{cases}$$

Clearly $A_i \subseteq \cup_i A_i$ *i.e.* $A_i \subseteq C$ and $\forall \alpha \in A_i$, $i \in I$, $F_i(\alpha) \le \vee_i \Delta_i(\alpha, A_i)$ *i.e.* $F_i(\alpha) \le H(\alpha)$

Thus $\forall\, i \in I$, $(F_i, A_i) \tilde{\subseteq} \tilde{\vee}_i (F_i, A_i)$

Theorem: 1.37

Let $\Im = \{(F_i, A_i) / i \in I\}$ be a family of fuzzy soft sets in a fuzzy soft class (U, E). Then $\forall\, i \in I$, $\tilde{\wedge}_i (F_i, A_i) \tilde{\subseteq} (F_i, A_i)$

Proof:

Suppose that, $\tilde{\wedge}_i (F_i, A_i) = (H, C)$, where $C = \cap_i A_i$ and $\forall \alpha \in C, i \in I$

$H(\alpha) = \wedge_i F_i(\alpha)$

Now, $C = \cap_i A_i \subseteq A_i$ and $\forall \alpha \in C, i \in I$, $H(\alpha) = \wedge_i F_i(\alpha) \le F_i(\alpha)$

Thus $H(\alpha) \le F_i(\alpha)$ $\forall \alpha \in C$ and hence the result follows.

Theorem: 1.38

Let $\Im = \{(F_i, A_i) / i \in I\}$ be a family of fuzzy soft sets in a fuzzy soft class (U, E). Then one has the following

1. $\tilde{\wedge}_i (F_i, A)^C = \left(\tilde{\vee}_i (F_i, A)\right)^C$
2. $\left(\tilde{\wedge}_i (F_i, A)\right)^C = \tilde{\vee}_i (F_i, A)^C$

Proof:

1. We have, $\wedge_i (F_i, A)^C \;=\; \wedge_i (F_i^C, A) \;=\; (H, A)$, say

   Where $\forall\, \alpha \in A$, $H(\alpha) = \cap_i F_i^C(\alpha)$ (13)

   Again suppose that $\tilde{\vee}_i (F_i, A) = (I, A)$.

   Then $\left(\vee_i (F_i, A)\right)^C = (I, A)^C = (I^C, A)$, where $I^C(\alpha) = [I(\alpha)]^C = \left[\cup_i F_i(\alpha)\right]^C$

   $\forall \alpha \in A$, we have $I^C(\alpha) = \left[\cup_i F_i(\alpha)\right]^C = \cap_i F_i^C(\alpha)$ (14)

   From (13) and (14), we get the desired result.

2. We have, $\underset{i}{\vee}(F_i, A)^C = \underset{i}{\vee}(F_i^C, A) = (H, A)$, say

Where $\forall \alpha \in A,\ H(\alpha) = \underset{i}{\cup} F_i^C(\alpha)$ (15)

Again suppose that $\underset{i}{\tilde{\wedge}}(F_i, A) = (I, A)$.

Then $\left(\underset{i}{\wedge}(F_i, A) \right)^C = (I, A)^C = (I^C, A)$, where

$I^C(\alpha) = [I(\alpha)]^C = \left[\underset{i}{\cap} F_i(\alpha) \right]^C$

$\forall \alpha \in A$, we have $I^C(\alpha) = \left[\underset{i}{\cap} F_i(\alpha) \right]^C = \underset{i}{\cup} F_i^C(\alpha)$ (16)

From (15) and (16), we get the desired result.