BULK ARRIVAL ADMISSIBILITY RESTRICTED QUEUEING SYSTEMS WITH MULTI OPTIONAL SECOND SERVICE AND BERNOULLI VACATION

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RAJESWARI, B
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A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM INSTITUTE FOR HOMESCIENCE AND HIGHER EDUCATION FOR WOMEN, COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

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Signature of the Guide

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CONTENTS

CHAPTER I

INTRODUCTION

“Delay is the enemy of efficiency and waiting is the enemy of utilization”

Congestion is a natural phenomenon in real responses. A service facility gets congested if there are more people than the server can possibly handle.

In designing queueing systems we need to aim for a balance between service to customers and economic considerations. Queueing theory was developed to provide models to predict the behavior of systems that attempt to provide service for randomly arising demands.

Queueing theory is the mathematical study of waiting lines or queues. In queuing theory a model is constructed so that queue lengths and waiting times can be predicted. A flow of customers from finite or infinite population towards the service facility forms a queue on account of lack of capability to serve them all at a time.

The arriving unit that requires some service to be performed is called ‘customer’. The customer may be persons, machines, vehicles and so on. Queue stands for the number of customers waiting to be served. This does not include the customer being serviced. The process or system that performs services to the customer is termed by service channel or service facility.

The objective of the queueing system is that the average waiting time of the customers is minimized and the percentage utilization of the server is maintained above the desired level.

APPLICATION AREAS OF QUEUEING SYSTEMS

- Bank counters
- Tollgate
- Ration shop
- Mainframe computer centre
- Library
- Traffic signal
- Final inspection station of television assembly line
- Airport runways
- Telephone booth
- Maintenance shop

CHARACTERISTICS OF QUEUEING PROCESS

Queueing system can be completely described by

1. Input pattern or arrival process of customers
2. Service time pattern

3. Service discipline
4. Number of servers
5. System capacity
6. Service stages

1. Input pattern or arrival process of customers

The arrival process describes the way in which the customers arrive and join the system. It is specified by the interarrival time between any two consecutive arrivals. Usually the interarrival times are assumed to follow a common distribution and are independent of each other. The input pattern indicates the behavior of the customers when arriving at the service system. Some customers may wait for a long time patiently, other customers are less patient after a while. It is also very important to know if the customers arrive in batches or one by one.

2. Service time patterns

The pattern of service times is the manner in which the service is rendered. It is specified by the time taken to complete a service. It is assumed usually that the service times follow a common distribution and are independent of each other and independent of the interarrival time. The most common distributions that the service times have are deterministic and exponential distributions. Service times may also be dependent on the queue length.

3. Service discipline

The service discipline indicates the manner in which the units are taken from the queue and allowed into service. Customers may also be served in groups or one by one. The most known disciplines are:

- **First In First Out (FIFO)**

The usual queue discipline is First Come First Served (FCFS), a customer that finds the service centre busy goes to the end of the queue.

- **Last In First Out (LIFO)**

Last Come First Served (LCFS), a customer that finds the service Centre busy proceeds immediately to the head of the queue, this customer will be served next, given that no further customers arrive.

- **Service In Random Order (SIRO)**

Also called (SIRO), customers are selected for service at random, irrespective of their arrivals in the service system. Every customer in the queue is equally likely to be selected. In this case the time of arrival of the customers is therefore, of no relevance.

- **Priority disciplines**

Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency according to some identifiable characteristic and FCFS rule is used within each class to provide service.

Treatment of VIP's in preference to other patients in a hospital is an example of priority service.

4. Number of servers

A system may have single server or a group of servers providing service to the customers. Increasing the number of service channels helps to decrease the waiting time. Given a number of service channels they may operate in parallel being able to serve customers simultaneously. It is generally assumed that the service mechanisms of the parallel channels operate independently of each other. An arrival who finds more than one free server may choose any one of them for receiving service. If he finds all the servers busy, he joins a queue common to all the servers, the first customer in the common queue goes to the server who becomes free first.

5. System capacity

This is the maximum number of customers allowed at any time in the system. A system may have an infinite capacity that is the queue in front of the server(s) may grow to any length; in this case the system is called a delay system. In some queuing processes there is a physical limitation to the amount of waiting room, so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available as a result of a service completion. These are referred to as finite queuing situations. The system has to be specified by the number of customers available, so that an arrival may not be able to join the system when the system is full; in this case the system is called a loss system.

6. Service stages

The customers may proceed through one stage or several stages to complete their service before departing the system. In the case of multistage queueing systems, the customer enter a queue waits for service, gets service and departs the service station to enter a new queue for another service, and so on. In some multistage queueing systems recycling or feedback may be allowed, this case is common in manufacturing processes, where parts that do not meet quality standards are sent back for reprocessing.

KENDALL'S NOTATION

Kendall's notation describe the arrival process to the queue, the nature of service process, the number of servers, maximum number in the queue and some basic queue disciplines. The notation has been considerably extended to allow it to represent a wide variety of queues. A queue is represented by a sequence A/B/C/D/E with the following meaning attached to the letters A to E.

A : This symbolically represents the nature of arrival process to the queue. Special letters are used to symbolize the nature of interarrival time distribution as follows:

M Exponentially distributed Markovian interarrival times

D Deterministic interarrival time

E_k Erlang distribution of order k for the interarrival time

G General distribution for the interarrival time

B : This symbolically represents the nature of service time distribution for the customers getting served in the queue. Here, the symbols are similar to interarrival time distribution

C : Number of servers in the queue.

D : System capacity. The Default is infinity.

E : Queue discipline. Default is First Come First Served (FCFS).

VACATION

After a busy period, the server goes on vacation of random length. It examines the queue once again when it returns from the vacation. Queueing system with server vacations arises as models of many diverse fields such as computer, communication and production systems. The non-availability of a server at the system is termed as server's vacation. The purpose of leaving the system is manifold. The server may want to utilize his idle time for another task or if the server is a machine, it may need some repair after completing a job. An exhausted bank teller may like to take a coffee break before getting ready for the next customer. There are many type of vacations.

Single vacation

After a busy period ends, server goes on only one vacation. If system is still empty when in returns, the server stays and waits for a job to arrive. In N-policy system a server on completion of a service will start servicing again only if the system has at least the minimum number of customers required to start the service. Otherwise the server will withdraw from the system for a vacation. When the server returns from a vacation and finds less than the minimum number of customer required for service, stay in the system until the queue length reaches the minimum number. This is known as single vacation.

Multiple vacations

After completing the service, if the queue length is less than predetermined value say ‘a’, the server may leave for a vacation. After returning from vacation if the queue length is still less than ‘a’, he may leave for another vacation so on, until he finally finds at least ‘a’ customers wait for service. This type of vacation is known as multiple vacations.

Random Vacation

The non-availability of server irrespective of the number in the waiting line is considered as server’s vacation.

REVIEW OF LITERATURE

The Queueing theory had its origin in 1909, when Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic. After this early work, many authors like Feller, Kendall, Clarke, Bailey, Morse, Satty have contributed much to the growth of this literature.

The classical M/G/1 queueing system received a good deal of attention in the literature since Kendall (1953). He was the first to study such a system through an imbedded Markov chain technique by using the concept of regeneration points. Later, Cox (1955) provided an analytical treatment of the same system using supplementary variables technique. Considerable efforts have been devoted to study this system by Saaty (1962), Takacs (1963), Prabhu (1965), Kleinrock (1975), Cohen (1982) and Medhi (1994), Keilson and Kooharian (1960), Bhat (1964), Chaudhry and Templeton (1981), Choi and Park (1990) and Madan (1991,1992).

Queueing systems with server vacations have been attracted much attention to numerous researchers since Levy and Yechiali (1976). One of excellent surveys of queueing systems with server vacations can be referred to

Doshi (1986) and Takagi (1991), which includes some applications. Bernoulli vacations were studied by many authors including Keilson and Servi (1986), Ramasamy and Servi (1988), Doshi (1986) and Takagi (1991)..

Gelenbe and Mitrani (1980) used vacation queueing models to analyse some real life systems such as digital communications, computer network. Choudhury (2002) modeled a batch arrival $M^x/G/1$ queueing system with a single vacation policy which extended the results of Doshi and Takagi. Batch arrival M/G/1 queueing systems with multiple vacations were first studied by Baba (1986). The variations and extensions of these models can be referred to Rosenberg and Yechilai (1993), Lee et al., (1995). Choudhury (2002), Shomrony and Yechiali (2001), Tang and Tang, Tang et al (2000) and many others. Recently, Ke (2003) analysed the optimal policy for M/G/1 queueing system with different vacation types and a startup time. Yechiali (2004) examined an $M^x/G/1$ queueing system with a waiting server and vacations, in which the server, upon finding an empty system at the end of a vacation, activates a timer of duration T and waits dormant. Meanwhile the server operates the following policy: if a batch arrives during the dormant period T and then a new busy period starts, but if no arrivals occur, the server waits no more and takes another vacation.

In day to day life there are numerous examples arise in M/G/1 queue with second optional service, where the server provides the first essential service to all the arriving customers. As soon as the first essential service of a customer is completed it may leave the system with certain probability or may immediately opt for a second optional service with complementary probability. Medhi (2002) have considered the M/G/1 model with second optional service. Choudhury (2003), Kalyanaraman and Pazhani Bala Murugan (2008) have analyzed batch arrival queueing system with an additional service channel.

The following are few examples for the system with second optional service.

- At a Barber's shop every one may need a hair-cut but only a part of the customers may need a shave after their hair-cut.
- In a small town one finds many shops which sell coffee beans and grains of various kinds. All such shop-keepers normally have a grinding machine. All customers coming to such a shop buy grains or coffee beans but only some of these customers want to utilize the grinding facility.
- All ships arriving as a port may need unloading service on arrival but only some of them may require re-loading service soon after the unloading.
- All clients who come to meet a lawyer discuss their cases with her/ him but only some of them actually hire her/ him to file their cases in a court of law.

Queueing systems with batch arrival are common in number of real situations. In computer communication systems, message which is to be transmitted could consist of a random number of packets. Choudhury and Templeton (1983) provide a comprehensive review on bulk queue and their application. Baba (1986), Choudhury (2000), Choudhury and Borthakur

(2000), Lee and Srinivasan (1989), Lee et al (1994), Rosenberg and Yechiali (1993), and Teghem (1990) and many others have studied batch arrival. In batch arrival queues another condition such as admissibility restricted policy was studied by some authors in which not all arriving batches are allowed to join the system at all times. Madan and Abu-Dayyeh (2002, 2003), Madan and Choudhury (2004), Alnowibet and Tadj (2007), Choudhury (2008) have discussed bulk queue with restricted admissibility of arriving batches with different vacation policies.

PROFILE OF WORK

Introduction and Review of literature are presented in chapter 1.

The paper “An M^x / (G_1, G_2) / G (BS) / V_s Optional Second Service and Admissibility Restricted” presented by A. Badamchi Zadeh (2009) is extended to multi optional second service in chapter 2.

In chapter 3, Steady state analysis of an $M^X / \begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{1B} \end{pmatrix} / 1$ queue with restricted admissibility of arriving batches and modified Bernoulli schedule server vacations based on a single vacation policy presented by Madan (2010) is analysed.

CHAPTER 2

BATCH ARRIVAL ADMISSIBILITY RESTRICTED QUEUEING SYSTEM WITH VACATION AND MULTI OPTIONAL SECOND SERVICE

In this chapter a single server queue with batch arrival Poisson input and two phases of heterogeneous services is studied. The first phase of service is essential for all customers and the second phase is optional. There is a restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. After completion of first phase or second phase of service, the server either goes for a vacation or may continue to serve the next unit. The steady- state equations and probability generating functions of the system are obtained. The Mean queue size, Mean waiting time and the actual arrival rate are derived.

Model Description

Customers arrive at the system in a compound Poisson process with batch of random size X and mean rate $\lambda>0$. Let a_k = Prob (X = k) and $E(X^k)$ be the k^{th} factorial moment of X. The server provides two phases of heterogeneous service in succession. The service discipline is assumed to be on the basis of first come, first served. The first essential service is needed to all arriving customers, the service time has general distribution. Its distribution function, density function, the first two moments are B(x), b(x), E(B) and $E(B^2)$ respectively.

The hazard rate function of B is $\mu(x) = \frac{b(x)}{1-B(x)}$ (1)

The second phase has m options. Each customer can choose first option

with probability of r_1 and the second option with probability r_2 and so on. As soon

as the first service is completed, the customer opts for second phase with probability $\sum_{I=1}^{m} r_i$ or leaves the system with probability $r_0 = 1 - \sum_{i=1}^{m} r_i$. The m-option second service time follows an arbitrary distribution, and its distribution function, density function, the first two moments are $B_i(x)$, $b_i(x)$,$E(B_i)$ and $E(B_i^2)$ for i=1,2…m respectively.

The hazard rate function of B is $\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$ (2)

Under the policy of restricted admissibility, not all batches are allowed to join the system at all times. Let α and β are respectively the probabilities that an arriving batch will be allowed to join the system during the period of server's non-vacation and vacation period respectively. As soon as the completion of each phase, the server may go for a vacation of random length V with probability θ or may continue to serve the next customer, if any, with probability (1- θ), otherwise the server remains in the system and waits for a new arrival. The vacation service time follows an arbitrary distribution and its distribution function, density function, the first two moments are V(x), v(x), E(V) and $E(V^2)$.

The hazard rate function of V is $\nu(x) = \dfrac{v(x)}{1 - V(x)}$ (3)

Let N_S (t) to be the system size at time t. Define the following probabilities

1. $R_0(t)$ is the probability that the server is idle and there is no customers in the system.

2. $Q_n(x,t)$ is the probability that at time t the server is on vacation and there are n customers in the system with elapsed vacation time between x and dx.

3. $P_{1,n}(x,t)$ is the probability that the server is being served in the first phase service and the elapsed service time is between x and dx.

4. $P_{2,i,n}(t)$ is the probability that at time t there are n customers in the system, the server is being served in the second phase i^{th} optional service and the elapsed service time is between x and dx.

Governing equations

The set of differential-difference equations that governs the system are given below.

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)P_{1,n}(x,t)+\left(\lambda\alpha+\mu(x)\right)P_{1,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k\,P_{1,n-k}(x,t) \qquad n \geq 1$$

(4) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)P_{2,i,n}(x,t)+\left(\lambda\alpha+\mu(x)\right)P_{2,i,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k P_{2,i,n-k}(x,t) \qquad n \geq 1$$

(5) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)Q_n(x,t)+\left(\lambda\beta+\nu(x)\right)Q_n(x,t)=\lambda\beta\sum_{k=1}^{n}a_k\,Q_{n-k}(x,t)$$

n ≥ 1 (6) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)Q_0(x,t)+\left(\lambda\beta+\nu(x)\right)Q_0(x,t)=0$$

(7)

$$\frac{d}{dt}R_0(t)=-\lambda\alpha R_0+(1-\theta)\,r_0\int_0^\infty P_{1,1}(x,t)\mu(x)dx+(1-\theta)\int_0^\infty\sum_{i=1}^{m}P_{2,i,1}(x,t)\mu_i(x)dx + \int_0^\infty Q_0(x,t)\nu(x)dx \qquad (8)$$

with the boundary conditions

$$P_{1,n}(0,t)=\lambda\alpha a_n R_0+(1-\theta)\,r_0\int_0^\infty P_{1,n+1}(x,t)\,\mu(x)dx+(1-\theta)\int_0^\infty\sum_{i=1}^{m}P_{2,i,n+1}(x,t)\mu_i(x)dx + \int_0^\infty Q_n(x,t)\nu(x)dx \qquad n \geq 1 \qquad (9)$$

$$P_{2,i,n}(0,t) = r_i \int_0^{\infty} P_{1,n}(x,t)\mu(x)\,dx \quad n \geq 1, \qquad i=1,2,\ldots,m \qquad (10)$$

$$Q_n(0,t) = \theta\, r_0 \int_0^{\infty} P_{1,n+1}(x,t)\,\mu(x)\,dx + \theta \int_0^{\infty} \sum_{i=1}^{m} r_i\, P_{2,\,i\,,n+1}(x\,,t)\,\mu_i(x)\,dx \quad n \geq 1 \qquad (11)$$

Steady- State Probability Generating Functions

The steady state equations corresponding to equations (4) to (11) are

$$\frac{\partial}{\partial x}P_{1,n}(x) = -\left(\lambda\alpha + \mu(x)\right)P_{1,n}(x) + \lambda\alpha\sum_{k=1}^{n} a_k\, P_{1,\,n\ \,k}(x) \qquad n \geq 1$$

$$(12)\ \frac{\partial}{\partial x}P_{2,i,n}(x) = -\left(\lambda\alpha + \mu_i(x)\right)P_{2,i,n}(x) + \lambda\alpha\sum_{k=1}^{n} a_k\, P_{2,i,n\ \,k}(x) \qquad n \geq 1$$

$$(13)\ \frac{\partial}{\partial x}Q_n(x) = -\left(\lambda\beta + \nu(x)\right)Q_n(x) + \lambda\beta\sum_{k=1}^{n} a_k\, Q_{n\ \,k}(x) \qquad n \geq 1$$

$$(14)\ \frac{\partial}{\partial x}Q_0(x) = -\left(\lambda\beta + \nu(x)\right)Q_0(x) \qquad (15)$$

$$\lambda\alpha R_0 = (1-\theta)\, r_0 \int_0^{\infty} P_{1,1}(x)\,\mu(x)\,dx + (1-\theta)\int_0^{\infty}\sum_{i=1}^{m} P_{2,i,1}(x)\mu_i(x)\,dx + \int_0^{\infty} Q_0(x)\,\nu(x)\,dx \qquad (16)$$

$$P_{1,n}(0) = \lambda\alpha a_n R_0 + (1-\theta)\, r_0 \int_0^{\infty} P_{1,n+1}(x)\,\mu(x)\,dx + (1-\theta)\int_0^{\infty}\sum_{i=1}^{m} P_{2,i,n+1}(x)\mu_i(x)\,dx + \int_0^{\infty} Q_n(x)\nu(x)\,dx, \quad n \geq 1 \qquad (17)$$

$$P_{2,i,n}(0) = r_i \int_0^\infty P_{1,n}(x)\mu(x)dx \qquad n \geq 1 \tag{18}$$

$$Q_n(0) = \theta\, r_0 \int_0^\infty P_{1,n+1}(x)\mu(x)\, dx + \theta \int_0^\infty \sum_{i=1}^{m} P_{2,i,n+1}(x)\mu_i(x)dx \qquad n \geq 0 \tag{19}$$

The normalizing condition is

$$R_0 + \sum_{n=0}^{\infty} \int_0^\infty P_{1,n}(x)\, dx + \sum_{i=1}^{m} \int_0^\infty P_{2,i,n}(x)\, dx + \sum_{n=0}^{\infty} Q_n(x)\, dx = 1 \tag{20}$$

By assuming the existence of the steady state, define

$$R_0 = \lim_{t\to\infty} R_0(t)$$

$$P_{1,n}(x) = \lim_{t\to\infty} P_{1,n}(x,t), \qquad n \geq 0$$

$$P_{2,i,n}(x) = \lim_{t\to\infty} P_{2,i,n}(x,t), \qquad n \geq 0,\ i = 1, 2\ldots, m$$

$$Q_n(x) = \lim_{t\to\infty} Q_n(x,t), \qquad n \geq 0$$

Define the probability generating functions

$$P_1(x,z) = \sum_{n=1}^{\infty} P_{1,n}(x) z^n$$

$$P_{2,i}(x,z) = \sum_{n=1}^{\infty} P_{2,i,n}(x) z^n \qquad i = 1, 2, \ldots, m$$

$$Q(x,z)=\sum_{n=0}^{\infty} Q_n(x) z^n$$

$$A(z)=\sum_{n=1}^{\infty} a_n z^n$$

Multiplying equation (12) by z^n and summing over for n=1 to ∞, we get

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} P_{1,n}(x) z^n = -\sum_{n=1}^{\infty}(\lambda\alpha+\mu(x))P_{1,n}(x)\, z^n + \lambda\alpha\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_k\, P_{1,n-k}(x) z^n$$

$$\frac{\partial}{\partial x}P_1(x,z) + P_1(x,z)\,(\lambda\alpha+\mu(x)-\lambda\alpha\, A(z))=0$$

Solving the above equation we get

$$P_1(x,z) = A\ e^{-\lambda\alpha\, [(1-A(z))x + \log(1-B(x)]}$$

$$= A\ (1-B(x))\ e^{-\lambda\alpha(1-A(z))\, x}$$

Putting x=0, we get, $A=P_1(0, z)$

Hence, $P_1(x,z) = P_1(0,z)(1-B(x))\ e^{-\lambda\alpha(1-A(z))x}$ (21)

Multiplying equation (13) by z^n and summing over for n=1 to ∞, we get

$$\frac{\partial}{\partial x}P_{2,i}(x,z)+P_{2,i}(x,z)(\lambda\alpha+\mu_i(x)-\lambda\alpha\, A(z))=0$$

Solving the above equation we get,

$$P_{2,i}(x,z) = P_{2,i}(0,z)(1-B_i(x))\,e^{-\lambda\alpha(1-A(z))x} \qquad (22)$$

Multiplying equation (14) by z^n and summing over for n=0 to ∞, we have

$$\frac{\partial}{\partial x}Q(x,z) + Q(x,z)\,[\lambda\beta + \nu(x) - \lambda\beta\, A(z)] = 0$$

Solution of equation is given by

$$Q(x, z) = Q\,(0, z)\,(1-\nu(x))\,e^{-\lambda\beta(1-A(z))\,x}, \qquad (23)$$

Multiplying equation (17) by z^n and summing over for n=1 to ∞, we obtain

$$\sum_{n=1}^{\infty} P_{1,n}(0)\,z^n = \sum_{n=1}^{\infty} \lambda\alpha a_n\, z^n R_0 + (1-\theta)\, r_0 \sum_{n=1}^{\infty}\int_0^{\infty} P_{1,n+1}(x)\,\mu(x)\,z^n\,dx$$

$$+ (1-\theta)\int_0^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{m} P_{2,i,n+1}(x)\,\mu_i(x)\,dx + \sum_{n=1}^{\infty}\int_0^{\infty} Q_n(x)\,\nu(x)\,z^n\,dx \qquad n \ge 1$$

Hence

$$z\,P_1(0,z) = z\,\lambda\alpha A(z)\,R_0 + (1-\theta)\,r_0\int_0^{\infty}\left(P_1(x,z) - z\,P_{1,1}(x)\right)\mu(x)\,dx$$
$$+ (1-\theta)\int_0^{\infty}\left(P_{2,i}(x,z) - z\,P_{2,i,1}(x)\right)\mu_i(x)\,dx + z\int_0^{\infty}\left(Q(x,z) - Q_0(x)\right)\nu(x)\,dx \qquad (24)$$

Multiplying equation (18) by z^n and summing over for n=1 to ∞ we obtain

$$P_{2,i}(0,z) = r_i\int_0^{\infty} P_1(x,z)\,\mu(x)\,dx \qquad (25)$$

Multiplying equation (19) by z^n and summing over for n=1 to ∞, we get

$$Q(0,z) = \theta\, r_0 \int_0^\infty P_1(x,z)\,\mu(x)\,dx + \theta \sum_{i=1}^{m} \int_0^\infty P_{2,i}(x,z)\,\mu_i(x)\,dx \tag{26}$$

Using equation (20) in equation (24), we get

$$z P_1(0,z) = z\lambda\alpha R_0\,(A(z)-1) + (1-\theta)\, r_0 \int_0^\infty P_1(x,z)\,\mu(x)\,dx$$

$$+ (1-\theta)\int_0^\infty \sum_{i=1}^{m} P_{2,i}(x,z)\,\mu_i(x)\,dx + z\int_0^\infty Q(x,z)\,\nu(x)\,dx$$

Substituting $P_1(x,z)$, $P_{2,i}(x,z)$ and $Q(x,z)$ in the above equation, we obtain

$$z P_1(0,z) = (1-\theta)\, r_0 \int_0^\infty P_1(0,z)\, e^{-\lambda\alpha(1-A(z))x}(1-B(x))\mu(x)dx$$

$$+ (1-\theta)\int_0^\infty \sum_{i=1}^{m} P_{2,i}(0,z)\, e^{-\lambda\alpha(1-A(z))x}(1-B_i(x))\mu(x)\,dx$$

$$+\int_0^\infty Q(0,z)\, e^{-\lambda\beta(1-A(z))x}(1-V(x))\nu(x)dx - \lambda\alpha(1-A(z))R_0$$

Using equations (1), (2) and (3) in the above equation, we get

$$z P_1(0,z) = (1-\theta)\, r_0 P_1(0,z)\, B^*[\lambda\alpha\,(1-A(z))] + (1-\theta)\sum_{i=1}^{m} P_{2,i}(0,z)\, B_i^*[\lambda\alpha\,(1-A(z))]$$

$$+ z\, Q(0,z)\, V^*[\lambda\beta\,(1-A(z))] - \lambda\alpha\,(1-A(z))\, R_0 \tag{27}$$

Substituting $P_1(x, z)$ in the equation (25), we have

$$P_{2,i}(0,z) = r_i \int_0^\infty P_1(0,z)\, e^{-\lambda\alpha(1-A(z))x} \left(1-B(x)\right) \mu(x)\, dx$$

Using equation (1) and integrating we get

$$P_{2,i}(0,z) = r_i\, P_1(0,z)\, B^*\left[\lambda\alpha\,(1-A(z))\right] \quad (28)$$

Substituting $P_1(x,z)$ and $P_{2,i}(x,z)$ in the equation (26) we get

$$Q(0,z) = \theta\, r_0 \int_0^\infty P_1(0,z)\, e^{-\lambda\alpha(1-A(z))x} (1-B(x))\, \mu(x) dx$$

$$+\theta \int_0^\infty \sum_{i=1}^{m} P_{2,i}(0,z) e^{-\lambda\alpha(1-A(z))x} (1-B_i(x))\, \mu_i(x) dx$$

$$= \theta\, r_0 P_1(0,z) B^*\left[\lambda\alpha\left(1-A(z)\right)\right]$$

$$+\theta \sum_{i=1}^{m} P_{2,i}(0,z) B^*\left[\lambda\alpha\left(1-A(z)\right)\right] B_i^*\left[\lambda\alpha\left(1-A(z)\right)\right] \quad (29)$$

Let us denote $B^*\left[\lambda\alpha\left(1-A(z)\right)\right]$ as B^*

$B_i^*\left[\lambda\alpha\left(1-A(z)\right)\right]$ as B_i^*

and $V^*\left[\lambda\beta(1-A(z))\right]$ as V^* for i=1,2.

Substituting equations (28) and (29) in the equation (27) we get,

$$P_1(0,z)\left[z - r_0\left((1-\theta)+\theta V^*\right) B^* - \sum_{i=1}^{m} r_i \left((1-\theta)+\theta V^*\right) B^* B_i^*\right] = -z\lambda\alpha\left(1-A(z)\right)R_0$$

Hence,

$$P_1(0,z) \;=\; \frac{z\,\lambda\alpha\big(A(z)-1\big)R_0}{z-\left[r_0\Big(\big((1-\theta)+\theta V^{*}\big)B^{*}+\sum_{i=1}^{m} r_i\,\big((1-\theta)+\theta V^{*}\big)B^{*}\,B_i^{*}\right]} \tag{30}$$

By setting $v^*(z) = (1- \theta) + \theta V^*$ and $b^*(z) = r_0 B^* + \sum_{i=1}^{m} r_i\, B^*\, B_i^*$

The equation (30) becomes

$$P_1(0,z) \;=\; \frac{z\,\lambda\alpha\big(A(z)-1\big)R_0}{z-b^{*}(z)v^{*}(z)} \tag{31}$$

Using equations (21) and (30), we get

$$P_1(z) = \int_0^{\infty} P_1(x,z)\,dx$$

$$= P_1(0,z)\int_0^{\infty}\big(1 \quad B(x)\big)\,e^{\lambda\alpha\left(1 \quad A(z)\right)x}\,dx$$

$$= R_0\,\frac{z\left(1-B^{*}\right)}{b^{*}(z)v^{*}(z)-z} \tag{32}$$

Using equations (28) and (30) in (22), we get

$$P_{2,i}(z) = \int_0^{\infty} P_{2,i}(x,z)\,dx$$

$$=\; r_i B^{*} P_1(0,z)\int_0^{\infty}\big(1-B_i(x)\big)\,e^{-\lambda\alpha(1-A(z))x}\,dx$$

$$= R_0 \frac{z\, r_i B^* (1 - B_i^*)}{b^*(z) v^*(z) - z} \tag{33}$$

Using equations (28), (29) and (30) in (23), we get

$$Q(z) = \int_0^\infty Q(x,z)\, dx$$

$$= Q(0,z) \int_0^\infty (1 \quad V(x))\, e^{\;\lambda\beta(1 \quad A(z))x}\, dx$$

$$= R_0 \frac{\theta\, \alpha\, b^*(z)(1 - V^*)}{\beta\, (b^*(z) v^*(z) - z)} \tag{34}$$

The normalizing condition

$R_0 + \lim_{z \to 1} P_1(z) + \sum_{i=1}^{m} \lim_{z \to 1} P_{2,i}(z) + \lim_{z \to 1} Q(z) = 1$ is equivalent to

$$R_0 + P_1(1) + \sum_{i=1}^{m} P_{2,i}(1) + Q(1) = 1 \tag{35}$$

Now by using equations (32), (33) and (34) and L'Hopital rule, we have the following results

Probability that the server is busy with first phase of service is given by

$$P_1(1) = \lim_{z \to 1} P_1(z)$$

$$= R_0 \frac{\lambda \alpha E(X) E(B)}{1 - \lambda E(X) \left[\alpha \left(E(B) + \sum_{i=1}^{m} r_i\, E(B_i) \right) + \theta \beta E(V) \right]}$$

Probability that the server is busy with second phase of service is given by

$$P_2(1) = R_0 \frac{r_i \lambda \alpha E(X) E(B_i)}{1 - \lambda E(X)\left[\alpha \left(E(B) + \sum_{i=1}^{m} r_i E(B_i) \right) + \theta \beta E(V) \right]}$$

Probability that the server is on vacation is given by

$$Q(1) = R_0 \frac{\theta \lambda \alpha E(X) E(V)}{1 - \lambda E(X)\left[\alpha \left(E(B) + \sum_{i=1}^{m} r_i E(B_i) \right) + \theta \beta E(V) \right]}$$

Using the values of $P_1(1)$, $P_2(1)$ and $Q(1)$, in equation (35), we get

$$R_0 = \frac{1 - \rho}{1 - \lambda \theta E(X) E(V) (\beta - \alpha)}$$

where $\rho = \lambda E(X)\left[\alpha \left(E(B) + \sum_{i=1}^{m} r_i E(B_i) + \theta \beta E(V) \right) \right]$ (36)

R_0 is the steady- state probability that the server is idle but available in the system. Hence $\rho < 1$ is the stability condition under which the steady state solution exists.

The probability generating function of the queue size distribution at a random epoch is

$$P_Q(z) = P_1(z) + \sum_{i=1}^{m} P_{2,i}(z) + Q(z)$$

$$= \frac{R_0\left[b^*(z)\left(z+\left(\frac{\alpha}{\beta}\right)\left(v^*(z)-1\right)\right)-z\right]}{z-G(z)} \tag{37}$$

where $G(z) = b^*(z)v^*(z)$

Mean Queue size

$P_Q(z)$ has the form $R_0\frac{f(z)}{g(z)}$

Let L_Q be the mean number of customers in the queue then we have

$$L_Q = \left.\frac{dP_Q(z)}{dz}\right|_{z=1} \tag{38}$$

Since $\lim_{z\to1} f(z)=0$ and $\lim_{z\to1} g(z)=0$, By using L' Hopital rule we have

$$L_Q = R_0\frac{f''(1)\,g'(1)-g''(1)f'(1)}{2\left(g'(1)\right)^2}$$

where

$$f(z) = \left(\frac{\alpha}{\beta}\right)G(z) + b^*(z)\left[z-\left(\frac{\alpha}{\beta}\right)\right]-z$$

$$f'(z) = b^*(z)\left[1+\left(\frac{\alpha}{\beta}\right)v^{*\prime}(z)\right]+\left[z+\left(\frac{\alpha}{\beta}\right)\left(v^*(z)-1\right)\right]b^{*\prime}(z)-1$$

$$f'(1) = \lambda E(X)\left[\alpha\left(E(B)+\sum_{i=1}^{m} r_i\, E(B_i)\right)+\left(\frac{\alpha}{\beta}\right)\theta\beta E(V)\right]$$

$$f''(z) = b^*(z)\left[\left(\frac{\alpha}{\beta}\right)v^{*''}(z)\right] + b^{*'}(z)\left[1+\left(\frac{\alpha}{\beta}\right)v^{*'}(z)\right] + b^{*''}(z)\left[z+\left(\frac{\alpha}{\beta}\right)\left(v^*(z)-1\right)\right]$$

$$+b^{*'}(z)\left[1+\left(\frac{\alpha}{\beta}\right)v^{*'}(z)\right]$$

$$f''(1)=\left(\frac{\alpha}{\beta}\right)\theta\lambda^2\beta^2E(X^2)E(V^2)+\left[1+\left(\frac{\alpha}{\beta}\right)\theta\lambda\beta E(X)E(V)\right]\left[\lambda\alpha E(X)\left(2\sum_{i=1}^{m}r_i\,E(B_i)+E(B)\right)\right]$$

$$+\lambda^2\alpha^2E(B_0^2)$$

$$g(z)=z-\left(r_0\,B^* + \sum_{i=1}^{m} r_i\,B^*\,B_i^*\right)(1-\theta+\theta V^*)$$

$$g'(z)=1-\left\{\theta V^{*'}\left[r_0\,B^* + \sum_{i=1}^{m} r_i\,B^*\,B_i^*\right]\right\}+\left(1-\theta+\theta V^*\right)\left[r_0\,B^{*'} + \sum_{i=1}^{m} r_i\,B^*\,B_i^{*'} + \sum_{i=1}^{m} r_i\,B^{*'}\,B_i^*\right]$$

$$g'(1)=1-\rho$$

$$g''(z) = -\left\{r_0\ \theta\,B^*\ V^{*''} + \sum_{i=1}^{m} r_i\ \theta\,B^*B_i^*\,V^{*''} + (1-\theta)r_0\,B^{*''} + (1-\theta)\sum_{i=1}^{m} r_i B^*\,B_i^{*''}\right.$$

$$+ (1-\theta)\sum_{i=1}^{m} r_i\ B^{*'}\,B_i^{*'} + (1-\theta)\sum_{i=1}^{m} r_i\ B^{*''}\,B_i^{*} + (1-\theta)\sum_{i=1}^{m} r_i\ B^{*'}\,B_i^{*'} + \theta\,r_0\ B^{*''}\ V^*$$

$$+ \sum_{i=1}^{m} r_i\,\theta B^*\,B^{*''}\,V^* + \sum_{i=1}^{m} r_i\,\theta B^{*'}\,B_i^{*'}\,V^* + \sum_{i=1}^{m} r_i\,\theta B_i^*\,B^{*''}\,V^* + \sum_{i=1}^{m} r_i\,\theta B^{*'}\,B_i^{*'}\,V^*$$

$$\left.+2\left(\sum_{i=1}^{m} r_i\,\theta B_i^*\,B^{*'}\,V^{*'} + \theta r_0\ B^{*'}\ V^{*'} + \sum_{i=1}^{m} r_i\,\theta B^*\,B_i^{*'}V^{*'} + (1-\theta)\sum_{i=1}^{m} r_i\ B^{*'}\,B_i^{*'}\right)\right\}$$

$$g''(1) = -\left(\lambda E(X)\right)^2 \left(2\alpha^2 \sum_{i=1}^{m} r_i E(B)E(B_i) + \alpha^2 E(B^2) + \sum_{i=1}^{m} r_i \alpha^2 E(B_i^2) \right)$$

$$+ 2\theta\left(\alpha\beta E(B)E(V) + \sum_{i=1}^{m} r_i \alpha\beta E(B)E(V) \right) + \theta\beta^2 E(V^2)$$

Mean Waiting Time

Let W_Q be the mean waiting time in the queue. Using Little's formula, we get

$$W_Q = \frac{L_Q}{\lambda_x}$$

Actual Arrival Rate

$\lambda_x = \lambda\alpha$ (proportion of non- vacation time) $+\lambda\beta$ (proportion of vacation time).

But the proportion of vacation time = Q (1) = $\theta\lambda\beta E(X)\, E(V)$

Hence the proportion of non- vacation time including the first and second service times and idle time is equal $1-\theta\lambda\beta E(X)\, E(V)$

Consequently, $\lambda_x = \lambda\alpha + (\beta-\alpha)\, \theta\lambda^2\beta E(X)\, E(V)$. (39)

Particular cases

Case 1

If $\theta \to 0$, there is no vacation in the system , then v*(z) =1 and from (37), we have

$$P_Q(z) = \frac{zR_0\left(b^*(z)-1\right)}{z-b^*(z)} \tag{40}$$

Case 2

If $\sum_{i=1}^{m} r_i \to 1$ a tagged customer surely accept the second phase of service, then $b^*(z)=B^*(r_0+B_i^*)$ and

$$P_Q(z) = \frac{R_0\left[B^*(r_0+B_i^*)\left(z+\left(\frac{\alpha}{\beta}\right)\left(v^*(z)-1\right)\right)-z\right]}{z-v^*(z)\,B^*(r_0+B_i^*)} \tag{41}$$

Case3

If $\sum_{i=1}^{m} r_i \to 0$, there is no second phase service, then $b^*(z) = r_0B^*$, we get results for $M^X/G/1$ system with vacation and admissibility restriction.

Case 4

If α, $\beta \to 1$, there is no restricted admissibility then

$$P_Q(z) = \frac{R_0 \left\{ b^*(z) \left[z + \left(v^*(z) - 1 \right) \right] - z \right\}}{z - G(z)} \quad (42)$$

Case 5

If $r_i = 0$ for i=3, 4…m, then the model reduces to the model discussed by Badamchi Zadeh [3].

CHAPTER 3

STEADY STATE ANALYSIS OF AN $M^X / \begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{pmatrix} /1$ QUEUE WITH RESTRICTED ADMISSIBILITY OF ARRIVING BATCHES AND MODIFIED BERNOULLI SCHEDULE SERVER VACATIONS BASED ON A SINGLE VACATION POLICY

A batch arrival queue with single server providing two- stages of heterogeneous service is considered. Each customer have the option to choose one of the two types of first stage service followed by one of the two types of second stage service. After completion of the two stages of service in succession to each customer, the server has the option to take a vacation of a random length with probability p or to continue staying in the system with probability (1-p). Further, the batches arriving at the system have restricted admissibility into the system. The policy of restriction differs when the server is available in the system and when he is away on vacation. The steady state queue size distribution at a random epoch and some important performance measures for this model are derived. Many queueing systems are deduced as particular cases.

Model Description

Consider a batch arrival queuing system, where arrivals occur according to a compound Poisson process with the batch size random variable X and mean arrival rate $\lambda > 0$. Let a_k = Prob (X = k) and $E(X^k)$ be the k^{th} factorial moment of X.

The server provides two stages of heterogeneous service in succession. The service discipline is assumed to be first come, first served for both stages of service. Let α be the probability that an arriving batch will be allowed to join the system during the period of time when the server is busy or idle and β be the probability that an arriving batch will be allowed to join the system during the period of time when the server is on the vacation. The first stage service has two options 1A and 1B and the second stage service has two options 2A and 2B. When a customer, turn for service, he opts 1A with probability r_1 or 1B with probability $(1-r_1)$. After completion of the first stage service the customer enters second stage service and he may choose 2A with probability r_2 or 2B with probability $(1-r_2)$. The service times of 1A and the service times of 1B follow a general distribution. Its distribution functions are $S_{1A}(x)$ and $S_{1B}(x)$ and the first two moments are $E\left(S_{1A}^k\right)$ and $E\left(S_{1B}^k\right)$ for k = 1, 2.The hazard rate functions are

$$\mu_{1A}(x)dx = \frac{dS_{1A}(x)}{1-S_{1A}(x)} \quad \text{and} \tag{1}$$

$$\mu_{1B}(x)dx = \frac{dS_{1B}(x)}{1-S_{1B}(x)} \tag{2}$$

Similarly, the second stage service times have distribution functions $S_{2A}(x)$ and $S_{2B}(x)$ and the first two moments $E\left(S_{2A}^k\right)$ and $E\left(S_{2B}^k\right)$ for k = 1, 2. with the hazard rate functions

$$\mu_{2A}(x)dx = \frac{dS_{2A}(x)}{1-S_{2A}(x)} \quad \text{and} \tag{3}$$

$$\mu_{2B}(x)dx = \frac{dS_{2B}(x)}{1-S_{2B}(x)} \tag{4}$$

As soon as the second stage service of a customer is completed, the server may go for a vacation of random length V with probability p or he may stay in the system with probability 1-p. The vacation time of the server follows a general probability distribution with distribution function V(x), and the two moments E(V) and $E\left(V^k\right)$ where k =1, 2.

The hazard rate functions of V is

$$\nu(x)dx = \frac{dV(x)}{1-V(x)} \tag{5}$$

After returning from a vacation, the server does not find any customers in the system, he joins the system without taking any further vacations and this policy is termed as single vacation (V_s) with Bernoulli schedule (BS).

Let $N_s(t)$ be the system size at time t. Define the following probabilities

5. $R_0(t)$ is the probability that the server is idle and there is no customers in the system.

6. $Q_n(x,t)$ is the probability that at time t, the server is on vacation and there are n customers in the system with elapsed vacation time between x and dx.

7. $P_{1i,n}(x,t)$ is the probability that the server is being served in the first phase service for i=A, B and the elapsed service time is between x and dx.

8. $P_{2i,n}(x,t)$ is the probability that the server is being served in the second phase service for i=A, B and the elapsed service time is between x and dx.

Governing equations

The set of differential- difference equation that governs the system are given below

$$\left(\frac{d}{dx}+\frac{d}{dt}\right)P_{1A,n}(x,t)+(\lambda\alpha+\mu_{1A}(x))P_{1A,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k P_{1A,n-k}(x,t),\quad n\geq 1$$

(6) $$\left(\frac{d}{dx}+\frac{d}{dt}\right)P_{1B,n}(x,t)+(\lambda\alpha+\mu_{1B}(x))P_{1B,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k P_{1B,n-k}(x,t),\quad n\geq 1$$

(7) $$\left(\frac{d}{dx}+\frac{d}{dt}\right)P_{2A,n}(x,t)+(\lambda\alpha+\mu_{2A}(x))P_{2A,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k P_{2A,n-k}(x,t),\quad n\geq 1$$

(8) $$\left(\frac{d}{dx}+\frac{d}{dt}\right)P_{2B,n}(x,t)+(\lambda\alpha+\mu_{2B}(x))P_{2B,n}(x,t)=\lambda\alpha\sum_{k=1}^{n}a_k P_{2B,n-k}(x,t),\quad n\geq 1$$

(9) $$\left(\frac{d}{dx}+\frac{d}{dt}\right)Q_n(x,t)+(\lambda\beta+\nu(x))Q_n(x,t)=\lambda\beta\sum_{k=1}^{n}a_k Q_{n-k}(x,t)\quad n\geq 1 \qquad (10)$$

$$\left(\frac{d}{dx}+\frac{d}{dt}\right)Q_0(x,t)+(\lambda\beta+\nu(x))Q_n(x,t)=0$$

(11) $$\frac{d}{dt}R_0(t)=-\lambda\alpha R_0(t)+(1-p)\int_0^\infty \mu_{2A}(x)P_{2A,1}(x,t)\,dx+(1-p)\int_0^\infty \mu_{2B}(x)P_{2B,1}(x,t)\,dx + \int_0^\infty \nu(x)Q_0(x,t)\,dx$$

(12)

$$P_{1A,n}(0,t)=\lambda\alpha r_1 a_n R_0(t)+(1-p)r_1\int_0^\infty \mu_{2A}(x)P_{2A,n+1}(x,t)\,dx + (1-p)r_1\int_0^\infty \mu_{2B}(x)P_{2B,n+1}(x,t)\,dx + r_1\int_0^\infty \nu(x)Q_n(x,t)\,dx \quad n\geq 1 \qquad (13)$$

$$P_{1B,n}(0,t) = \lambda\alpha(1-r_1)a_n R_0(t) + (1-p)(1-r_1)\int_0^\infty \mu_{2A}(x) P_{2A,n+1}(x,t)\,dx$$

$$+(1-p)(1-r_1)\int_0^\infty \mu_{2B}(x) P_{2B,n+1}(x,t)\,dx + (1-r_1)\int_0^\infty \nu(x) Q_n(x,t)\,dx \quad n \geq 1$$

(14) $$P_{2A,n}(0,t) = r_2\int_0^\infty \mu_{1A}(x) P_{1A,n}(x,t)\,dx + r_2\int_0^\infty \mu_{1B}(x) P_{1B,n}(x,t)\,dx \qquad n \geq 1$$

(15) $$P_{2B,n}(0,t) = (1-r_2)\int_0^\infty \mu_{1A}(x) P_{1A,n}(x,t)\,dx + (1-r_2)\int_0^\infty \mu_{1B}(x) P_{1B,n}(x,t)\,dx \quad n \geq 1$$

(16) $$Q_n(0,t) = p\int_0^\infty \mu_{2A}(x) P_{2A,n+1}(x,t) + p\int_0^\infty \mu_{2B}(x) P_{2B,n+1}(x,t)\,dx \qquad n \geq 0$$

(17)

Steady- State Probability Generating Functions

The steady state equations corresponding to equations (6) to (17) are

$$\frac{d}{dx}P_{1A,n}(x) + (\lambda\alpha + \mu_{1A}(x))P_{1A,n}(x) = \lambda\alpha\sum_{k=1}^{n} a_k P_{1A,n-k}(x), \quad n \geq 1$$

(18) $$\frac{d}{dx}P_{1B,n}(x) + (\lambda\alpha + \mu_{1B}(x))P_{1B,n}(x) = \lambda\alpha\sum_{k=1}^{n} a_k P_{1B,n-k}(x), \quad n \geq 1$$

(19) $$\frac{d}{dx}P_{2A,n}(x) + (\lambda\alpha + \mu_{2A}(x))P_{2A,n}(x) = \lambda\alpha\sum_{k=1}^{n} a_k P_{2A,n-k}(x), \quad n \geq 1$$

(20) $$\frac{d}{dx}P_{2B,n}(x) + (\lambda\alpha + \mu_{2B}(x))P_{2B,n}(x) = \lambda\alpha\sum_{k=1}^{n} a_k P_{2B,n-k}(x), \quad n \geq 1$$

(21) $$\frac{d}{dx}Q_n(x) + (\lambda\beta + \nu(x))Q_n(x) = \lambda\beta\sum_{k=1}^{n} a_k Q_{n-k}(x) \quad n \geq 1$$

(22)

$$\frac{d}{dx}Q_0(x) + (\lambda\beta + \nu(x))Q_n(x) = 0 \tag{23}$$

$$\lambda\alpha R_0 = (1-p)\int_0^\infty \mu_{2A}(x)\, P_{2A,1}(x)\,dx + (1-p)\int_0^\infty \mu_{2B}(x) P_{2B,1}(x)\,dx + \int_0^\infty \nu(x) Q_0(x)\,dx \tag{24}$$

with boundary conditions

$$\begin{aligned} P_{1A,n}(0) = {} & \lambda\alpha r_1 a_n R_0 + (1-p) r_1 \int_0^\infty \mu_{2A}(x) P_{2A,n+1}(x)\,dx \\ & + (1-p)\, r_1 \int_0^\infty \mu_{2B}(x) P_{2B,n+1}(x)\,dx + r_1 \int_0^\infty \nu(x) Q_n(x)\,dx, \quad n \geq 1 \end{aligned} \tag{25}$$

$$\begin{aligned} P_{1B,n}(0) = {} & \lambda\alpha(1-r_1) a_n R_0 + (1-p)(1-r_1)\int_0^\infty \mu_{2A}(x) P_{2A,n+1}(x)\,dx \\ & + (1-p)(1-r_1)\int_0^\infty \mu_{2B}(x) P_{2B,n+1}(x)\,dx + (1-r_1)\int_0^\infty \nu(x) Q_n(x)\,dx, \quad n \geq 1 \end{aligned} \tag{26}$$

$$P_{2A,n}(0) = r_2 \int_0^\infty \mu_{1A}(x) P_{1A,n}(x)\,dx + r_2 \int_0^\infty \mu_{1B}(x) P_{1B,n}(x)\,dx, \qquad n \geq 1$$

(27) $P_{2B,n}(0) = (1-r_2)\int_0^\infty \mu_{1A}(x) P_{1A,n}(x)\,dx + (1-r_2)\int_0^\infty \mu_{1B}(x) P_{1B,n}(x)\,dx, \qquad n \geq 1$ (28)

$$Q_n(0) = p\int_0^\infty \mu_{2A}(x) P_{2A,n+1}(x)\,dx + p\int_0^\infty \mu_{2B}(x) P_{2B,n+1}(x)\,dx, \; n \geq 0 \tag{29}$$

The normalizing condition is

$$R_0 + \sum_{n=1}^{\infty}\int_0^\infty P_{1A,n}(x)\,dx + \sum_{n=1}^{\infty}\int_0^\infty P_{1B,n}(x)\,dx + \sum_{n=1}^{\infty}\int_0^\infty P_{2A,n}(x)\,dx$$

$$+\sum_{n=1}^{\infty}\int_0^{\infty} P_{2B,n}(x)\,dx + \sum_{n=0}^{\infty}\int_0^{\infty} Q_n(x)\,dx = 1 \qquad (30)$$

By assuming the existence of the steady state, define

$$R_0 = \lim_{t\to\infty} R_0(t)$$

$$P_{1A,n}(x) = \lim_{t\to\infty} P_{1A,n}(x,t),\ n \geq 1$$

$$P_{1B,n}(x) = \lim_{t\to\infty} P_{1B,n}(x,t),\ n \geq 1$$

$$P_{2A,n}(x) = \lim_{t\to\infty} P_{2A,n}(x,t),\ \ n \geq 1$$

$$P_{2B,n}(x) = \lim_{t\to\infty} P_{2B,n}(x,t),\ \ n \geq 1$$

$$Q_n(x) = \lim_{t\to\infty} Q_n(x,t),\ \ n \geq 0$$

Define the probability generating functions

$$P_{iA}(x,z) = \sum_{n=1}^{\infty} P_{iA,n}(x) z^n \qquad i = 1, 2$$

$$P_{iB}(x,z) = \sum_{n=1}^{\infty} P_{iB,n}(x) z^n \qquad i = 1, 2$$

$$Q(x,z) = \sum_{n=0}^{\infty} Q_n(x) z^n$$

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

Multiplying equation (18) by z^n with all admissible values of n, we get

$$\frac{d}{dx}\sum_{n=1}^{\infty}P_{1A,n}(x)z^n = -(\lambda\alpha+\mu_{1A}(x))\sum_{n=1}^{\infty}P_{1A,n}(x)z^n + \lambda\alpha\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_k P_{1A,n-k}(x)z^n$$

$$\frac{d}{dx}P_{1A}(x,z)+P_{1A}(x,z)\left(\lambda\alpha+\mu_{1A}(x)-\lambda\alpha A(z)\right)=0$$

Solving the above equation we get

$$P_{1A}(x,z)=A\ e^{-\lambda\alpha(1-A(z))x+\log(1-S_{1A}(x))}$$

$$= A\,e^{-\lambda\alpha(1-A(z))x}\left(1-S_{1A}(x)\right)$$

Putting x=0, we get, A = P_{1A} (0, z)

Hence, $P_{1A}(x,z)=P_{1A}(0,z)e^{-\lambda\alpha(1-A(z))}(1-S_{1A}(x))$ (31)

Multiplying equation (19) to (23) by z^n, and summing over n, we get

$$\frac{d}{dx}P_{1B}(x,z)+P_{1B}(x,z)\left(\lambda\alpha+\mu_{1B}(x)-\lambda\alpha A(z)\right)=0$$

$$\frac{d}{dx}P_{2A}(x,z)+P_{2A}(x,z)\left(\lambda\alpha+\mu_{2A}(x)-\lambda\alpha A(z)\right)=0$$

$$\frac{d}{dx}P_{2B}(x,z)+P_{2B}(x,z)\left(\lambda\alpha+\mu_{2B}(x)-\lambda\alpha A(z)\right)=0$$

$$\frac{d}{dx}Q(x,z)+Q(x,z)\left(\lambda\beta+\nu(x)-\lambda\beta A(z)\right)=0$$

Solving the above equations, we get

$$P_{1B}(x,z)=P_{1B}(0,z)e^{-\lambda\alpha(1-A(z))}(1-S_{1B}(x))$$

$$(32)\, P_{2A}(x,z)=P_{2A}(0,z)\,e^{-\lambda\alpha(1-A(z))}(1-S_{2A}(x))$$

$$(33)\, P_{2B}(x,z)=P_{2B}(0,z)\,e^{-\lambda\alpha(1-A(z))}(1-S_{2B}(x))$$

$$(34)\, Q(x,z)=Q(0,z)\,e^{-\lambda\beta(1-A(z))}(1-V(x)) \qquad (35)$$

Multiplying equation (25) by z^n with all admissible values of n, we get

$$\sum_{n=1}^{\infty}P_{1A,n}(0)z^n=\sum_{n=1}^{\infty}\lambda\alpha r_1 a_n R_0 z^n + (1-p)r_1\sum_{n=1}^{\infty}\int_0^{\infty}\mu_{2A}(x)P_{2A,n+1}(x)z^n dx$$

$$+(1-p)r_1\sum_{n=1}^{\infty}\int_0^{\infty}\mu_{2B}(x)P_{2B,n+1}(x)z^n dx + r_1\sum_{n=1}^{\infty}\int_0^{\infty}\nu(x)Q_n(x)\, z^n dx$$

$$P_{1A}(0,z)=\frac{1}{z}\Bigg\{z\lambda\alpha r_1\sum_{n=1}^{\infty}a_n R_0 z^n + (1-p)r_1\sum_{n=1}^{\infty}\int_0^{\infty}\mu_{2A}(x)P_{2A,n+1}(x)z^{n+1}dx$$

$$+(1-p)r_1\sum_{n=1}^{\infty}\int_0^{\infty}\mu_{2B}(x)P_{2B,n+1}(x)z^{n+1}dx + z r_1\sum_{n=1}^{\infty}\int_0^{\infty}\nu(x)Q_n(x)z^n\, dx\Bigg\}$$

$$zP_{1A}(0,z)=z\,\lambda\alpha r_1\, A(z)\, R_0+(1-p)r_1\int_0^{\infty}\left(\sum_{n=1}^{\infty}P_{2A,n}(x)z^n - zP_{2A,1}(x)\right)\mu_{2A}(x)\, dx$$

$$+(1-p)r_1\int_0^{\infty}\left(\sum_{n=1}^{\infty}P_{2B,n}(x)z^n - zP_{2B,1}(x)\right)\mu_{2B}(x)\, dx$$

$$+z\, r_1\int_0^{\infty}\left(\sum_{n=0}^{\infty}Q_n(x)z^n - Q_0(x)\right)\nu(x)\, dx$$

Substituting equation (24), we get

$$zP_{1A}(0,z) \quad = \quad z\lambda\alpha r_1 R_0\big(A(z)-1\big)+(1-p)r_1 P_{2A}(0,z)S_{2A}^{*}\big(\lambda\alpha(1-A(z))\big)$$

$$+(1-p)r_1 P_{2B}(0,z) S^*_{2B}(\lambda\alpha(1-A(z))) + r_1 z Q(0,z) V^*(\lambda\beta(1-A(z))) \quad (36)$$

Multiplying equations (26) to (29) by z^n with all admissible values of n and using equation (24), we get

$$z P_{1B}(0,z) = z\lambda\alpha(1-r_1) R_0 (A(z)-1) + (1-p)(1-r_1) P_{2A}(0,z) S^*_{2A}(\lambda\alpha(1-A(z)))$$

$$+(1-p)(1-r_1) P_{2B}(0,z) S^*_{2B}(\lambda\alpha(1-A(z)))$$

$$+(1-r_1) z Q(0,z) V^*(\lambda\beta(1-A(z))) \quad (37)$$

$$P_{2A}(0,z) = r_2 P_{1A}(0,z) S^*_{1A}(\lambda\alpha(1-A(z))) + r_2 P_{1B}(0,z) S^*_{1B}(\lambda\alpha(1-A(z))) \quad (38)$$

$$P_{2B}(0,z) = (1-r_2) P_{1A}(0,z) S^*_{1A}(\lambda\alpha(1-A(z))) + (1-r_2) P_{1B}(0,z) S^*_{1B}(\lambda\alpha(1-A(z))) \quad (39)$$

$$z Q(0,z) = p P_{2A}(0,z) S^*_{2A}(\lambda\alpha(1-A(z))) + p P_{2B}(0,z) S^*_{2B}(\lambda\alpha(1-A(z))) \quad (40)$$

Let us denote $S^*_{1A}(\lambda\alpha(1-A(z)))$ as S^*_{1A}

$S^*_{1B}(\lambda\alpha(1-A(z)))$ as S^*_{1B}

$S^*_{2A}(\lambda\alpha(1-A(z)))$ as S^*_{2A}

$S^*_{2B}(\lambda\alpha(1-A(z)))$ as S^*_{2B} and

$V^*(\lambda\beta(1-A(z)))$ as V^*

Substituting results in equations (38) and (39) in equation (37), we get

$$z\, P_{1B}(0,z) \;=\; z\,\lambda\,\alpha\,(1-r_1)\, R_0\,(A(z)-1)$$

$$+(1-p)(1-r_1)\left(r_2\, S_{1A}^{*}\, P_{1A}(0,z) + r_2\, S_{1B}^{*}\, P_{1B}(0,z)\right) S_{2A}^{*}$$

$$+(1-p)(1-r_1)\left((1-r_2)\, S_{1A}^{*}\, P_{1A}(0,z) + (1-r_2)\, S_{1B}^{*}\, P_{1B}(0,z)\right) S_{2B}^{*}$$

$$+(1-r_1)\left(p\, S_{1A}^{*}\, P_{1A}(0,z)\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right)\right) V^{*}$$

$$+(1-r_1)\left(p\, S_{1B}^{*}\, P_{1B}(0,z)\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right)\right) V^{*}$$

Simplifying the above equation, we get

$$P_{1B}(0,z)\, D_1(z) \;=\; z\,\lambda\,\alpha(1-r_1)\, R_0\,(A(z)-1)$$

$$+\, P_{1A}(0,z)\,(1-r_1)\, S_{1A}^{*}\left(1-p+pV^{*}\right)\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right)$$

where $D_1(z) = z - (1-r_1)\, S_{1B}^{*}\left(1-p+pV^{*}\right)\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right)$. (41)

Substituting equations (38), (39) and (40) in equation (36), we obtain

$$z\, P_{1A}(0,z) \;=\; z\,\lambda\,\alpha\, r_1 R_0\,(A(z)-1) + (1-p) r_1\left(r_2\, S_{1A}^{*}\, P_{1A}(0,z) + r_2\, S_{1B}^{*}\, P_{1B}(0,z)\right) S_{2A}^{*}$$

$$+\,(1-p)\, r_1\left((1-r_2)\, S_{1A}^{*}\, P_{1A}(0,z) + (1-r_2)\, S_{1B}^{*}\, P_{1B}(0,z)\right) S_{2B}^{*}$$

$$+\, r_1\left(p\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right) S_{1A}^{*} P_{1A}(0,z)\right) V^{*}$$

$$+\, r_1\left(p\left(r_2\, S_{2A}^{*} + (1-r_2)\, S_{2B}^{*}\right) S_{1B}^{*}\, P_{1B}(0,z)\right) V^{*}$$

Simplifying the above equation, we obtain

$$P_{1A}(0,z)\,D_2(z) = z\lambda\alpha r_1 R_0(A(z)-1) + P_{1B}(0,z)S^*_{1B}\,r_1\,\left(r_2\,S^*_{2A} + (1-r_2)S^*_{2B}\right)\left(1-p+pV^*\right)$$

where $D_2(z) = z - S^*_{1A}r_1\left(r_2\,S^*_{2A} + (1-r_2)\,S^*_{2B}\right)\left(1-p+pV^*\right)$

$$P_{1A}(0,z) = \frac{z\,\lambda\alpha r_1 R_0(A(z)-1)}{D_2(z)} + \frac{P_{1B}(0,z)S^*_{1B}r_1\left(r_2\,S^*_{2A} + (1-r_2)S^*_{2B}\right)\left(1-p+pV^*\right)}{D_2(z)} \qquad (42)$$

Substituting the equation (42) in (41), we get

$$P_{1B}(0,z)D_1(z)D_2(z) = z\,\lambda\alpha(1-r_1)R_0(A(z)-1)\;\;D_2(z)$$

$$+(1-r_1)\,S^*_{1A}r_1\,\left(1-p+pV^*\right)^2\left(r_2\,S^*_{2A} + (1-r_2)S^*_{2B}\right)^2 S^*_{1B}\,P_{1B}(0,z)$$

$$+(1-r_1)S^*_{1A}\,(1-p+pV^*)\,\left(r_2 S^*_{2A} + (1-r_2)S^*_{2B}\right)z\,\lambda\alpha\,r_1\,R_0(A(z)-1)$$

Hence

$$P_{1B}(0,z)\left\{\;D_1(z)D_2(z) - S^*_{1A}\,S^*_{1B}\,r_1(1-r_1)\left(1-p+pV^*\right)^2\;\left(r_2\,S^*_{2A} + (1-r_2)S^*_{2B}\right)^2\right\}$$

$$= z\,\lambda\alpha(1-r_1)\,R_0(\;A(z)-1)$$

Substituting the values of $D_1(z)$ and $D_2(z)$ in the above equation and simplifying we get

$$P_{1B}(0,z)\left\{\,z^2 - z\left(1-p+pV^*\right)\left(r_2\,S^*_{2A} + (1-r_2)\,S^*_{2B}\right)\left(r_1\,S^*_{1A} + (1-r_1)\,S^*_{1B}\right)\right\}$$

$$= z^2\,\lambda\alpha(1-r_1)R_0(A(z)-1)$$

$$P_{1B}(0,z)=\frac{z\lambda\alpha(1-r_1)(1-A(z))\,R_0}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)\,S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_1)\,S^*_{2B}\right)-z} \tag{43}$$

Substituting equation (43) in (42), we obtain

$$P_{1A}(0,z)=\frac{z\lambda\alpha r_1(1-A(z))R_0}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_1)S^*_{2B}\right)-z} \tag{44}$$

Substituting equation (38), (39), (43) and (44) in (40), we have

$$Q(0,z)=\frac{\lambda\alpha p(A(z)-1)R_0\left(r_2 S^*_{2A}+(1-r_2)S^*_{2B}\right)\left(r_2 S^*_{1A}+(1-r_1)S^*_{1B}\right)}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_1)S^*_{2B}\right)-z} \tag{45}$$

Using equation (43) and (44) in (38), we get

$$P_{2A}(0,z)=\frac{z\,r_2\,\lambda\alpha(1-A(z))\,R_0\left(r_1 S^*_{1A}+(1-r_1)\,S^*_{1B}\right)}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_1)S^*_{2B}\right)-z} \tag{46}$$

Substituting equation (43) and (44) in (39), we obtain

$$P_{2B}(0,z)=\frac{z(1-r_2)\lambda\alpha(1-A(z))R_0\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_1)S^*_{2B}\right)-z} \tag{47}$$

Integrating the equations (43) to (47), we have

$$P_{1A}(z)=\frac{z\,r_1\left(1-S^*_{1A}\right)R_0}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_2)S^*_{2B}\right)-z} \tag{48}$$

$$P_{1B}(z)=\frac{z(1-r_1)\left(1-S^*_{1A}\right)R_0}{\left(1-p+pV^*\right)\left(r_1 S^*_{1A}+(1-r_1)S^*_{1B}\right)\left(r_2 S^*_{2A}+(1-r_2)S^*_{2B}\right)-z} \tag{49}$$

$$P_{2A}(z)=\frac{z r_2 \left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(1-S_{2A}^*\right) R_0}{\left(1-p+pV^*\right)\left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right)-z} \tag{50}$$

$$P_{2B}(z)=\frac{z(1-r_2) \left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(1-S_{2B}^*\right) R_0}{\left(1-p+pV^*\right)\left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right)-z} \tag{51}$$

$$Q(z)=\frac{p\alpha \left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right)\left(1-V^*\right) R_0}{\beta\left(1-p+pV^*\right)\left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right)-z} \tag{52}$$

The normalizing condition

$$R_0 + \sum_{n=1}^{\infty}\int_0^{\infty} P_{1A,n}(x)\,dx + \sum_{n=1}^{\infty}\int_0^{\infty} P_{1B,n}(x)\,dx + \sum_{n=1}^{\infty}\int_0^{\infty} P_{2A,n}(x)\,dx$$

$$+ \sum_{n=1}^{\infty}\int_0^{\infty} P_{2B,n}(x)\,dx + \sum_{n=0}^{\infty}\int_0^{\infty} Q_n(x)\,dx = 1$$

is equivalent to $R_0 + P_{1A}(1) + P_{1B}(1) + P_{2A}(1) + P_{2B}(1) + Q(1) = 1$ (53)

Now by using equations (48) to (52) and L'Hopital rule, we have the following results

Probability that the server is busy with first phase, 1A service is given by

$$P_{1A}(1) = \frac{r_1 \lambda \alpha E(X) E(S_{1A}) R_0}{1-\rho} \tag{54}$$

Probability that the server is busy with first phase, 1B service is given by

$$P_{1B}(1) = \frac{(1-r_1) \lambda \alpha E(X) E(S_{1B}) R_0}{1-\rho} \tag{55}$$

Probability that the server is busy with second phase, 2A service is given by

$$P_{2A}(1) = \frac{r_2 \lambda \alpha E(X) E(S_{2A}) R_0}{1-\rho} \tag{56}$$

Probability that the server is busy with second phase, 2B service is given by

$$P_{2B}(1) = \frac{(1-r_2) \lambda \alpha E(X) E(S_{2B}) R_0}{1-\rho} \tag{57}$$

Probability that the server is on vacation is given by

$$Q(1) = \frac{p \lambda \alpha E(X) E(V) R_0}{1-\rho} \tag{58}$$

where

$$\rho = \lambda p \beta E(X) E(V) + \lambda \alpha E(X) \left(r_1 E(S_{1A}) + (1-r_1) E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2) E(S_{2B}) \right) \tag{59}$$

is the utilization factor of this system.

Using the values of $P_{1A}(1)$, $P_{1B}(1)$, $P_{2A}(1)$, $P_{2B}(1)$ and $Q(1)$ in equation (53), we get

$$R_0 = \frac{1-\rho}{1-\lambda p E(X) E(V) (\beta - \alpha)} \tag{60}$$

R_0 is the steady- state probability that the server is idle but available in the system. Hence ρ <1 can be the stability condition under which the steady state solution exists.

The probability generating function of the queue size distribution at a random epoch is

$$P_Q(z) = P_{1A}(z) + P_{1B}(z) + P_{2A}(z) + P_{2B}(z) + Q(z)$$

$$= R_0 \frac{\left[r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right]\left[r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right]\left[\left(\frac{p\alpha}{\beta}\right)(1-v^*) - z\right]}{(1-p+pV^*)\left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right)\left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right) - z} \quad (61)$$

Mean Queue size

P_Q (z) has the form $R_0 \frac{f(z)}{g(z)}$

Let L_Q be the mean number of customers in the queue then we have

$$L_Q = \left.\frac{dP_Q(z)}{dz}\right|_{z=1} \quad (62)$$

Since $\lim_{z \to 1} f(z) = 0$ and $\lim_{z \to 1} g(z) = 0$, Then, by using L' Hopital rule we have

$$L_Q = R_0 \frac{f''(1)\, g'(1) - g''(1) f'(1)}{2\left(g'(1)\right)^2}$$

where

$$f(z) = \left[r_1 S^*_{1A} + (1-r_1) S^*_{1B}\right]\left[r_2 S^*_{2A} + (1-r_2) S^*_{2B}\right]\left[\left(\frac{p\alpha}{\beta}\right)\left(1-V^*\right) - z\right]$$

$$f'(z) = \left[r_1\, S^*_{1B} + (1-r_1) S^*_{1B}\right]\left[\left(\frac{p\alpha}{\beta}\right)\left(1-V^*\right) - z\right]\left[r_2\, S^{*\prime}_{2A} + (1-r_2) S^{*\prime}_{2B}\right]$$

$$+ \left[r_1\, S^{*\prime}_{1A} + (1-r_1) S^{*\prime}_{1B}\right]\left[r_2\, S^*_{2A} + (1-r_2) S^*_{2B}\right]\left[\left(\frac{p\alpha}{\beta}\right)\left(1-V^*\right) - z\right]$$

$$+ \left[r_1\, S^*_{1A} + (1-r_1) S^*_{1B}\right]\left[r_2\, S^*_{2A} + (1-r_2) S^*_{2B}\right]\left[-\left(\frac{p\alpha}{\beta}\right) V^{*\prime} - 1\right]$$

$$f'(1) = -\left\{1 + \lambda\alpha E(X)\left[r_1 E(S_{1A}) + (1-r_1) E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2) E(S_{2B}) + pE(X)E(V)\right]\right\}$$

$$f''(z) = 2\left\{\left[r_1\, S^{*\prime}_{1A} + (1-r_1)\, S^{*\prime}_{1B}\right]\left[\frac{p\alpha}{\beta}\left(1-V^*\right) - z\right]\left[r_2\, S^{*\prime}_{2A} + (1-r_2)\, S^{*\prime}_{2B}\right]\right.$$

$$+ \left[r_1 S^*_{1A} + (1-r_1) S^*_{1B}\right]\left[\frac{-p\alpha V^{*\prime}}{\beta} - 1\right]\left[r_2\, S^{*\prime}_{2A} + (1-r_2) S^{*\prime}_{2B}\right]$$

$$\left. + \left[r_1\, S^{*\prime}_{1A} + (1-r_1)\, S^{*\prime}_{1B}\right]\left[r_2\, S^*_{2A} + (1-r_2)\, S^*_{2B}\right]\left[\frac{-p\alpha V^{*\prime}}{\beta} - 1\right]\right\}$$

$$+ \left[r_1 S^*_{1A} + (1-r_1) S^*_{1B}\right]\left[\frac{p\alpha}{\beta}\left(1-V^*\right) - z\right]\left[r_2\, S^{*\prime\prime}_{2A} + (1-r_2) S^{*\prime\prime}_{2B}\right]$$

$$+ \left[r_1 S^{*\prime\prime}_{1A} + (1-r_1) S^{*\prime\prime}_{1B}\right]\left[r_2\, S^*_{2A} + (1-r_2) S^*_{2B}\right]\left[\frac{p\alpha}{\beta}\left(1-V^*\right) - z\right]$$

$$+\left[r_1 S^{*}_{1A} + (1-r_1) S^{*}_{1B}\right]\left[r_2 S^{*}_{2A} + (1-r_2) S^{*}_{2B}\right]\left[\frac{-p\alpha V^{*''}}{\beta}\right]$$

$$f''(1) = \quad 2\Big\{\left[r_1 \lambda\alpha E(X)E(S_{1A}) + (1 \quad r_1)\lambda\alpha E(X)E(S_{1B})\right]$$

$$\times\left[r_2 \lambda\alpha E(X)E(S_{2A}) + (1 \quad r_2)\lambda\alpha E(X)E(S_{2B})\right]$$

$$+\left[r_1 \lambda\alpha E(X)E(S_{1A}) + (1 \quad r_1)\lambda\alpha E(X)E(S_{1B})\right] \quad \frac{p\alpha}{\beta} \quad \left(\lambda\beta E(X)E(V) \quad 1\right)$$

$$-\left\{\left[r_2 \lambda^2\alpha^2 E(X^2)E(S^2_{2A}) + (1-r_2)\lambda^2\alpha^2 E(X^2)E(S^2_{2B})\right]\right.$$

$$\left. + \left[r_1 \lambda^2\alpha^2 E(X^2)E(S^2_{1A}) + (1-r_1)\lambda^2\alpha^2 E(X^2)E(S^2_{1B})\right] + \left(\frac{p\alpha}{\beta}\right)\lambda^2\beta^2 E(X^2)E(V^2)\right\}$$

$$g(z) = \left(1-p+pV^{*}\right)\left(r_1 S^{*}_{1A} + (1-r_1)S^{*}_{1B}\right)\left(r_2 S^{*}_{2A} + (1-r_1) S^{*}_{2B}\right) - z$$

$$+\left(1-p+pV^{*}\right)\left(r_1 S^{*'}_{1A} + (1-r_1)S^{*'}_{1B}\right)\left(r_2 S^{*}_{2A} + (1-r_1)S^{*}_{2B}\right)$$

$$+\left(1-p+pV^{*}\right)\left(r_1 S^{*}_{1A} + (1-r_1)S^{*}_{1B}\right)\left(r_2 S^{*'}_{2A} + (1-r_1)S^{*'}_{2B}\right) - 1$$

$$g'(1) = p\lambda\beta E(X)E(V) + \lambda\alpha E(X)\left(r_1 E(S_{1A}) + (1-r_1)E(S_{1B})\right)$$

$$+ \lambda\alpha E(X)\left(r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\right) - 1$$

$$= -(1-\rho)$$

$$g''(1) = pV^{*''} + 2pV^{*'}\left(r_1 S^{*'}_{1A} + (1-r_1)S^{*'}_{1B}\right) + 2pV^{*'}\left(r_2 S^{*'}_{2A} + (1-r_2) S^{*'}_{2B}\right)$$

$$+\left(r_1 S^{*''}_{1A} + (1-r_1)S^{*''}_{1B}\right) + 2\left(r_1 S^{*'}_{1A} + (1-r_1)S^{*'}_{1B}\right)\left(r_2 S^{*'}_{2A} + (1-r_2)S^{*'}_{2B}\right)$$

$$+\left(r_2 S_{2A}^{*''} + (1-r_2) S_{2B}^{*''}\right)$$

Mean waiting time in the Queue

Let W_Q be the mean waiting time of an arbitrary customer. Using Little's formula in equation (62), we get

$$W_Q = \frac{L_Q}{\lambda_a}$$

where $\lambda_a =$ actual arrival rate

Special cases

Case 1

If $p \to 0$, there is no vacation in the system, then from (61), we have

$$P_Q(z) = R_0 \frac{z\left(r_1 S_{1A}^{*} + (1-r_1) S_{1B}^{*}\right)\left(r_2 S_{2A}^{*} + (1-r_2) S_{2B}^{*}\right)}{z - \left(r_1 S_{1A}^{*} + (1-r_1) S_{1B}^{*}\right)\left(r_2 S_{2A}^{*} + (1-r_2) S_{2B}^{*}\right)} \qquad (63)$$

Case 2

If $r_2 \to 1$, the system has two option in phase 1 and only one second option. Then equation (61) becomes

$$P_Q(z) = R_0 \frac{\left[r_1 S_{1A}^* + (1-r_1)\, S_{1B}^*\right]\left[\left(\frac{p\alpha}{\beta}\right)(1-v^*) - z\right] S_{2A}^*}{\left(1-p+pV^*\right)\,\left(r_1\, S_{1A}^* + (1-r_1) S_{1B}^*\right) S_{2A}^* - z} \tag{64}$$

Case 3

If $r_1 \to 1$, the system has only one first phase service and two option in second phase service. Then

$$P_Q(z) = R_0 \frac{\left[r_2\, S_{2A}^* + (1-r_2) S_{2B}^*\right]\left[\left(\frac{p\alpha}{\beta}\right)(1-v^*) - z\right] S_{1A}^*}{\left(1-p+pV^*\right)\,\left(r_2\, S_{2A}^* + (1-r_2)\, S_{2B}^*\right) S_{1A}^* - z} \tag{65}$$

Case 4

If $r_1 \to 1$ and $r_2 \to 1$, the system has two phases of service and only one second phase service, then equation (61) becomes

$$P_Q(z) = R_0 \frac{\left[\left(\frac{p\alpha}{\beta}\right)(1-v^*) - z\right] S_{1A}^* S_{2A}^*}{\left(1-p+pV^*\right) S_{1A}^*\, S_{2A}^* - z} \tag{66}$$

Case 5

If $\alpha, \beta \to 1$, there is no restricted admissibility. In this case

$$P_Q(z) = R_0 \frac{\left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right) \left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right) \left(p(1-v^*) - z\right)}{\left(1-p+pV^*\right) \left(r_1 S_{1A}^* + (1-r_1) S_{1B}^*\right) \left(r_2 S_{2A}^* + (1-r_2) S_{2B}^*\right) - z} \quad (67)$$

The result of this Case (5) agrees with model studied by Madan and Chaudhry [32].

SUMMARY AND CONCLUSION

In this dissertation queueing system with restricted admissibility is considered. Batch arrival admissibility restricted queueing system with vacation and multi optional second service is discussed in chapter 2. Mean number of customers in the queue and mean waiting time are derived. Some particular cases are also discussed.

In chapter 3, Steady state analysis of an $M^X / \begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{pmatrix} / 1$ queue with restricted admissibility of arriving batches and modified Bernoulli schedule server vacations based on a single vacation policy is discussed. The mean number of customers in the queue and the mean waiting time in the queue are derived. Some special cases are also discussed.

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