Chapter III

CHAPTER III

PRIME AND SEMIPRIME MATRICES OVER SEMIRINGS

Definition: 3.1

A nonzero nonunit matrix $A \in M_n(S)$ is a **prime matrix** in $M_n(S)$ provide that whenever $A = BC$ ($B, C \in M_n(S)$) B or C is a unit.

Definition: 3.2

A nonzero non-monomial matrix $A \in M_n(S)$ is a **semiprime matrix** in $M_n(S)$ provide that whenever $A = BC$ ($B, C \in M_n(S)$) B or C is a monomial matrix.

Note: 3.3

For the semiring of all $n \times n$ matrices with integer entries, the units are the matrices with determinant ± 1 and the primes are the matrices with determinant a prime number.

Remark: 3.4

Let S be a chain semiring. Then the units of $M_n(S)$ are $n \times n$ permutation (0, 1)-matrices. Thus a non zero non-permutation matrix $A \in M_n(S)$ is a prime provide that whenever $A = BC$ ($B, C \in M_n(S)$) B or C is a permutation matrix.

Definition: 3.5

The **zero-one pattern map π** from $M_n(S)$ to $M_n(\mathscr{B})$ is such that the (i, j)-entry $\pi(A)_{ij}$ of $\pi(A)$ is 1 if and only if the (i, j)-entry A_{ij} of A is not zero for i, j and $A \in M_n(S)$.

Note: 3.6

The pattern map π becomes a semiring homomorphism only when S is such an antinegative semiring with no zero divisors as $\mathscr{B}$ or $\mathscr{F}$.

Remark: 3.7

Even though $\pi(A)$ is prime in $M_n(\mathscr{B})$, A is not always prime but A is always semiprime in $M_n(S)$. A necessary condition for $A \in M_n(S)$ to be prime in $M_n(S)$ is that the maximum of the entries of each row and column of A is 1

Theorem: 3.8

Let R be chain semiring and let the maximum of the entries of each row and column of $A \in M_n(S)$ be 1. If $\pi(A)$ is prime in $M_n(\mathscr{B})$, then A is prime in $M_n(S)$. But the converse dose not always hold when S has more than three elements.

Proof:

Since S is a chain semiring. $\pi(BC) = \pi(B)\pi(C)$ for B and C in $M_n(S)$. Therefore, A is prime in $M_n(S)$ if $\pi(A)$ is prime in $M_n(\mathscr{B})$. Next, consider the following matrix.

$$A = \begin{pmatrix} 1 & \alpha & \beta & 0 \\ 0 & 1 & \alpha & \beta \\ \beta & 0 & 1 & \alpha \\ \alpha & \beta & 0 & 1 \end{pmatrix}, \text{ where } 0 < \pi < \beta < 1.$$

Even though A is prime in $M_n(S)$, $\pi(A)$ is not prime in $M_n(\mathscr{B})$. Therefore, the converse is not always true when there are more than three elements in S.

Example: 3.9

Let S be a semiring and let $S^n = \{(v_1, v_2, \ldots\ldots v_n) / v_i \in S\}$ denote an n-dimensional space over S. For A$\in$ $M_n(S)$ and integers i and j, A_{i*} and A_{*j} denote respectively the i^{th} row and j^{th} column of A. The **row space** R(A) of A is the subspace of S^n generated by the row $\{A_{i*}\}$ of A. The **row rank** of A is the smallest possible size of a spanning set for R(A). Column space C(A) and **column rank** of A are defined in dual fashion.

Definition: 3.10

Let S be a semiring and A,B $\in$ $M_n(S)$ •R(A) is a **maximal row space** in S^n provided that R(A) $\neq$ S^n, and R(A) $\subset$ R(B) implies R(B) = R(A) or R(B) = S^n.

Note: 3.11

(1) If R(A) is a maximal row space in S^n, then the semiring rank of A is n.

(2) The semiring rank of an $n \times n$ semiprime matrix or a prime matrix is n.

Remark: 3.12

For A,B$\in$ $M_n(S)$, we can see that R(π(A)) may not be a maximal row space in $\mathscr{B}^n$ even through R(A) is a maximal row space in S^n from Theorem 3.1, and there exists a matrix X$\in$ $M_n(S)$ such that A = XB if and only if R(A) $\subset$ R(B). Similarly, there exists a matrix Y$\in$ $M_n(S)$ such that A = BY if and only if C(A) $\subset$ C(B).

Definition: 3.13

A matrix $E \in M_n(S)$ is called an **elementary matrix** if E is permutationally equivalent to a direct sum of $\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$ and an (n-2) $\times$ (n-2) monomial matrix, where the a_i's are non-zero elements of S.

Definition: 3.14

An elementary (monomial) matrix $E \in M_n(S)$ is a **principal elementary (monomial) matrix** if R(E) is a maximal row space in S^n respectively.

Theorem: 3.15

Let S be a chain semiring. There exists a principal elementary matrix in $M_n(S)$ if and only if the infimum of the non-zero elements of S is not 0. There exists a principal monomial matrix in $M_n(S)$ if and only if the supremum of the nonunit elements of S is neither 0 nor 1.

Proof:

Let $A = [a_{ij}]$ be an elementary matrix. We may assume that A has a main diagonal whose entries are all nonzero. Let $\tilde{A} = [\tilde{a}_{ij}]$ be the matrix obtained from A by replacing the main diagonal elements with 1's. Then $\tilde{A}$ is also an elementary matrix, and there exist i, j ($i \neq j$) such that $a_{ij} = \tilde{a}_{ij} > 0$. Note that $R(A) \subset R(\tilde{A})$ since $A_{i*} = a_{ii}\tilde{A}_{i*} + a_{ij}\tilde{A}_{j*}$. Let β be an element of S with $0 < \beta < \tilde{a}_{ij}$, and let B be the matrix obtained from $\tilde{A}$ by replacing $\tilde{a}_{ij}$ with β. Then $R(\tilde{A}) \subsetneq R(B)$ and $R(B) \neq S^n$. Note that if there is no such β, then $\tilde{A}$ is a principal elementary matrix. Therefore a necessary and sufficient condition for the existence of a principal elementary matrix is that there is a non-zero element $\delta \in S$ such that $\delta \leq \alpha$ for all non-zero $\alpha \in S$.

Now let $A = [a_{ij}]$ be an identify matrix, and let $\tilde{A} = [\tilde{a}_{ij}]$ be the matrix obtained from A by replacing a diagonal entry a_{ii} with a nonunit element δ. Clearly, a necessary and sufficient condition for $\tilde{A}$ to be a principal monomial matrix is that $\delta \neq 0$ and $\alpha \leq \delta$ for all nonunit $\alpha \in S$. Thus the second statement can be proved using this observation.

Note: 3.16

$M_n(\mathscr{F})$ does not have a principal elementary matrix but $M_n(\mathscr{B})$ does by Theorem 3.15. Also note that both $M_n(\mathscr{B})$and $M_n(\mathscr{F})$ do not have a principal monomial matrix since the supremum of the non unit elements is zero or one.

If S be a chain semiring and if $\delta\,(\neq 0)$ is the infimum of the non-zero elements of S, then by Theorem 3.15, $E \in M_n(S)$ is a principal elementary matrix if and only if E is permutationally equivalent to a direct sum of $\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$ and an $(n\text{-}2) \times (n\text{-}2)$ permutation matrix.

Definition: 3.17

$A \in M_n(S)$ is **dominated** by $B \in M_n(S)$, denoted by $A \leq B$, if the (i, j) – entry A_{ij} of A is less than or equal to B_{ij} for positive integers i, j($\leq$ n).

Note: 3.18

If $A \in M_n(S)$ is prime, then no row (column) of $\pi(A)$ is dominated by another row (column) respectively.

Theorem: 3.19

Let S be a chain semiring and $A \in M_n(S)$. R(A) is a maximal row space in S^n if and only if A is a prime matrix or a principal elementary matrix or a principal monomial matrix.

Proof:

($\Leftarrow$) Let A be prime in $M_n(S)$ and $R(A) \subset R(B) \neq S^n$ for sum $B \in M_n(S)$. Then A = XB for sum $X \in M_n(S)$. Since B is not a permutation matrix, X is a unit (permutation matrix) in $M_n(S)$. Thus R(A) = R(B) and R(A) is a maximal row space in S^n. By definition R(A) is a maximal row space if A is a principal elementary matrix or a principal monomial matrix.

($\Rightarrow$) Let $A = [a_{ij}] \in M_n(S)$ and R(A) be a maximal row space in S^n. Note that A dominates a monomial matrix. If A is an elementary (monomial) matrix, then A must be a principal elementary (monomial) matrix respectively. Thus in what follows we assume that A is neither an elementary matrix nor a monomial matrix, and we will show that A is a prime matrix. First of all, we claim that no row of $\pi(A)$ dominates another row and argue as follows. Suppose a row of $\pi(A)$ dominates another row, say $\pi(A)_{1*} \geq \pi(A)_{2*}$. We have two cases: A_{1*} dominates A_{2*}, or A_{1*} does not dominates A_{2*}. We will show that both cases lead to a contradiction.

Case1: A_{1*} dominates A_{2*}. Let $a_{2k} = \max\{a_{2j} \mid 1 \leq j \leq n\}$ for some k, and let $\tilde{A} = [\tilde{a}_{ij}]$ be the matrix obtained from A by replacing a_{1k} and a_{2k} with 0 and 1 respectively. Note that $R(A) \subset R(\tilde{A})$ due to the fact that $A_{1*} = \tilde{A}_{1*} + a_{1k}\tilde{A}_{2*}$ and $A_{2*} = a_{2k}\,\tilde{A}_{2*}$. Suppose $R(A) = R(\tilde{A})$, then

$\tilde{A}_{1*} = \sum_{i=1}^{n} \alpha_i A_{i*} = \sum_{i=3}^{n} \alpha_i A_{i*} = \sum_{i=3}^{n} \alpha_i \tilde{A}_{i*}$ since $a_{1k} \geq a_{2k} > 0$ and $\tilde{a}_{1k} = 0$. Thus $\tilde{A}_{1*}$ is generated by the other rows of $\tilde{A}$. But none of the rows of A can be generated by the other rows since R(A) is maximal row space in S^n. Therefore $R(\tilde{A}) = S^n$ and $\tilde{A}$ is permutation matrix due to the maximality of R(A). Thus A is an elementary matrix, a contradiction.

Case2: A_{1*} does not dominate A_{2*}. Suppose $a_{1k} = \min \{a_{1j} | a_{1j} < a_{2j}$ and $1 \leq j \leq n\}$, and let $\tilde{A}$ be the matrix obtained from A by replacing from a_{1k} with 0. If $a_{1k} = 0$, then $a_{2k} = 0$ since $\pi(A)_{1*} \geq \pi(A)_{2*}$, since $a_{2k} > a_{1k}$, we have $a_{1k} > 0$ and $A_{1*} = \tilde{A}_{1*} + a_{1k}\tilde{A}_{2*}$. Note that if $R(A) = R(\tilde{A})$, then $\tilde{A}_{1*} = \sum_{i=1}^{n} \alpha_i A_{i*} = \sum_{i=3}^{n} \alpha_i A_{i*}$ since $a_{1k} > a_{2k} > 0$. Therefore $R(\tilde{A}) = S^n$ and A is permutation matrix. Since A is not an elementary matrix, this is a contradiction. In summary, we show that no row of π(A) dominates another row if A is neither an elementary matrix nor a monomial matrix and if R(A) is a maximal row space in S^n.

Second, Let A = XB for some X, B$\in M_n(S)$. If B is not a permutation matrix, then $R(A) \subset R(B) \neq S^n$. Then by the maximality of R(A), R(A) = R(B) and B = YA for some Y$\in M_n(S)$. Thus A = ZA where Z = XY, and $\pi(A) = \pi(Z)\, \pi(A)$ where $\pi(Z) = \pi(X)\, \pi(Y)$ since R is chain semiring. Note that $\pi(Z)$, $\pi(X)$, and $\pi(Y)$ are permutation matrices since no row of $\pi(A)$ dominates another row and A has a full semiring rank. Thus X, Y and Z are monomial matrices. We now claim that the maximum of the entries of each row of A is 1. Suppose the maximum of a row of A is not 1, say $a_{1k} = \max \{a_{1j} | 1 \leq j \leq n\} \neq 1$ for some k. Let $\tilde{A}$ be the matrix obtained from A by replacing a_{1k} with 1. Note that if $\widetilde{A}_{1*} = \sum_{i=1}^{n} \alpha_i A_{i*}$, then $\alpha_j \neq 0$ for some $j \neq 1$. Then $\pi(A)_{j*} \leq \pi(A)_{1*}$ since $\alpha_j \neq 0$ and $\alpha_j A_{j*} \leq \widetilde{A}_{1*}$, and $R(A) \not\subset R(\widetilde{A})$ holds. This is a contradiction since A is not a monomial matrix and $\widetilde{A}$ can't be a permutation matrix. Thus the maximum of the entries of each row of A is 1. Hence X is a permutation matrix since A = ZA = XYA, and A is a prime matrix in $M_n(S)$.

Definition: 3.20

Let S be a semiring and A, B $\in M_n(S)$. R(A) is a **semimaximal row space** in S^n provided that $R(\pi(A)) \neq \mathscr{B}^n$, and $R(A) \subset R(B)$ implies $R(\pi(B)) = R(\pi(A))$ or $R(\pi(B)) = \mathscr{B}^n$.

Remark: 3.21

For a non-monomial matrix $A \in M_n(S)$, if $R(\pi(A))$ is a maximal row space in $\mathscr{B}^n$, then R(A) is a semimaximal row space in S^n. But the converse is not always true as shown in the following example.

Example: 3.22

For the matrix $A = \begin{pmatrix} 1 & \alpha & \beta & 0 \\ 0 & 1 & \alpha & \beta \\ \beta & 0 & 1 & \alpha \\ \alpha & \beta & 0 & 1 \end{pmatrix}$, where $0 < \alpha < \beta < 1$ we see that R(A) is a semimaximal row space in S^n even though $R(\pi(A))$ is not a maximal row space in $\mathscr{B}^n$.

Theorem: 3.23

If $A \in M_n(S)$ is a semiprime matrix or an elementary matrix, then R(A) is a semimaximal row space in S^n.

Proof:

Let A be a semiperime matrix in $M_n(S)$. Then, $R(A) \subset R(B)$ for some non-monomial matrix $B \in M_n(S)$ implies that A = XB and X is a monomial matrix. Thus $R(\pi(A)) = R(\pi(B))$ and R(A) is a semimaximal row space in S^n. If A is an elementary matrix, then $R(\pi(A))$ is a maximal row space in $\mathscr{B}^n$ and R(A) is a semimaximal row space in S^n.

Note: 3.24

A is not a monomial matrix if R(A) is a semimaximal row space in S^n.

Theorem: 3.25

Let S be a chain semiring and $A \in M_n(S)$. If R(A) is a semimaximal row space in S^n, then the following statements hold.

(i) The semiring rank of A is n.

(ii) If no row of $\pi(A)$ dominates another row, then A is a semiprime matrix.

(iii) If a row of $\pi(A)$ dominates another row, then $A = E_1........E_d$ $(d \geq 1)$ or $A = E_1...........E_dP$ $(d \geq 1)$, where each E_i denotes an elementary matrix and P denotes a semiprime matrix.

Proof:

(i) Suppose the semiring rank r of A is less than n, and let A = BC, where B and C are $n \times r$ and $r \times n$ matrices respectively. If there is exactly one nonzero entry in each row of C, then there is a row vector $v = (v_1, v_2 \ldots\ldots\ldots v_n) \in \mathscr{B}^n (\subset S^n)$ such that there are exactly two nonzero entries in v and $v \notin R(\pi(C))$ since $r<n$. If there is a row of C that has more than one nonzero entry, then there is a row vector $v = (v_1, v_2 \ldots\ldots\ldots v_n) \in \mathscr{B}^n$ such that there is exactly one nonzero entry in v and $v \notin R(\pi(C))$ since $r<n$. Now consider a $n \times n$ matrix D obtained from C by attaching the two row vector v n-r times. Note that $R(A) \subset R(C) \subset R(D)$, and $R(\pi(D)) \neq \mathscr{B}^n$. Thus $R(\pi(A)) = R(\pi(D))$, and we have a contradiction since $\pi(v) \notin R(\pi(A))$.

(ii) Suppose no row of $\pi(A)$ dominates another row. We claim that if $A = XB$ for some X, $B \in M_n(S)$, then $X = [x_{ij}]$ or B is a non monomial matrix. If B is not a monomial matrix and we may assume $x_{ii} > 0$ for all i. Then $\pi(A) \geq \pi(B)$ holds since $\pi(A) = \pi(X)\,\pi(B)$. If $\pi(A)_{i*} > \pi(B)_{i*}$ for some i, then $\pi(A)_{i*}$ dominates another row of $\pi(A)$ since $\pi(B)_{i*}$ is generated by the rows of $\pi(A)$. Thus we have $\pi(A) = \pi(B)$ and $\pi(A) = \pi(X)\,\pi(A)$ holds. Since no row of $\pi(A)$ dominates another row, $\pi(X)$ must be a permutation matrix and consequently X is a monomial matrix. Hence A is semiprime matrix in $M_n(S)$.

(iii) Suppose now that a row of $\pi(A)$ dominates another row say $\pi(A)_{i*} \geq \pi(B)_{i*}$, and we assume A is not an elementary matrix to consider the nontrivial cases. We have two cases: A_{1*} dominates A_{2*} or A_{1*} does not dominate A_{2*}. For each case, following the proof of

Theorem 3.19, we can construct a matrix $A_1(=\tilde{A})$ such that $A = E_1A_1$. Here, E_1 is an elementary matrix obtained from the identity matrix $E = [e_{ij}]$ by replacing e_{12} and e_{22} with a_{1k} and a_{2k} respectively if $A_{1*} \geq A_{2*}$, and by replacing e_{12} with a_{1k} if not ($\tilde{A}$ and k are defined in the proof of Theorem 3.19). Note that $w(A) = w(A_1) + 1$. If there is a row domination among the rows of $\pi(A)_1$ and $w(A_1) > n$, repeating the above procedure, we can construct an elementary matrix E_2 and a matrix A_2 such that $A_1 = E_2 A_2$ and $w(A_1) = w(A_2) + 1$. Note that the right – hand – side factor A_i must dominate a monomial matrix by the previous (i). Therefore, by repeating this argument, we eventually arrive at the situation where the final A_i is an elementary matrix, or there is no row domination among the rows of $\pi(A_i)$ and $w(A_1) > n$. By (ii), A_i is a semiprime matrix if there is no row domination among the rows of $\pi(A_i)$ and $w(A_i) > n$ since $R(A_i)$ is also a semimaximal row space in S^n.

Definition: 3.26

Let $A \in M_n(\mathscr{B})$ is called a **Hall matrix** if A dominates an $n \times n$ permutation matrix.

Note:-3.27

The set $H_n(\mathscr{B})$ of all $n \times n$ Hall matrices forms a subsemiring of $M_n(\mathscr{B})$.

Definition: 3.28

When $A \in H_n(\mathscr{B})$ dominates a permutation matrix P, P is called an **identifying permutation matrix of A** if $P_{*i} \leq A_{*j}$ implies $A_{*i} \leq A_{*j}$ for each i and j.

Theorem: 3.29

Let $A \in H_n(\mathscr{B})$. Then the following statements are equivalent;

(i) A is regular in $H_n(\mathscr{B})$.

(ii) A dominates an identifying permutation matrix.

Theorem: 3.30

Let S be a chain semiring, and let $A \in M_n(S)$ be a non-monomial matrix with full semiring rank. Then $A = P_d P_{d-1} \ldots\ldots P_1$ $(d \geq 1)$, where each P_i is an elementary matrix or a semiprime matrix in $M_n(S)$.

Proof:

If A is elementary or semiprime in $M_n(S)$, then the Theorem holds by getting P_1 be A. Thus in what follows we consider the case when $A = [a_{ij}]$ is neither elementary nor semiprime in $M_n(S)$. We claim that we can choose B and M in $M_n(S)$ so that A = BM, w(A) >w(B), and M can be expressed as a product using elementary matrices and (or) a semiprime matrix. Note that every $n \times n$ factor F of A dominates a monomial matrix since the semiring rank of F is also n.

First, suppose a column of $\pi(A)$ dominates another column, say let $\pi(A)_{*i} \leq \pi(B)_{*i}$, we have two cases: A_{*1} dominated by A_{*2}, or A_{*1} is not dominated by A_{*2}.

Case 1: A_{*1} dominated by A_{*2}. Suppose for some k, $a_{k1} = \max\{a_{i1} \mid 1 \leq i \leq n\}$. By letting B be the matrix obtained from A by replacing a_{k1} and a_{k2} with 1 and 0 respectively, and M be the matrix obtained from identity matrix $E = [e_{ij}]$ by replacing e_{11} and e_{12} with $a_{k1}$and a_{k2} respectively, we have A = BM where w(A) >w(B), and M is an elementary matrix.

Case 2: A_{*1} not dominated by A_{*2}. Suppose for some k, $a_{k2} = \min\{a_{i2} \mid a_{i1} > a_{i2}$ and $1 \le i \le n\}$. By letting B be the matrix obtained from A by replacing a_{k2} with 0, and M be the matrix obtained from identity matrix $E = [e_{ij}]$ by replacing e_{12} with a_{k2}, we have A = BM where w(A) >w(B), and M is an elementary matrix.

Second, suppose now that no column of $\pi(A)$ dominates another column. Since A is not a monomial matrix there is a matrix $M \in M_n(S)$ such that R(M) is a semimaximal row space in S^n and $R(A) \subset R(M)$. Thus there is a matrix $B \in M_n(S)$ such that A = BM. Note that for a permutation matrix Q no column of $\pi(AQ)$ dominates another column, and we say assume that M has a positive main diagonal. Then $\pi(A) \geq \pi(B)$ and $w(A) \geq w(B)$ hold since $\pi(A) = \pi(B)\,\pi(M)$. We now claim that w(A) >w(B) in this case. Suppose w(A) = w(B) (i.e, $\pi(A) = \pi(B)$). Note that $\pi(A) = \pi(B)\,\pi(M) = \pi(A)\,\pi(M) = \pi(A)\,\pi(M)^g$ for any positive integer g in this case. Consider the trace $T = \{\pi(M)^g \mid g$ is a positive integer$\}$. Then $\pi(M)^i = \pi(M)^j$ for some positive integer i and j (i<j) since $H_n(\mathscr{B})$ is finite set. Note that $G = \{\pi(M)^i, \pi(M)^{i+1}, \ldots\ldots\ldots \pi(M)^j\}$ forms a cyclic group with identity $\pi(M)^P$ for some P. Note that $\pi(M)^P$ is an idempotent matrix, and thus a regular matrix. Since $\pi(M)^P$ is regular in $H_n(\mathscr{B})$, $\pi(M)^P$, has an identifying permutation matrix Q by Theorem 3.29. Since $\pi(M)^P$ is not a permutation matrix, some i-th column of $\pi(M)^P$ dominates some j-th column $(i \neq j)$ of Q, so i-th column of $\pi(M)^P$ dominates j-th column of $\pi(M)^P$. Then from $\pi(A) = \pi(A)\,\pi(M)^P$ i-th column of

$\pi(A)$ dominates j-th column of $\pi(A)$, a contradiction. Therefore we have $w(A) < w(B)$. Furthermore, by Theorem 3.25 (ii) & (iii), M can be expressed as a product using elementary matrices and (or) a semiprime matrix since R(M) is a semimaximal row space in S^n.

In summary, if a non-monomial matrix A will full semiring rank is neither elementary nor semiprime in $M_n(S)$, then there exist B and M in $M_n(S)$ such that A = BM with $w(A) > w(B)$, and M satisfies the claim. If this matrix B is a monomial matrix, then we prove the theorem. The reason is that the first factor of M is either an elementary matrix or a semiprime matrix by Theorem 3.25 (ii) & (iii), and the product of a monomial matrix and an elementary (semiprime) prime is an elementary (semiprime) matrix respectively. Now consider the case when the above B is a non-monomial matrix. By the previous reasoning, if this matrix B is neither elementary nor semiprime in $M_n(S)$, then there exist C and N in $M_n(S)$ such that $w(B) > w(C)$ and B = CN, where N can be expressed using elementary matrices and (or) a semiprime matrix. Thus A = BM = (CN)M and $w(A) > w(B) > w(C)$, and repeating this argument, we eventually arrive at the situation where the left-hand-side factor is monomial or elementary or semiprime in $M_n(S)$ since the number of nonzero entries of left-hand-side factor is strictly decreasing.

Corollary: 3.31

Let S be a chain semiring, and let $A \in M_n(S)$ be a non-monomial matrix. If $\pi(A)$ is a regular matrix of $M_n(\mathscr{B})$ with full semiring rank, then A can be written as a product of elementary matrices.

Proof:

We now apply the above theorem to prove the following corollary. Given S be a chain semiring and $A \in M_n(S)$. The perment per(A) of A is number of $n \times n$ permutation submatrices of $\pi(A)$. Assume that $\pi(A)$ is non-monomial regular matrix of $M_n(\mathscr{B})$ having a full semiring rank. Then

per(A) = 1. Since $\pi(A)$ has a full semiring rank and the semiring rank of $\pi(A)$ is less than or equal to that of A, A has a full semiring rank too. Therefore A can be written as $P_d P_{d-1} \ldots\ldots\ldots P_1$ with P_i elementary or semiprime in $M_n(S)$ by Theorem 3.30. Since $per(\pi(A)) \geq per(\pi(P_i))$ for each i and the permanent of any semiprime matrix in $M_n(S)$ is greater than one, none of the P_i's can be a semiprime matrix in $M_n(S)$. Therefore A can be written as $P_d P_{d-1} \ldots\ldots\ldots P_1$, where each P_i is an elementary matrix in $M_n(S)$.