Chapter 1

Preliminaries

Basic definitions and results in topological spaces that are used to accomplish the present study are given in this chapter.

1.1 Topological Spaces

Definition 1.1.1: Throughout the thesis $(X,\tau),(Y,\sigma)$ and (Z,η) denote topological spaces on which no separation axioms are mentioned unless or otherwise stated.

If A is non-empty subset of (X,τ) then the union of all open sets contained in A is called interior of A and it is denoted by $Int(A)$. The intersection of all closed sets containing A is called closure of A and it is denoted by $Cl(A)$.

Definition 1.1.2: A subset A of a space X is said to be

a) Regular open [Stone, 1937] if A = Int (Cl(A))

b) Semi-open [Levine, 1963] if A ⊆ Cl (Int(A))

c) δ-open [Veliko, 1968] if for each x ϵ A, there exists an open set G such that x ∈ G ⊆ IntClG ⊆ A

d) Clopen if A is both open and closed

e) θ-semi-open [Joseph, 1980] if for each x ϵ A, there exists an semi-open set G such that x ∈ G ⊆ ClG ⊆ A

f) Preopen [Mashhour, 1982] if A ⊆ Int (Cl(A))

g) β-open [Abd EI-Monsef, 1983] if A ⊆ Cl (Int (Cl(A)))

h) semi-θ-open [Di Maio, 1987] if for each x ϵ A, there exists an semi-open set G such that x ∈ G ⊆ sClG ⊆ A

i) α-open [Masshour, 1987] if A ⊆ Int(Cl(Int(A)))

j) δ-preopen [Rayachaudhuri, 1993] if A ⊆ Int(δCl(A))

k) δ-semi-open [Park, 1997] if $A \subseteq Cl(\delta - Int(A))$

l) η-open [Dontchev, 1998] if A is a union of δ-closed sets.

m) θ-open [Yunis, 2001] if for each x ϵ A there exists an open set G such that x ∈ G ⊆ ClG ⊆ A

n) e*-open [Ekici, 2009] if $A \subseteq Cl(Int(\delta - Cl(A))$

o) δ-semi-θ-open [Zanyar, 2019] if for each $x \in A$, there exists a δ-semiopen such that $x \in U \subseteq \delta\text{-} scl(U) \subseteq A$

p) δ-semiregular [Zanyar, 2019] if $A = \delta\ sInt(\delta\ sCl(A))$

- The closure and interior of A with respect to X are denoted by Cl(A) and Int(A) respectively.
- The family of all preopen (resp. Semi-open, regular open, δ-open, θ-open, δ-preopen, θ-semiopen, β-open, η-open, e*-open, δ-semiopen, α-open, δ-semi-θ-open, δ-semiregular open) subsets of X is denoted by PO(X) (resp. SO(X), RO(X), δO(X), θO(X), δPO(X), θSO(X), SθO(X), βO(X), ηO(X), e*O(X), δSO(X), αO(X), δSθO(X), δSRO(X)
- The complement of a preopen (resp. Semi-open, regular open, δ-open, θ-open, δ-preopen, θ-semiopen, semi-θ-open, β-open, η-open, e*-open, δ-semiopen, α-open, δ-semi-θ-open, δ-semiregular open) is said to be preclosed (resp. Semi-closed, regular closed, δ-closed, θ-closed, δ-preclosed, θ-semiclosed, semi-θ-closed,β-closed, η-closed, regular semiclosed, e*-closed, δ-semiclosed, α-closed, δ-semi-θ-closed, δ-semiregular closed).
- The family of all preclosed (resp. resp. Semi-closed, regular closed, δ-closed, θ-closed, δ-preclosed, θ-semiclosed, β-closed, η-closed, regular semiclosed, e*-closed, δ-semiclosed, α-closed, δ-semi-θ-closed, δ-semiregular closed) subsets of X is denoted by PC(X) (resp. SC(X), RC(X), δC(X), θC(X), δPC(X), θSC(X), SθC(X), βC(X), ηC(X), e*C(X), δSC(X), αC(X), δSθC(X), δSRC(X)).
- The family of all α-open sets in a topological space (X, τ) is a topology on X finer than τ denoted by τ_α
- The intersection of particular class of closed sets of X containing A is called the corresponding closure of A.
- The union of particular class of open sets of X contained in A is called the corresponding interior of A.

Definition 1.1.3 [Navalalagi 1998]. A subset A of a space X is said to be preregular if A is both preopen and preclosed.

Definition 1.1.4 [Ahmed, 1990]. A space X is s-regular if for each $x \in X$ and each open set G containing x, there exists a semi-open set H such that $x \in H \subseteq sClH \subseteq G$.

Definition 1.1.5 [Maheshwari, 1975]. A space X is called semi-T_1 if for each pair of distinct points x, y in X, there exists a pair of semi-open sets, one containing x but not y and the other containing y but not x.

Definition 1.1.6 [Di Maio, 1987]. A space (X, τ) is said to be extremally disconnected if $ClU \in \tau$ for every $U \in \tau$.

Definition 1.1.7 [Dontchev, 1998]. A space X is said to be hyperconnected if every non-empty open subset of X is dense.

Definition 1.1.8 [Dontchev, 1998]. A space X is called locally indiscrete if every open subset of X is closed.

Definition 1.1.9[Khalaf, 2009]. A subset A of a space X is called P_S-open if for each $x \in A \in PO(X)$, there exists a semi-closed set F such that $x \in F \subseteq A$. The family of all P_S-open sets of a topological space(X,τ) is denoted by $P_SO(X,\tau)$ or $P_SO(X)$.

Theorem 1.1.10 [Maheshwari, 1975]. A space X is semi-T_1 if for any point $x \in X$, the singleton set $\{x\}$ is semi-closed.

Theorem 1.1.11 [Guo, 1981]. A space X is extremally disconnected if and only if $RO(X) = RC(X)$.

Theorem 1.1.12 [Yunis, 2001]. A space X is extremally disconnected if and only if $\delta O(X) = \theta SO(X)$.

Lemma 1.1.13 [Dlaska, 1992]. A space (X, τ) is hyperconnected if and only if $RO(X) = \{\emptyset, X\}$.

Lemma 1.1.14[Dontchev, 1999]. If X is locally indiscrete space, then

a) Each semi-open subset of X is closed and

b) Each semi-closed subset of X is open.

Theorem 1.1.15 [Ahmed, 1990]. Let (Y,τ_Y) be a subspace of a space (X,τ). Then, the following statements are true:

a) If $A \in PO(X,\tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$.

b) If $F \in SC(X,\tau)$ and $F \subseteq Y$, then $F \in SC(Y, \tau_Y)$.

c) If $F \in SC(Y, \tau_Y)$ and $Y \in SC(X,\tau)$, then $F \in SC(X,\tau)$.

Theorem 1.1.16 [Rayachaudhuri, 1993]. a) In a topological space (X,τ), if A ϵ δPO(X), B ϵ δO(X) then $A \cap B \in \delta PO(X)$.

b) In a topological space (X, τ), if A ∈ δP O(X), B ∈ δO(X), then A ∩ B ∈ δPO(B).

c) Let A, Y be subsets of a topological space (X, τ) such that A ⊆ Y ∈ δO(X). If A ∈ δPO(Y), then A ∈ δPO(X).

Lemma 1.1.17[Ganster, 2000]. If Y is an open subspace of a space X and F ∈ SC(X), then F ∩ Y ∈ SC(Y).

Lemma 1.1.18[Jankovic, 1985]. Let A be a subset of a space (X,τ). Then A ∈ PO(X,τ) if and only if sClA = IntClA.

Theorem 1.1.19. Let A be a subset of a topological space (X,τ). Then, we have:

a) If A ∈ SO(X), then pClA = ClA [Dontchev, 1998].

b) If A ∈ βO(X), then $Cl_{\delta}A$ = ClA [Ramprasad Paul, 1999].

c) If $A \in \beta O(X)$, then $\alpha Cl(A) = Cl(A)$.[Abdulla, 1986].

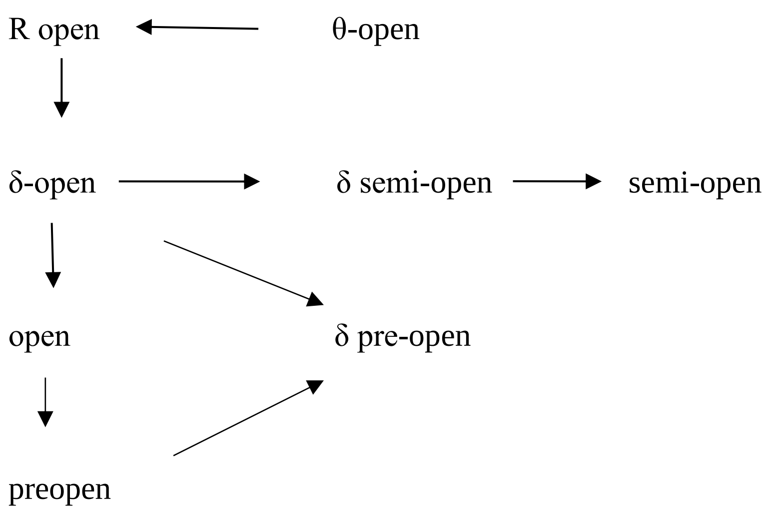
Theorem 1.1.20 [Zanyar, 2019]. Let A, Y be subsets of a topological space (X,τ) and let A ⊆ Y ϵ δPO(X). Then A ϵ δPO(X) if and only if A ϵ δPO(Y).

Lemma 1.1.21 [Caldas, 2005]. Let A be a subset of a topological space (X, τ). Then the following properties hold:

(a) If A is preopen in (X, τ), then it is δ-preopen in (X, τ),

(b) A is δ-preopen in (X, τ) if and only if it is preopen in (X, τ_s).

Remark 1.1.22. From the following figure we have:



Lemma 1.1.23. Each clopen is regular open.

Lemma 1.1.24 [Rayachaudhuri, 1993]. Union of δ-preopen sets is δ-preopen.

Lemma 1.1.25 [Zanyar, 2019]. In a topological space (X, τ), if $A \in \delta PO(X), B \in aO(X)$, then $A \cap B \in \delta PO(X)$

Lemma 1.1.26 [Khalaf, 2009]. Every δ-open set is P_S-open set.

Lemma 1.1.27 [Caldas, 2005]. If A is preopen then A is δ-preopen in (X, τ).

Lemma 1.1.28 [Zanyar, 2019]. Let A, Y be subsets of a topological space (X, τ) and let $A \subseteq Y \in \delta PO$. Then $A \in \delta PO(X)$ if and only if $A \in \delta PO(Y)$.

Lemma 1.1.29 [Zanyar, 2019]. Let A, B be subsets of topological space (X, τ). If $A \in \delta SO(X)$ and $B \in \delta PO(X)$, then $A \cap B \in \delta PO(A)$.

Lemma 1.1.30[Dontchev, 1999]: Let A be regular open by then A is semiclosed and preopen.

Proposition 1.1.31[Khalaf 2009]: For any subset A of a space X. If A ∈ δO(X), then A ∈ P_SO(X).

Theorem 1.1.32[Benchalli 2017]. For a subset A of a space X the following are equivalent:

a) A is clopen

b) A is δ-open and δ-closed

c) A is regular-open and regular-closed.

Corollary 1.1.33[Khalaf 2009]. For any subset A ⊆ X. The following conditions are equivalent:

a) A is clopen.

b) A is P_S-open and closed.

c) A is α-open and closed.

d) A is preopen and closed.

Corollary 1.1.34 [Khalaf 2009]. For any subset A ⊆ X. The following statements are equivalent:

a) A is regular open.

b) A is P_S-open and semi-closed.

c) A is open and semi-closed.

d) A is α-open and semi-closed.

e) A is preopen and semi-closed.

Theorem 1.1.35 [Zanyar 2019]. For any subset A of a space (X, τ), the following statements are equivalent:

a) A is regular open.
b) A is δ-open and δ-semiregular.
c) A is δ-open and δ-semi-θ-closed.
d) A is δ-open and δ-semiclosed.
e) A is a-open and δ-semiclosed.
f) A is δ-preopen and δ-semiclosed.
g) A is a-open and e* -closed.

Proposition 1.1.36 [Dontchev 1998]. The family of all η-open sets of a topological space (X, τ) is a topology τ_η on X containing the co-semi-regularization.

θ-semi-open ⇒ η-open ⇒ V-set

Definition 1.1.37 [Ramprasad Paul, 1999]. A space (X, τ) is said to have the property P if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.

From the above Definition 1.1.37 Paul and Bhattacharyya, 1999 pointed out the following remark:

Remark 1.1.38 [Ramprasad Paul, 1999]. If a space X has the property P, then the intersection of any two preopen sets is preopen, as a consequence of this, PO (X, τ) is a topology for X and it is finer than τ.

Theorem 1.1.39 [Abd El-Monsef, 1983]. Let A ⊆ X, then A ∈ βO(X) if and only if ClA = ClIntClA.

Theorem 1.1.40 [Levine, 1963]. Let A ⊆ X, then A ∈ SO(X) if and only if ClA = Cl IntA.

Theorem 1.1.41 [Tadros, 1998]. Every regular semi-open set is ssemi-closed.

Lemma 1.1.42 [Stone, 1937]: For an open set δ-closure and closure coincide.

1.2 Separation Axioms

Definition 1.2.1. A topological space (X, τ) is called

a) **almost weakly Hausdorff [Dontchev, 2004]** if its semi-regularization is $T_{1/2}$. In other words, g-closed sets of (X, τ_s) are δ-closed in (X, τ).
b) **weakly Hausdorff [Dontchev, 2004]** iff each singleton is δ-closed.

c) $T_{3/4}$-space [Dontchev, 1996] if every δg-closed set is δ-closed in (X, τ).

d) $_{\delta}T_{3/4}$-space [Benchalli, 2012] if every gδs-closed set is δ-closed in (X, τ).

e) T_{δ}-space [Dontchev, 2004] if every gδ-closed set is δ-closed in (X, τ).

f) δ-T_0 [Georgiou, 2004] if for any distinct pair of points in X, there is a δ-open set U containing one of the points but not the other.

g) δ-T_1 [Georgiou, 2004] if for any distinct pair of points x and y in X, there is a δ-open U in X containing x but not y and a δ-open set V in X containing y but not x.

h) δ-T_2 [Georgiou, 2004] if for any distinct pair of points x and y in X, there are δ-open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.

i) semi-regular [Dontchev, 2000] if its regular open sets form a base. In a semi-regular space, open sets.

j) $T_{1/2}$-space [Levine, 1970] if every g-closed set is closed.

Definition 1.2.2: A filter base $\mathfrak{J}$ is said to be

a) **p-converges [Jafari, 1998]** to a point $x \in X$ if for every preopen set V containing x, there exists an $F \in \mathfrak{J}$ such that $F \subseteq V$
b) **pre-θ-converges [Di Maio, 2000]** to a point $x \in X$ if for every preopen set V containing x, there exists an $F \in \mathfrak{J}$ such that $F \subseteq pCl(V)$.
c) **δ-converges [Velicko, 1968]** to a point $x \in X$ if for every open set V containing x, there exists an $F \in \mathfrak{J}$ such that $F \subseteq Int\ Cl(V)$.
d) **p-accumulates [Di Maio, 1987]** to a point $x \in X$ if $F \cap V \neq \emptyset$ for every preopen set V containing x and every $F \in \mathfrak{J}$.
e) **pre-θ-accumulates [Jafari, 1998]** to a point $x \in X$ if $F \cap pCl(V) \neq \emptyset$ for every preopen set V containing x and every $F \in \mathfrak{J}$.
f) **δ-accumulates [Velicko, 1968]** to a point $x \in X$ if $F \cap Int\ Cl(V) \neq \emptyset$, for every open set V containing x and every $F \in \mathfrak{J}$.

Definition 1.2.3: A topological space (X,τ) is said to be

a) **nearly compact [Singal, 1969]** if X is N-closed relative to X.
b) **N-closed [Noiri,1978]** relative to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup \{Int\ Cl(V_\alpha) : \alpha \in \Delta_0\}$
c) **quasi- H-closed [Porter, 1969]** relative to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup \{Cl(V_\alpha) : \alpha \in \Delta_0\}$.

d) **strongly compact [Masshour, 1984]** if every preopen cover of X has a finite subcover.

e) **α-compact [Di Maio, 2000]** if every α-open cover of X has a finite subcover.

Lemma 1.2.4 [Baravan A.Asaad, 2016]: If a topological space (X,τ) is P_S-compact, then it is nearly compact.

Example 1.2.5 [Di Maio, 2000]: (a) The unit interval [0,1] with the usual topology is compact, hence QHC, but not p-closed since it is resolvable.

(b) Let $X = \mathfrak{R}, \tau = \{X, \emptyset, \{0\}\}$. Then, X is p-closed and s-closed but not α-compact and hence not strongly compact.

1.3 Functions

Definition 1.3.1. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be

a) **continuous** (Levine, 1970) if $f^{-1}(V)$ is a closed set in (X, τ) for every closed set V in (Y, σ).

b) **semi-continuous** (Levine, 1963) if $f^{-1}(V)$ is a semi-closed set in (X, τ) for every closed set V in (Y, σ)

c) **precontinuous** [Masshour, 1982] if the inverse image of each open subset of Y is preopen in X

d) **α-continuous** [Reilly, 1985], if the inverse image of each open subset of Y is α-open in X

e) **θs-continuous** [Khalaf, 1999], if the inverse image of each open subset of Y is θ-semi-open in X

f) **perfectly continuous** [Noiri, 1976], if the inverse image of each open subset of Y is clopen in X

g) **complete continuous** [Arya, 1974] if the inverse image of each open subset of Y is regular open in X

h) **super continuous** [Munshi, 1982]) if the inverse image of each open subset of Y is δ-open in X.

i) **δ-continuous** (Noiri, 1980) if $f^{-1}(V)$ is δ-open in (X, τ) for every δ-open set V in (Y, σ).

j) **θ-continuous** [Fomin, 1943]) if for each x ∈ X and each open set V of Y containing f(x), there exists an open set U of X containing x such that $f(Cl(U)) \subseteq Cl(V)$.

Definition 1.3.2. A function $f:(X,\tau) \to (Y,\sigma)$is said to be

a) **almost precontinuous** [Nasef, 1997], if the inverse image of each regular open subset of Y is preopen in X.

b) **almost continuous** [Singal and Singal, 1968], if the inverse image of each regular open subset of Y is open in X.

c) **almost θs-continuous** [Khalaf, 2001], if the inverse image of each regular open subset of Y is θ-semi-open in X.

d) **contra-continuous** [Dontchev, 1996], if the inverse image of every open subset of Y is closed in X.

e) **contra-semi-continuous** [Dontchev, 1999] if the inverse image of every open subset of Y is closed semiclosed in X.

f) **contra-δ-continuous** [Zainab] if $f^{-1}(V)$ is δ-closed in (X, τ) for every open set V in (Y, σ).

g) **δ-precontinuous** if for each x ∈ X and each δ-preopen set V containing f(x), there is a δ-preopen set U in X containing x such that f(U) ⊆ V.

h) **contra e*-continuous** [Ekici, 2008] if $f^{-1}(V)$ is e^*-closed in (X, τ) for every open set V in (Y, σ).

i) **δ*-almost-continuous** [Dontchev, 1999], if $f^{-1}(V) \in \delta PO(X,\tau)$ for each $V \in \delta PO(Y,\tau)$.

j) **Somewhat continuous** (Gentry et.al., 1971) if for $U \in (Y,\sigma)$ and $f^{-1}(U) \neq \emptyset$, there exists $V \in (X,\tau)$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$.

Definition 1.3.3. A function f : (X, τ) → (Y, σ) is called a

a) **Semi closed** (resp. semi open) **function** (Noiri, 1973) if f(V) is semi closed (resp. semi open) in (Y, σ) for every closed (resp. open) set V of (X, τ).

b) **δ-closed** (resp. δ-open) **function** (Noiri, 1978) if f(V) is δ-closed (resp. δ-open) in (Y,σ) for every closed (resp. open) set V of (X, τ).

c) **δ-semi closed** (resp. δ-semi open) **function** (Park, 1997) if f(V) is δ-semi closed (resp. δ-semi open) in (Y, σ) for every closed (resp. open) set V of (X, τ).

d) **Somewhat open** (Gentry et.al., 1971) if for U∈(X, τ) and U ≠ ϕ, there exists V ∈ (Y,σ) such that $V \neq \emptyset$ and $V \subseteq f(U)$.

Definition 1.3.4 [Khalaf, 2009]. A function $f:(X,\tau) \to (Y,\sigma)$is called Ps-continuous at a point x ∈ X if for each x ∈ X and each open set V of Y containing f (x), there exists a Psopen set U of X containing x such that $f(U) \subseteq V$. If f is Ps-continuous at every point of X, then it is called Ps-continuous.

Proposition 1.3.5 [Khalaf, 2009]. If $f:(X,\tau) \to (Y,\sigma)$ is a continuous and open function and V is a preopen set of Y, then $f^{-1}(V)$ is a preopen set of X.

Proposition 1.3.6 [Khalaf, 2009]. If $f:(X,\tau) \to (Y,\sigma)$is a continuous and open function and F is a semi-closed set of Y, then $f^{-1}(F)$ is a semi-closed set of X.

Definition 1.3.7 [Noiri, 1980]. A function f : (X, τ) → (Y, σ) is called δ-closed if f(V) is δ-closed in Y for every closed set V in X.

Lemma 1.3.8 [Jafari and Noiri 1999, 4.1]: The following statements hold for subsets A and B of a space X:

(a) $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for every closed subset F of X containing x.
(b) $A \subseteq \ker(A)$ and $A = \ker(A)$ if A is open in X.
(c) If $A \subseteq B$, then $\ker(A) \subseteq \ker(B)$

Proposition 1.3.9 [Khalaf 2012, 2.3]: If a function $f:(X,\tau) \to (Y,\sigma)$ is δ-continuous, then f is almost P_S-continuous.

Lemma TP 1.3.10 [Park 1997]. Let A be a subset of a topological space (X, τ). Then δ-sCl(δ-Int(A)) = Int(Cl(δ-Int(A)), or equivalently, δ-sCl(U) = Int(Cl(U)) for each δ-open set U of X.

Lemma 1.3.11 [Andrijevic D 1986]: A subset A of a space (X, τ) is β-open if and only if Cl A is regular closed.

Lemma 1.3.12 [2.7(a), Jankovic 1985]: Let A be a subset of a space (X, τ). Then A ∈ PO (X, τ) if and only if sCl A = IntCl A.

Lemma 1.3.13 [Nasef 1997]. The following results can be proved easily:

a) If $f:(X,\tau) \to (Y,\sigma)$ is almost precontinuous and Y is semi-regular, then f is precontinuous.

b) If $f:(X,\tau) \to (Y,\sigma)$ is almost continuous and Y is semi-regular, then f is continuous.

c) A function $f:(X,\tau) \to (Y,\sigma)$ is almost precontinuous if and only if $f^{-1}(V)$ is preopen set in X, for every δ-open set V in Y.

Lemma 1.3.14 [Prakash and Srivatsava 1977]. Let Y be a dense subspace of X. If O is regular open in Y, then $O = Y \cap Int(cl(O))$.

Lemma 1.3.15 [Dontchev 1998]: If R ∈ RO(X) and P ∈ PO(X), then R ∩P ∈ RO(P).

Theorem 1.3.16 [Singal 1968, 2.1(a)]: In a mapping $f: (X, \tau) \to (Y, \sigma)$, f is almost continuous at $x \in X$.

Theorem 1.3.17 [Jankovic 1985]: A function $f: (X, \tau) \to (Y, \sigma)$ is preopen if and only if $f^{-1}(ClV) \subseteq Cl(f^{-1}(V))$, for each semi-open set V of Y.

Proposition 1.3.18 [Veliko 1968]: Let A be a subset of a topological space (X, τ), then if $A \in \tau$, then (a) $Cl_{\theta}(A) = Cl(A)$.

(b) $\delta Cl(V) = Cl(V)$

Theorem 1.3.19 [Khalaf 2010, 1.16]: If $f: (X, \tau) \to (Y, \sigma)$ is almost continuous and Y is semi-regular, then *f* is continuous.

Theorem 1.3.20 [Khalaf 2010, 1015]. If $f: (X, \tau) \to (Y, \sigma)$ is almost δ-precontinuous and Y is semi-regular, then f is precontinuous.

Theorem 1.3.21 [Rose 1984, 7]. A function $f: (X, \tau) \to (Y, \sigma)$ is weakly continuous if and only if $Clf^{-1}(V) \subseteq f^{-1}(ClV)$ for each open subset V of Y.

Theorem 1.3.22 [Rose 1984, 11]. A function $f: (X, \tau) \to (Y, \sigma)$ is almost-open if and only if $f^{-1}(ClV) \subseteq Clf^{-1}(V)$ for each open subset V of Y.

Theorem 1.3.23 [Rose 1984, 14]. A function $f: (X, \tau) \to (Y, \sigma)$ is almost-open and almost continuous if and only if $Clf^{-1}(V) = f^{-1}(ClV)$ for each open subset V of Y.

Corollary 1.3.24 [Khalaf 2009 2.20]. For any subset A of a space X. If A ∈ θSO(X) and A ∈ PO(X), then A ∈ $P_SO(X)$.

Lemma 1.3.25 [Khalaf 2009, 3.2]: Every complete continuous and hence every super continuous function is P_S- continuous and every P_S-continuous function is precontinuous.

Definition 1.3.26 [Raja, 2014]: Let A be a subset of a topological space (X, τ). A point x ∈ A is said to be a δ-limit point of A if for each δ-open set U containing x, $U \cap (A\{x\}) \neq \emptyset$. The set of all δ-limit points of A is called the δ-derived set of A and is denoted by $D_{\delta}(A)$.

Definition 1.3.27 [Raja, 2014]: Let A be a subset of a topological space (X, τ). δ-frontier of A is defined by $Fr_{\delta}(A) = cl_{\delta}(A) \setminus int_{\delta}(A)$.

Definition 1.3.28 [Raja, 2014]: Let A be a subset of a topological space (X, τ). δ-exterior of A is defined by $Ext_{\delta}(A) = X \setminus Cl_{\delta}(A)$.

Definition 1.3.29 [Choquet, 1947]: A grill G on a topological space X is a non-empty collection of non-empty subsets of X satisfying two conditions:

a) A ∈ G and A ⊆ B ⊆ X ⇒ B ∈ G.

b) A, B ⊆ X and A ∪ B ∈ G ⇒ A ∈ G or B ∈ G.

Definition 1.3.30 [Roy, 2009]: Let $X \neq \emptyset$, the principal grill generated by a non-empty subset A of X is defined as $[A] = \{B \subseteq X | A \cap B \neq \emptyset\}$. If (X, τ) is a topological space with a grill G on X. Then, we call it a grill topological space and denote it by (X, τ, G). (Roy, 2007)

1.4 Bitopological Spaces

Definition 1.4.1: Let τ_i and τ_j with $i \neq j$ be topologies defined on a non empty set X. Then the triplet (X, τ_i, τ_j) is called a bitopological space.

Let A be a subset of (X, τ_i, τ_j). Then the union of all τ_i-open sets contained in A is called τ_i-interior of A and it is denoted by $\tau_i Int(A)$. The intersection of all τ_j-closed sets containing A is called τ_j closure of A and it is denoted by $\tau_j Cl(A)$ (Kelley, 1963).

Definition 1.4.2: For $(i, j) = 1,2$ and $i \neq j$, a subset A of a bitopological space (X, τ_i, τ_j) is called

a) (i, j)-semi-open (Maheswari, et.al., 1977) if $A \subseteq \tau_j Cl(\tau_i Int(A))$.

b) (i, j)-regular open (Bose, 1981) if $A = \tau_i\ Int(\tau_j\ Cl(A))$.

c) (i, j)-α-open (Jelic, 1990) if $A \subseteq \tau_i\ Int(\tau_j\ Cl(\tau_i Int(A)))$.

The complement of the above mentioned sets are called (i, j)-semi-closed, (i, j)-regular closed and (i, j)-α-closed sets respectively.

Definition 1.4.3: For $(i, j) = 1,2$ and $i \neq j$, a subset A of a bitopological space (X, τ_i, τ_j) is called

a) (i, j)-g-closed (Fukutake, 1986) if $\tau_j \delta Cl(A) \subseteq U$, whenever $A \subseteq U$ where U is τ_i -open.

b) (i, j)-πgs-closed (Arya et.al., 1990) if $\tau_j sCl(A) \subseteq U$, whenever $A \subseteq U$ where U is τ_i –π-open.

c) (i, j)-πg-closed (Julian Dontchev, 2000) if $\tau_j Cl(A) \subseteq U$, whenever $A \subseteq U$ where U is τ_i –π-open.

d) (i,j)-gpr-closed (Fukutake et.al., 2002) if $\tau_j\ pCl(A) \subseteq U$, whenever $A \subseteq U$ where U is τ_i -regular open.

e) (i,j)-πgp-closed (Park, 2004) if $\tau_j\ p\ Cl(A) \subseteq U$, whenever $A \subseteq U$ where U is τ_i–π-open.

Definition 1.4.4 [Khalaf, 2010]: A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(i,j)\ P_S$-open, if A is a j-pre-open set and for all x in A, there exists an i-semi-closed set F such that $x \in F \subseteq A$. A subset B of X is called $(i,j)\ P_S$-closed if and only if Bc is $(i,j)\ P_S$-open. The family of $(i,j)\ P_S$-open (resp., $(i,j)\ P_S$-closed) subset of x is denoted by $(i,j)\ P_SO(X)$ (resp., $(i,j)\ P_SC(X)$).

Definition 1.4.5: For $i,j,k = 1{,}2$ and $i \neq j$, a mapping $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is called

a) $\tau_j\sigma_k$ -continuous (Maki et.al., 1991) if $f^{-1}(V) \in \tau_j$ for every $V \in \sigma_k$.

b) (i,j)-πgs-σ_k -continuous (Arya et.al., 1990) if $f^{-1}(V)$ is (i,j) πgs -closed for each σ_k -closed set V in (Y, σ_i, σ_j), $i \neq j$ and $(i,j) = 1{,}2$.

c) (i,j)-g-σ_k -continuous (Maki et.al., 1991) if $f^{-1}(V)$is (i,j)-g-closed for each σ_k-closed set V in (Y, σ_i, σ_j), $i \neq j$ and $(i,j) = 1{,}2$.

d) (i,j)-πg-σ_k-continuous (Julian Dontchev, 2000) if $f^{-1}(V)$ is (i,j)-πg -closed for each σ_k-closed set V in (Y, σ_i, σ_j), $i \neq j$ and $(i,j) = 1{,}2$.

e) (i,j)-gpr–σ_k-continuous (Fukutake et.al., 2002) if $f^{-1}(V)$ is (i,j)-gpr -closed for each σ_k-closed set V in (Y, σ_i, σ_j), $i \neq j$ and $(i,j) = 1{,}2$.

f) R-map [Carnahan, 1973], if the inverse image of each regular open subset of Y is regular open in X

g) (i,j)-perfectly continuous [Ganster, 2000] if f $^{-1}$(V) is both j-closed and j-open in X for each i-open subset V of Y

Theorem 1.4.6 [Khalaf 2010, 3.7]: In a bitopological space (X, τ_1, τ_2). If the space (X, τ_i) is hyperconnected, then $(i,j)\ P_SO(X) = \{X, \emptyset\}$.

Lemma 1.4.7 [Ahmed 1990]: For any two topological spaces X and Y. If $A \subseteq X$ and $B \subseteq Y$ then,

a) $\delta pInt_{X\times Y}(A \times B) = \delta pInt_X(A) \times \delta pInt_Y(B)$

b) $sCl_{X\times Y}(A \times B) = sCl_X(A) \times sCl_Y(B)$