Chapter II

CHAPTER II

LINEAR OPERATORS PRESERVING ZERO-TERM RANK OF REAL MATRICES

Let $\mathbf{M}_{m,n}(\mathbb{R})$ denote the set of all $m\times n$ matrices with entries in $\mathbb{R}$, the real numbers. Let $\mathbf{M}_{m,n}(\mathbb{B})$ denote the set of all $m\times n$ matrices with zero entries in $\mathbb{B}$, where $\mathbb{B}=\{0,1\}$ is the Boolean algebra.

Definition : 2.1

A matrix $A\in\mathbf{M}_{m,n}(\mathbf{S})$ is said to be of **term rank k** ($t(A)=k$) if the least number of lines needed to include all nonzero elements of A is equal to k.

A matrix $A\in\mathbf{M}_{m,n}(\mathbf{S})$ is said to be of **zero-term rank k** ($z(A)=k$) if the least number of lines needed to include all zero elements of A is equal to k.

Definition : 2.2

Let $T : \mathbf{M}_{m,n}(\mathbb{R})\rightarrow \mathbf{M}_{m,n}(\mathbb{R})$ be a linear operator

(i) T **preserves zero-term rank k** if $z(T(A))=k$ whenever $z(A)=k$ for all A in $\mathbf{M}_{m,n}(\mathbb{R})$;

(ii) T **preserves zero-term rank** if it preserves zero-term rank for every $k\leq\min\{m,n\}$.

Definition : 2.3

If $T : \mathbf{M}_{m,n}(\mathbb{R})\rightarrow \mathbf{M}_{m,n}(\mathbb{R})$ is a linear operator, define $\overline{T}:\mathbf{M}_{m,n}(\mathbb{B})\rightarrow \mathbf{M}_{m,n}(\mathbb{B})$ by

$$\overline{T}(\overline{A})=\sum_{i=1}^{m}\sum_{j=1}^{n}\overline{T(a_{ij}E_{ij})}$$

for any $A\in\mathbf{M}_{m,n}(\mathbb{R})$.

Definition : 2.4

An operator T is called a **(P,Q,B)-operator** if there exist permutation matrices P and Q and a matrix B with no zero entries such that $T(X)=P(X\circ B)Q$ for all $X\in\mathbf{M}_{m,n}(\mathbf{S})$ or if for m=n, $T(X)=P(X\circ B)'Q$ for all $\mathbf{M}_{m,n}(\mathbb{F})$. A (P,Q,B)-operator is called a (P,Q)-operator if B=J, the matrix of all ones.

Theorem : 2.5 [9]

If **S** is any anti-negative semiring, and T is a linear operator on the $m\times n$ matrices with entries in **S**, then the following statements are equivalent:

(i) T preserves zero-term rank ;

(ii) T preserves zero-term ranks 0 and 1 ;

(iii) T is a (P,Q,B)-operator.

Notation : 2.6

Let $\|A\|$ denote the number of nonzero entries of A.

Lemma : 2.7

If T preserves zero-term rank, then there exists $C\in\mathbf{M}_{m,n}(\mathbb{R})$ such that $\|T(C)\|=mn$.

Proof :

Choose $C\in\mathbf{M}_{m,n}(\mathbb{R})$ such that $T(C)\geq T(A)$ for all $A\in\mathbf{M}_{m,n}(\mathbb{R})$. Suppose that $\|T(C)\|\neq mn$. Then, for some (s,t), $T(A)\circ E_{s,t}=0$, for all $A\in\mathbf{M}_{m,n}(\mathbb{R})$. By permuting rows and columns, we may assume that (s,t)=(1,1). Also we assume that $\overline{C}=J$, so that z(C)=0. Let E_{hk} be a cell such that $T(E_{hk})$ has nonzero (p,q) entry with $p,q\geq 2$. If no such cell existed, then we obtain that $z(T(C-c_{ij}E_{ij}))=1$ for every cell E_{ij} but

$$z(T(C-c_{ij}E_{ij}))=\min\{m,n\},$$

a contradiction. Now, for $T(E_{hk})=D=(d_{ij})$, we have that

$$z\left(T\left(C-\frac{T(C)_{pq}}{d_{pq}}\right)\right)\geq 2\text{, and } z\left(C-\frac{T(C)_{pq}}{d_{pq}}E_{hk}\right)\leq 1.$$

Thus, we must have $z\left(C-\frac{T(C)_{pq}}{d_{pq}}E_{hk}\right)=0$, since T preserves zero-term rank 1.

Let $F=(f_{ij})=C-\frac{T(C)_{pq}}{d_{pq}}E_{hk}$. If $T(E_{uv})_{pq}=0$ for some cell E_{uv}, then $z(F-f_{uv}E_{uv})=1$, while $z(T(F-f_{uv}E_{uv}))=z(T(F)-f_{uv}T(E_{uv}))\geq 2$, which is a contradiction.

Thus $T(E_{ij})_{pq}\neq 0$ for all cells E_{ij}.

If $T(E_{11})=X=(x_{ij})$ and $T(E_{12})=Y=(y_{ij})$, then

$$T\left(F-f_{11}E_{11}+\left(\frac{f_{11}x_{pq}}{y_{pq}}\right)E_{12}\right)$$ has zeros in the (1,1) and (p,q) entries,

and hence has zero term rank atleast 2, while

$$z\left(F-f_{11}E_{11}+\left(\frac{f_{11}x_{pq}}{y_{pq}}\right)E_{12}\right)=1,$$

a contradiction. Thus $\|T(C)\|=mn$.

Lemma : 2.8

If T preserves zero-term rank 1, then T maps each cell to a nonzero multiple of some cell which induces a bijection on the set of indices $\{1,2,\ldots\ldots,m\}\times\{1,2,\ldots\ldots,n\}$.

Proof :

By Lemma 2.7, there exist $C\in \mathbf{M}_{m,n}(\mathbb{R})$ such that $\|T(C)\|=mn$. Suppose that there is some cell E_{ij} such that $\|T(E_{ij})\|>1$. If $\|T(E_{ij})\|\neq mn$, then there exists a pair (h,k) such that $(h,k)\neq(i,j)$ and for some nonzero real number r_{hk}.

$$T(E_{ij}+r_{hk}E_{hk})>T(E_{ij}).$$

Let $D_1=E_{ij}+r_{hk}E_{hk}$. If $\|T(D_1)\| \neq mn$, then there is some cell E_{pq} such that for some nonzero real number r_{pq} , $T(D_1+r_{pq}E_{pq})>T(D_1)$. Continuing this process, we have a matrix $D=(d_{ij})$ such that $\|D\| = mn$ while $\|T(D)\|$=mn. Since $\|D\| < mn$, we may assume $d_{11}=0$ without loss of generality. Let F be the (0,1)-matrix in $\mathbf{M}_{m,n}(\mathbb{R})$ such that $f_{11}=0$ and for $(i,j)\neq(1,1)$, $f_{ij}=0$ if and only if $d_{ij}\neq 0$. Thus, for some sufficiently small positive real number r, we have

$$\|D+rF\| = mn-1 \text{ and } \|T(D+rF)\| = mn .$$

That is,

$$z(D+rF)=1 \text{ and } z(T(D+rF))=0.$$

This is a contradiction. If $\|T(E_{ij})\| = mn$, then we can take $D=E_{ij}$ in the above case and obtain the same contradiction. Thus $\|T(E_{ij})\| \leq 1$ for all cells E_{ij}. If $T(E_{ij})=0$ for some cell E_{ij} , then the fact that $\|T(C)\| = mn$ implies $\|T(E_{pq})\| \geq 2$ for some (p.q), which is a contradiction.

That is, T is bijective on the set of indices $\{1,2,\ldots\ldots,m\}\times\{1,2,\ldots\ldots,n\}$.

Theorem :2.9

If T preserves zero-term rank 1, the T is a (P,Q,B)-operator.

Proof :

By Lemma 2.8, T is bijective on the set of indices $\{(i,j) \mid i=1,2,\ldots\ldots,m, j=1,2,\ldots\ldots,n\}$. Thus, for any A in $\mathbf{M}_{m,n}(\mathbb{R})$,

$$\overline{T(A)} = \overline{\sum_{i=1}^{m}\sum_{j=1}^{n} T(a_{ij}E_{ij})} = \sum_{i-1}^{m}\sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})} = \overline{T}(\overline{A}) .$$

This shows that $\overline{T}$ preserves zero-term rank 1 since T does also. By Theorem 2.5, $\overline{T}$ is a (P,Q,B)-operator, where B=J. Thus, the mapping $\overline{A} \mapsto \overline{P^{t}T(A)Q^{t}}$ is the identity linear operator on $\mathbf{M}_{m,n}(\mathbb{B})$. That is, $P^{t}T(E_{ij})Q^{t}=b_{ij}E_{ij}$ for each pair (i,j) (or perhaps $P^{t}T(E_{ij})Q^{t}=b_{ij}E_{ji}$ in the case m=n).

Then, $T(C)=P(C \circ B)Q$ for all $C \in \mathbf{M}_{m,n}(\mathbb{R})$ or $m=n$ and $T(C)=P(C \circ B)^t Q$ for all $C \in \mathbf{M}_{m,n}(\mathbb{R})$.

Theorem : 2.10

For a linear operator $T : \mathbf{M}_{m,n}(\mathbb{R}) \to \mathbf{M}_{m,n}(\mathbb{R})$, the following are equivalent:

(i) T preserves zero-term rank ;

(ii) T preserves zero-term rank 1 ;

(iii) T is a (P,Q,B)-operator.

Proof :

Obviously (i) implies (ii) and (iii) implies (i). By Theorem 2.9, we have that (ii) implies (iii).

Theorem : 2.11

Let A be an $m \times n$ real matrix. Then the zero-term rank of A is equal to the maximal number of zeros in A with no two of the zeros on a line.

Proof :

We prove this equality by induction on the number of lines in A. For the case that $m=1$ or $n=1$, the equality holds. Hence we take $m>1$ and $n>1$. Let $z(A)=p$ and q denote the maximal number of zeros in A with no two of the zeros on a line. Then the definition of zero-term implies that $q \le p$. Hence it suffices to show that $q \ge p$. Consider two cases:

Case 1

Assume that A has a proper covering. Then we must have $p=\min\{m,n\}$. We permute the lines of A so that the permuted matrix B has a zero in the (1,1) position. We delete row 1 and column 1 of the permuted matrix B and denote the resulting matrix of size m-1 by n-1 by $B(1 \mid 1)$. The matrix $B(1 \mid 1)$ cannot have a covering composed of fewer than $p-1=\min\{m-1,n-1\}$ lines because such a covering of B(1 | 1) plus the two deleted lines would yield a proper covering for A. We now apply the induction hypothesis to B(1 | 1) and this allows us to conclude that B(1 | 1) has p-1 zeros with no two of the zeros on a line. But then A has p zeros with no two of the zeros on a line and it follows that $q \geq p$.

Case 2

Assume that A has a proper covering composed of e rows and f columns where p=e+f. We permute lines of A so that these e rows and f columns occupy the left-upper positions of the permuted matrix B. Then B assumes the following form

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

In this decomposition B_{22} is the (m-e)×(n-f) submatrix with all nonzero entries. The matrix B_{12} has e rows and cannot be covered by fewer than e lines and the matrix B_{21} has f columns and cannot be covered by fewer than f lines. This is the case because otherwise we contradict the fact that p=e+f is the minimal number of lines in A that cover all of the zeros on A. We may apply the induction hypothesis to both A_1 and A_2 and this allows us to conclude that $q \geq p$.