Some Interesting Results on Local Connectedness

By

Ananthi M. P.

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K. N. Meenakshi
Signature of the Head
of the Department
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G. K. Chandrika
Signature of the Guide

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Contents

CONTENTS

Introduction

INTRODUCTION

The main aim of this thesis is to discuss some interesting results on local connectedness in topological spaces, bitopological spaces and fuzzy topological spaces.

Fundamental results on local connectedness in topological spaces are collected from the well-known works of J.R. Munkres [15], N. Bourbaki [4], J.G. Hocking and G.S. Young [7] and K.D. Joshi [8]. Some weaker forms of local connectedness are studied by Groot and McDowell [6] in 1967. Some of the interesting results given in this paper are discussed in chapter 1.

Various authors have defined pairwise local connectedness in different ways. In this thesis we have discussed the definitions given by Dasgupta and Lahri [5], Birsan [3] and Lakshmi [11], which are referred to as DL- local connectedness, B-local connectedness and L-local connectedness. The papers taken for discussion in this connection are the following :

(1) Dasgupta and Lahri, Local connectedness in bitopological spaces [5].

(2) Mrsevic and Reilly , A note on local connectedness in bitopological spaces [14].

Regarding local connectedness in fuzzy topological spaces, the paper entitled "connectedness and local connectedness in fuzzy topological spaces and Heyting-algebra - valued sets" by Ajmal and Kohli [1], is taken for discussion .

In chapter 1 we have discussed some interesting properties of local connectedness in topological spaces. In section 1.1, we have collected the preliminary definitions and results needed for discussion. some of the important results discussed in sections 1.2 and 1.3 are as follows :

(1) Every component of a locally connected space is open .

(2) Every open subspace of a locally connected space is locally connected.

(3) A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(4) The boundary of a component of an open subset U of a locally connected space is contained in the boundary of U.

(5) In a locally connected space, quasicomponents and components coincide.

(6) A closed set S cuts (Definition 1.2.21) a connected locally connected regular space X between points a and b if and only if S separates a from b.

(7) Quotient space of a locally connected space is locally connected.

(8) Product of a family of locally connected spaces in which all but a finite number of components spaces is connected, is locally connected. Conversely, if the product of a family of non empty topological spaces is locally connected, then each component space is locally connected . Moreover all but a finite number of component spaces is connected.

Groot and McDowell [6] have studied the concepts of weakly locally connected, quasilocally connected and padded spaces and obtained interesting properties connecting these spaces and locally connected spaces. These results are discussed in section 1.4.

Relation between rimcompact and locally connected spaces is discussed in section 1.5. In a connected space X, it is true that X is padded at $x \Rightarrow$ X is locally connected at x. The converse implication holds good in a continuum (i.e. a compact connected Hausdorff space) and in a rimcompact connected Hausdorff space. Apart from the above mentioned property, Groot and McDowell [6] have proved that in a continuum, the property " components coincide with quasi components on every open subset " is equivalent to local connectedness.

Chapter 2 is devoted to the study of local connectedness in bitopological spaces. In section 2.1 we have discussed in detail, the contributions of Dasgupta and Lahri [5] to the study of pairwise local connectedness. They [5] have proved some basic properties of pairwise local connectedness and established with the help of an example that there are spaces which are locally connected with respect to the two topologies taken separately but the space considered as a bitopological space, is not locally connected.

In section 2.2 we have made a comparative study of DL - local connectedness, B-local connectedness and L-localconnectedness. Mrsevic and Reilly [14] have given examples to show that these three definitions are independent.

In the last section of chapter 2, we have discussed the behaviour of DL-local connectedness and B - local connectedness in the case of the particular bitopological space (X,τ, τ^{α}).

In chapter 3 we study the notion of local connectedness in fuzzy topological spaces. Ajmal and Kohli [1] have introduced four notions of local connectedness in fuzzy topological spaces, reffered to as C_i - local connectedness ($i = 1,2,3,4$). C_2 and C_4- local connectedness are proved to be "good extensions ". An example is given to show that C_3- local connectedness is not a "good extension". Some of the important results obtained are as follows:

(1) A fuzzy topological space X is C_i - locally connected ($i = 1,2,3,4$) if and only if every fuzzy open subspace of X is C_i-locally connected.

(2) A fuzzy topological space X is C_2 - locally connected if and only if every C_2 -component of every fuzzy open set in X is fuzzy open.

(3) If $f : X \rightarrow Y$ is a fuzzy quotient map of a C_2 -locally connected space on to a fuzzy topological space Y, then Y is C_2 -locally connected.

(4) If the fuzzy product πX_α of a family of fuzzy topological spaces is C_2 - locally connected then each coordinate space X_α is C_2 locally connected.

The authors [1] have also introduced the notions of C_4-quasicomponents and C_4-quasi local connectedness. The chapter is concluded by discussing some interesting results involving these concepts.

Review of Literature

REVIEW OF LITERATURE

Many authors have contributed to the study of loally connected spaces since it was introduced. Interesting articles have been published on various aspects of local connectedness. Quasi component approach to the study of local connectedness has been followed by Groot and Mc Dowell [6]. The works of Whyburn [24] and Kuratowski [10] provide a good information regarding hereditarily locally connected meteric continua. Nishiura and Tymchatyn [17] have obtained some metric characterisations of connected subsets of hereditarily locally connected metric continua. Contributions have also been made to the study of campactifications of locally connected spaces. We give here a brief survey of some of the articles published on some of the aspects of locally connected spaces.

Local connectedness of extension spaces

[B.Banaschewski , 1956] [2]

An extension E* of a topological space (i.e. a space containing E as a dense subspace) determines a family of filters, there $\mathfrak{I}$ (u) on E given by the traces $U \cap E$ of the neighbourhood $U \subset E^*$ of each $u \in E^* \setminus E$. The author has called there $\mathfrak{I}(u)$ as trace filter and has obtained the following result:

"If E* is an extension of E each of whose trace filters is connected, then E* is locally connected if and only if E is locally connected and each trace filter has a basis consisting of connected opensets".

As an application of this result to the stone - cech compactification βE (where E is a completely regular space having the property that there exist denumerably open sets $O_i \subset E$ whose closures are mutually disjoint and have a closed union) the author has proved that βE is not locally connected.

Compactification of hereditarily locally connected spaces.

[E.D. Tymchatyn, 1977] [23]

The author has obtained an interesting charactersation that a continuum is hereditarily locally connected if and only if every connected subset is locally connected. He has also obtained some charactersations of connected spaces which admit a hereditarily locally connected compactification.

Local connectedness in fuzzy setting.

[S.Saha, 1987] [20]

In this paper an analogue of the motion of local connectedness in the fuzzy seting is introduced and investigated. A neccesary and sufficient condition for the productivity of this notion is given.

$\beta(X-\{x\})$ for X not locally connected

[Smith , Michel] [21]

Let X be a metric continuum, $x \in X$ and $Y = \beta(X-\{x\}) \setminus X$. The author has proved that if Y is a continuum then Y is a decomposable continuum with one component whenever X is not weakly locally connected at x. A sufficient condition for Y to be indecomposable is given and it is shown that Y is never hereditarily indecomposable.

A locally connected rim - countable continuum which is the continuous image of no arc.

[J.Nikoiel, H.M.Tuncali, E.D.Tymchatyn, 1991] [16]

This paper sharpens our knowledge of which continua are images of ordered continua. The authors consider the separable, non metric continuum Y_β, which arises when each point of a certain uncountable dense subset β of the square I^2 is "replaced" by a circle. A quotient space of Y_β is the desired example.

Chapter I

CHAPTER 1

LOCAL CONNECTEDNESS IN TOPOLOGICAL SPACES

In this chapter we discuss some interesting properties of local connectedness in topological spaces. Fundamental results on local connectedness are collected from the well-known works of J.R. Munkres [15], N.Bourbaki [4], J.G. Hocking and G.S. Young [7], and K.D. Joshi [8], Groot and McDowell [6] have introduced some weaker forms of local connectedness, namely weakly local connectedness and quasilocal connectedness. Interesting results connecting these concepts with local connected spaces and padded spaces are discussed here. Using the quasicomponent approach the authors [6], have obtained simple proofs of the results on local connected spaces. Moreover this approach has led to the interesting result that in a connected compact Hausdorff space (i.e a continuum) the property "Components coincide with quasicomponents on every open subset" is equivalent to local connectedness. First let us give the preliminary definitions and results needed for discussion.

Section 1.1

Preliminary definitions and results

Definition 1.1.1

Let X be a topological space with topology τ. If Y is a subset of X, the collection $\tau_Y = \{Y \cap U \,/\, U \in \tau\}$ is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X ; its open sets consist of all intersections of open sets of X with Y.

Definition 1.1.2

Let (X, τ) be a topological space. A **neighbourhood** of a point p in X is a set containing p in its interior.

Notation

$\overline{A}$ denotes the closure of A in a topological space.

Definition 1.1.3

A subset A of a topological space is called **Clopen** if it is both open and closed.

Definition 1.1.4

If $A \subset X$, the **boundary of A** denoted by **Bd A** is defined by,

$\text{Bd } A = \overline{A} \cap (\overline{X - A})$

Definition 1.1.5

Let X and Y be topological spaces; let $p : X \rightarrow Y$ be a surjective map. The map p is said to be a **quotient map**, provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition 1.1.6

If X is a space and A is a set and if $p : X \rightarrow A$ is a surjective map, then there exists exactly one topology τ on A relative to which p is a quotient map : it is called the **quotient topology** induced by p.

Definition 1.1.7

Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $P : X \rightarrow X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

Definition 1.1.8

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Let S_β denote the collection. $S_\beta = \{\pi_{\beta^{-1}}(U_\beta) / U_\beta$ is open in $X_\beta\}$, and let S denote the union of these collections, $S = U_{\beta \in J}\ S_\beta$. The topology generated by the subbasis S is called the **product topology** on $X = \pi_{\alpha \in J} X_\alpha$. In this topology $\pi_{\alpha \in J} X_\alpha$ is called a **Product space.**

Definition 1.1.9

Let X be a topological space, A **separation** of X is a pair U, V of disjoint non empty open subsets of X whose union is X. The space X is said to be **Connected** if there does not exist a separation of X.

Properties of Connectedness 1.1.10

(i) Continuous image of a connected space is connected.

(ii) Product of connected spaces is connected.

(iii) The union of a collection of connected sets that have a point in common is connected.

Definition 1.1.11

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence classes are called the **components** of X.

Properties of Components 1.1.12

(i) Components are maximal connected subsets.

(ii) Components are closed, pairwise disjoint and their union is X.

Definition 1.1.13

In a space X, define a relation on X by setting $x \sim_q y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$. This is an equivalence relation. The equivalence classes are called the **quasicomponents** of X.

Theorem 1.1.14

The quasicomponent of x in X is the intersection of all the clopen subsets of X containing x.

Proof

Consider the quasicomponent $Q(x)$ of x. Let S be the intersection of all the clopen sets containing x.

Take $y \in Q(x)$. Let A be a clopen set containing x. Then $x \sim_q y$

Claim

$y \in A$.

Suppose $y \notin A$ then $y \in X - A$, which is again a clopen set. Now $X = A \cup (X - A)$.

Hence there exists a separation of X such that $x \in A$ and $y \in X - A$. Which is a contradiction, because $x \sim_q y$.

Therefore $y \in A$.

Therefore y belongs to all the clopen sets containing x.

Hence $y \in S$.

Therefore $Q(x) \subset S$.

Conversely, take $y \in S$.

Suppose $y \notin Q(x)$, then there is a separation $X = A \cup B$ with $x \in A$ and $y \in B$.

Since A and B are open sets, A is a clopen set containing x.

But $y \notin A$ which is a contradiction.

Therefore $y \in Q(x)$

Therefore $Q(x) = S$

Hence the result.

Theorem 1.1.15

Every component of X lies in a quasicomponent of X.

Proof

Let C be a component of X. Take x in C.

Consider the quasicomponent $Q(x)$ of x. Take $y \in C$

To prove $y \in Q(x)$

suppose $y \notin Q(x)$

Then there is a separation $X = A \cup B$ with $x \in A$ and $y \in B$.

Since C is a connected set $C \subset A$ or $C \subset B$.

If $C \subset A$ then $y \in A$ which is a contradiction.

If $C \subset B$ then $x \in B$ which is a contradiction.

Therefore $y \in Q(x)$

Therefore $C \subset Q(x)$

Theorem 1.1.16

(Quasi) Components are closed and pairwise disjoint, and every quasicomponent is a union of components.

Theorem 1.1.17

Every open quasicomponent is a component.

Proof

If Q is an open quasicomponent; then Q is clopen.

If $Q = A \cup B$, where A and B are clopen in Q, then as Q is clopen in X, A and B are clopen in X.

Take $x \in A$ and $y \in B$.

since $x \in Q$, $Q = Q(x)$, Also as $y \in Q = Q(x)$, $x \sim_q y$ which is a contradiction as

$Q = A \cup B$ with $x \in A$ and $y \in B$.

Hence the result.

Theorem 1.1.18

Components and quasicomponents coincide in a compact Hausdorff space.

Theorem 1.1.19

If $X \subset Y$, and C_X is a quasicomponent in X, there is a quasicomponent C_Y in Y such that $C_X \subset C_Y$.

Theorem 1.1.20

If Q is a quasicomponent in the space X, and K a compact set in X disjoint from Q, then there exists a clopen set S containing Q and disjoint from K; moreover Q is a quasicomponent of $X \setminus K$.

Proof

Given $K \cap Q = \phi$

Take $k \in K$ then $k \notin Q$.

Therefore there exists a clopen set S_k containing Q, with $k \notin S_k$

Then $k \in X - S_k$ which is an open set .

Therefore the collection $\{X - S_k / k \in K\}$ is an open cover of K and hence has a finite subcover, say, $\{X - S_{k_1}, X - S_{k_2}, \ldots\ldots\ldots\ldots\ldots X - S_{k_n}\}$.

Let $S = \bigcap_{i=1}^{n} S_{k_i}$ Since S is clopen and each $S_{ki} \supset Q$, We get $Q \subset S$.

Also $S \cap K = \phi$

Hence the result.

Theorem 1.1.21

If U is an open subset of the connected space X such that $\overline{U} \setminus U$ is compact and non - empty, then every quasicomponent Q of $\overline{U}$ meets $\overline{U} \setminus U$.

Proof

Suppose there exists a quasicomponent Q of $\overline{U}$, which does not intersect $\overline{U} \setminus U$. Then by theorem (1.1.20), there exists a clopen set S containing Q with $S \cap (\overline{U} \setminus U) = \phi$

This implies that $S \subset X \setminus (\overline{U} \setminus U)$.

Since $\overline{U} \setminus U \neq \phi$, S is a proper clopen subset of the connected space X, which is a contradiction.

Section 1.2

Local connectedness

Definition 1.2.1

A space X is said to be **locally connected at x** if for every neighbourhood U of x, there is an open connected neighbourhood V of x contained in U. If X is locally connected at each of its points , it is said to be **locally connected.**

Remark 1.2.2

X is locally connected if there is a basis for X consisting of connected sets.

Remark 1.2.3

Local connectedness and connectedness of a space are not related to one another. A space may possess one or both of these properties, or neither.

Example 1.2.4

Every discrete space is locally connected .

Example 1.2.5

Each interval and each ray in the real line is both connected and locally connected.

Example 1.2.6

The subspace $[-1,0) \cup (0,1]$ of R is not connected but it is locally connected

Example 1.2.7

A space which is connected but not locally connected.

Let X be the set $B \cup A$ where $B = \{(x,0) \in R^2 : 0 \leq x \leq 1\}$ and

$A = \{(x,y) \in R^2 : 0 \leq y \leq 1,\ x = 0$ or $x = \frac{1}{n}$ for some $n \in N\}$.

It consists of infinitely many vertical segments of unit length, including a segment on the y- axis and a horizontal segment along the x- axis.

Give X the relative topology induced by the usual topology on the plane. X is then called a **'comb space'.**

It is easy to show that X is connected, for each of the vertical segment is connected and meets the horizontal segment B which is also connected.

However X is not locally connected.

For, let V be the open ball (in X, with usual metric) centered at (0,1/2) and with radius 1/4.

The components of V will be portions of the vertical segments.

They will all be open (in X) except the one along the y-axis, namely the component $\{(0,y) \in R^2 : \frac{1}{4} < y < \frac{3}{4}\}$. This component is the only connected set containing $(0, \frac{1}{2})$ and contained in V but it is not open .

Hence the space is not locally connected at $(0,\frac{1}{2})$.

Example 1.2.8

The set Q of rationals is neither connected nor locally connected.

Example 1.2.9

A space which is locally connected at all points except at one point.

Consider the graph of the function $y = \sin\frac{1}{x}, 0 < x \leq 1$, together with the origin in R^2.

Consider any small circle C containing the origin.

Then C is open, but the only connected set containing the origin and lying within C is the origin itself and this one-point set is not open.

Any other point in this space lies in arbitrarily small open arcs, which are connected. Hence the space fails to be locally connected at only one point.

Example 1.2.10

The topologist's sine curve

It is the graph of the function $y = \sin\left(\frac{1}{x}\right), 0 < x \leq 1$ together with the interval $-1 \leq y \leq 1$ on the y-axis in R^2.

A small circle C, about a point p on the segment $-1 \leq y \leq 1$ defines an open set containing p.

The only connected set lying within C and containing p is the segment on the interval $-1 \leq y \leq 1$ which lies within C.

But this segment is not open in R^2.

Hence the topologist's sine curve fails to be locally connected at each point of the interval $-1 \leq y \leq 1$.

Properties of local connectedness 1.2.11

(i) Local connectedness is not a hereditary property.

(ii) Since every discrete space is locally connected and every topological space is the continuous image of discrete space (with the same underlying set) we get that local connectedness is not preserved under continuous functions.

(iii) If X is a locally connected regular space then given a point x in X and an open set U containing x, there exists an open connected set V such that $x \in V \subset \overline{V} \subset U$.

Theorem 1.2.12

Every component of a locally connected space is open.

Proof

Let C be a component of a locally connected space (X,τ) and let $x \in C$.

Since X is locally connected, each openset U containing x contains an open connected set G_x containing the point x.

By definition of component, C is the largest connected set containing x.

Therefore $x \in G_x \subset C$.

Hence C is open.

Theorem 1.2.13

Every open subspace of a locally connected space is locally connected.

Proof

Let Y be an open subset of X.

Take $y \in Y$ and an open set $U \in \tau_Y$ containing y.

Since Y is open in X, U is open in X.

As X is locally connected there exists an open connected set G such that

$y \in G \subset U \subset Y$.

Since $G \subset Y$, $G \cap Y = G$.

Therefore G is open in Y and also as Y is open in X, there is no separation for G in Y.

Therefore G is an open connected set in Y such that $y \in G \subset U$.

Therefore Y is locally connected at y.

Theorem 1.2.14

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof

Suppose that X is locally connected.

Let U be an open set in X and let C be a component of U.

If x is a point of C, choose a connected openset V of x such that $V \subset U$.

Since V is connected, it must lie entirely in the component C of U.

Therefore C is open in X.

Conversely, suppose that components of open sets in X are open.

Given a point x of X and a neighbourhood U of x, choose an open set V such that $x \in V \subset U$.

Let C be the component of V containing x, then C is connected.

Since C is open in X by assumption, we get that X is locally connected at x.

Theorem 1.2.15

Let U be an open subset of a locally connected space X, let V be a component of U. Then the boundary of V (relative to X) is contained in the boundary of U.

Proof

Since V is a component of U it is closed in U.

By theorem (1.2.14), V is also open in U.

As V is both open and closed in U, there is no boundary point of V relative to U .....(1)

Take $x \in V - V$. Then $x \in U$.

To prove $x \notin U$.

Suppose $x \in U$.

Take any open set in U containing x. It would be of the form $W \cap U$, with W open in

X. Since x is a boundary point of V, $W \cap U \cap V \neq \phi$ and $W \cap U \cap (X-V) \neq \phi$.

Hence x is a boundary point of V relative to U.

This is a contradiction to (1)

Hence the result.

Theorem 1.2.16

If X is locally connected, then the quasicomponents of X are the same as the components of X.

Definition 1.2.17

Let X be a topological space and S be a subset of X. S **separates** two subsets A and B of X − S if and only if there exists disjoint open sets U and V containing A and B respectively such that $A \subset U$, $B \subset V$ and $X - S = U \cup V$.

Theorem 1.2.18

If X is a connected locally connected space, and C is a component of an open set in X such that $X \setminus \overline{C}$ is not empty, then $\overline{C} \setminus C$ is not empty and separates C and $X \setminus \overline{C}$ in X .

Proof

If $\overline{C} \setminus C$ is empty, then C is closed.

By theorem 1.2.14, we get that C is open, so $X \setminus \overline{C} = X \setminus C$ is both open and closed.

Since $X \setminus \overline{C}$ is not empty by hypothesis, we get that X is not connected.

Therefore $\overline{C} \setminus C$ cannot be empty.

Hence, as $X \setminus (\overline{C} \setminus C) = C \cup (X \setminus \overline{C})$, we conclude that $\overline{C} \setminus C$ separates X.

Definition 1.2.19

Given two points a and b of a space X, a collection A_1 , A_2 ....An of sets is a **simple chain** from a to b provided that A_1 (and only A_1) contains a, A_n (and only A_n) contains b, and $A_i \cap A_j$ is non empty if and only if $|i-j| \leq 1$ i.e each **link** intersects just the one before it and the one after it.

Theorem 1.2.20

If a and b are two points of a connected space X, and $\{U_\alpha\}$ is a collection of open sets covering X, then there is a simple chain of elements of $\{U_\alpha\}$ from a to b.

Definition 1.2.21

A set S in a connected space X **cuts** X between two points a and b of X−S if S intersects every closed connected subset that contains both a and b, clearly, if S separates a and b, then S cuts X between a and b.

The converse implication holds for a closed set in a locally connected regular space as is seen from the following theorem :

Theorem 1.2.22

A closed set S cuts a connected locally connected regular space X between points a and b if and only if S separates a from b.

Proof

Suppose S cuts X between a and b.

Since S is closed the components of X−S are all open by theorem (1.2.14) .

Case (i)

a lies in a component U b does not lie in U.

In this case $X-S = U \cup ((X-S)-U)$ is a separation of a and b.

Case (ii)

a and b both lie in the same component U of X$-$S.

Since X is locally connected and regular each point x of U lies in an open connected set U_x whose closure lies in U (by property 1.2.11, (3)).

Now the collection $\{U_x / x \in U\}$ is an open covering of U.

Hence by theorem (1.2.20) there is a simple chain $U_1, U_2 \ldots\ldots\ldots U_n$ of sets in $\{U_x\}$ from a to b. The set $\cup_{i=1}^{n} \overline{U_i}$ is then a closed connected subset of X$-$S that contains a and b. Hence S could not cut X between a and b.

Therefore this case does not arise.

Hence the result.

As a consequence of the above theorem, we get the following corollary :

Corollary 1.2.23

If X is a locally connected regular space, and if U is a connected open set in X, then every pair of points of U lie in a closed connected subset C of X such that C is contained in U.

Section 1.3

Quotient and product spaces

Theorem 1.3.1

Every quotient space of a locally connected space is locally connected.

Proof

Let X be a locally connected space and R an equivalence relation on X.

Let $\phi : X \rightarrow X / R$ be canonical mapping.

To prove X /R is locally connected it is enough if we prove the components of an open set in X/R are open in X/R.

Let A be an open set of X/R and C a component of A.

Claim

ϕ^{-1} (C) is a union of components of ϕ^{-1} (A).

Take $x \in \phi^{-1}$ (C).

Let K be the component of x in ϕ^{-1} (A).

Since ϕ is the quotient map, it is continuous and hence ϕ(K) is connected.

Also ϕ (K) $\subset$A and $\phi(x) \in \phi$(K).

Since C is a component of A containing $\phi(x)$, we get ϕ(K)$\subset$C.

which implies that K $\subset \phi^{-1}$ (C).

Therefore ϕ^{-1} (C) is union of components of ϕ^{-1} (A).

Hence the claim.

Since X is locally connected, and ϕ^{-1} (A) is open in X, components of ϕ^{-1} (A) are open in X (by theorem 1.2.14).

Hence ϕ^{-1} (C) is open in X being union of open sets.

Therefore C is open in X/R.

Therefore every component of A is open in X/R.

Therefore, by theorem (1.2.14), X / R is locally connected.

Theorem 1.3.2

(a) Let $(X_t)_{t \in I}$ be a family of locally connected spaces such that X_t is connected for all but a finite number of indices $t \in I$. Then the product space $X = \prod_{t \in I} X_t$ is locally connected.

(b) Conversely, if the product of a family (X_t) of non empty topological spaces is locally connected, then each X_t is locally connected and X_t is connected for all but a finite number of indices.

Proof of (a)

Let J be the finite subset of I such that X_t is not connected if and only if $t \in J$.

Take $x = (x_t)_{t \in I}$ in X.

Let W be a neighbourhood of x. Then there exists a basis element $U = \prod_{t \in I} U_t$ such that $x \in U \subset W$.

Hence there exists a finite subset K of I such that $U_t \neq X_t$ if and only if $t \in K$.

If $t \notin J \cup K$, define $V_t = X_t$.

If $t \in J \cup K$, then as X_t is locally connected there exists an open connected set V_t such that $x_t \in V_t \subset U_t$.

Therefore $V = \prod_{t \in I} V_t$ will be a basis element containing x and also product of connected spaces is connected .

We get V is connected.

Also $x = (x_t) \in V \subset U$

Therefore X is locally connected.

Proof of (b)

Let $a = (a_t)$ be a point of X.

Since X is locally connected there exists a connected open set V containing a.

Let $\pi_t : X \rightarrow X_t$ be the projection map.

Then $\pi_t(V) = X_t$ except for a finite number of indices.

Since π_t is continuous and as continuous image of a connected set is connected,

we get X_t is connected except for a finite number of indices.

To prove each X_α is locally connected.

Take a point $a_\alpha \in X_\alpha$ and an open set V_α in X_α containing a_α.

Let $a \in X$ be such that, α^{th} coordinate of a is a_α. i.e. $\pi_\alpha(a) = a_\alpha$

Let $V = \underset{t\in I}{\pi} V_t$ where $V_t = X_t$ for $t \neq \alpha$. Then $a \in V$.

Since X is locally connected there is an open connected set W such that $a \in W \subset V$.

Therefore $\pi_\alpha(a) \in \pi_\alpha(W) \subset \pi_\alpha(V)$

i.e. $a_\alpha \in \pi_\alpha(W) \subset V_\alpha$

Since W is connected and continuous image of a connected set is connected, we get

$\pi_\alpha(W)$ is connected.

More over $\pi_\alpha(W)$ is open as projection maps are open.

Therefore X_α is locally connected.

Section 1.4

Relation among locally connected, weakly locally connected, quasilocally connected and padded spaces.

Definition 1.4.1

X is **weakly locally connected** at x if every neighbourhood of x contains a connected neighbourhood of x, equivalently for every neighbourhood U of x, the component of x in U is a neighbourhood of x. In regular spaces, this is equivalent to every neighbourhood of x contains a closed connected neighbourhood of x.

Definition 1.4.2

X is **quasilocally connected** at x if for every neighbourhood U of x, the quasicomponent of x in U is a neighbourhood of x.

Definition 1.4.3

X is **padded** at x if for every neighbourhood U of x, there exist open sets W and V such that $p \in W \subset \overline{W} \subset V \subset U$ and $V \backslash \overline{W}$ has only finitely many omponents.

Theorem 1.4.4

If X is locally connected at x, then X is weakly locally connected at x.

If X is weakly locally connected at x, then X is quasilocally connected at x.

Neither of these implications can be reversed.

Theorem 1.4.5

If the connected space X is padded at x, then X is locally connected at x.

Proof

Let U be a neighbourhood of x.

Since X is padded at x, there exist opensets W and V such that

$x \in W \subset \overline{W} \subset V \subset U$, and $V \backslash \overline{W}$ has only finitely many components,

say, C_1 , C_2 ....C_n .

Since every component is contained in a quasicomponent, for each i, $1 \leq i \leq n$,

there is a quasicomponent Q_i of V such that $C_i \subset Q_i$.

Claim

Each $v \in V$ is in some Q_i .

Suppose not, i.e. $v \notin Q_i$, for every i = 1,2................n.

Then there exists a clopen set T_i in V such that $v \notin T_i$ and $Q_i \subset T_i$.

Take $V_i = V - T_i$.

Then V_i is clopen in V, $v \in V_i$ and $V_i \cap Q_i = \phi$.

But $\bigcap_{i=1}^{n} V_i$ is open in V and closed in $\overline{W}$.

Therefore it is clopen in X which is a contradiction as X is connected.

Therefore for every $v \in V$, there exist Q_i such that $v \in Q_i$ i.e. $V \subset \bigcup_{i=1}^{n} Q_i$.

Hence $V = \bigcup_{i=1}^{n} Q_i$.

Since V has only finitely many quasicomponents and each of them is open, we get that each of them is a component by theorem (1.1.17).

Therefore that V is a union of we get finitely many components. Then the component of x in V is an open connected neighbourhood of x lying in U.

Hence the space X is locally connected. at x.

Theorem 1.4.6

In a locally connected space, components and quasicomponents coincide in every open subset.

Proof

Let X be a locally connected space and U an open set in X.

Let C be a component in U. Take $x \in C$.

Since X is locally connected, there exist an open connected neighbourhood V of x such that $V \subset U$.

Since C is closed and $x \in C$, $V \cap C \neq \phi$.

As C is a component and V is connected with $V \cap C \neq \phi$, we get $V \subset C$.

Therefore C is open.

Therefore every component in U is open.

Thus every quasicomponent in U, being a union of components is open.

Hence by theorem 1.1.17 every quasicomponent is a component.

Theorem 1.4.7

The following conditions on a space X are equivalent.

(i) X is quasilocally connected.

(ii) X is weakly locally connected .

(iii) X is locally connected.

(iv) (Quasi) components in every open subset of X are open.

Proof

First observe that the two assertions in four are equivalent by theorem 1.1.16 and theorem 1.1.17.

Then, since a set is open if and only if it is a neighbourhood of each of its points, (iv) is equivalent to (i), (ii) and (iii).

Corollary 1.4.8

A locally connected space is the topological union of its (quasi)components, i.e., each (quasi)component is clopen.

The section is concluded by stating the following two important properties:

Theorem 1.4.9

A space X fails to be weakly locally connected at the point x if and only if for some neighbourhood U of x, there is a collection of distinct components $\{C_\alpha\}$ in U such that every neighbourhood of x meets infinitely many of the C_α.

Theorem 1.4.10

If X is a dense subspace of Y and X is locally connected at $x \in X$, then Y is locally connected at x.

Section 1.5

Rim compact and locally connected spaces.

Definition 1.5.1

X is **rim - compact (Semi compact, or locally peripherally compact)** at x if every neighbourhood of x contains an open neighbourhood of x with compact boundary.

Remark 1.5.2

Every rim-compact Hausdorff space is completely regular.

Definition 1.5.3

A compact, connected Hausdorff space is called a **continuum.**

Theorem 1.5.4

If X is a continuum, or more generally a rim compact connected Hausdorff space, the following are equivalent:

(i) X is locally connected

(ii) X is padded

(iii) Components and quasicomponents coincide on every open subset of X.

Proof

(ii) $\Rightarrow$(i) follows from Theorem 1.4.5

(i) $\Rightarrow$ (iii)follows from theorem 1.4.6

Now we have to prove, (i) $\Rightarrow$(ii) and (iii) $\Rightarrow$(i)

(i) $\Rightarrow$ (ii):

Assume X is locally connected.

Let x be a point and U an open set containing x.

Since X is rim compact and regular (by remark 1.5.2), there exists an open neighbourhood W of x such that $\overline{W} \subset V$ and $\overline{W} \setminus W$ is compact.

Take w$\in \overline{W} \setminus W$.

Since $\{x\}$ is closed, $U - \{x\}$ will be open .

Also w $\in U \setminus \{x\}$

Since X is locally connected and regular there exist an open connected set U_w

such that $\overline{U}_w \subset U \setminus \{x\} \subset U$.

Consider the collection $\{U_w / w \in \overline{W} \setminus W\}$.

This is a covering of the compact set $\overline{W} \setminus W$. This has a finite sub collection, say,

$\{U_{w_1}, U_{w_2}, \ldots\ldots\ldots\ldots U_{w_k}\}$ covering $\overline{W} / W$. Therefore $\overline{W} \setminus W \subset \bigcup_{i=1}^{k} U_{w_i} \subset U$.

From $\{U_{w_1}, U_{w_2} \ldots\ldots\ldots\ldots U_{w_k}\}$ we get a finite subcollection say,

$U_1, U_2 \ldots\ldots U_n$, of disjoint connected open sets.

Let $O = \bigcup_{i=1}^{n} U_i$. Then $O \subset U$. Let $V = W \cup O$ and $G = V \setminus \overline{O}$.

Then $x \in G \subset \overline{G} \subset V \subset U$ and $V \setminus \overline{G}$ is a finite union of connected sets.

Hence X is padded at x.

(iii) $\Rightarrow$ (i):

Suppose X is not locally connected. By theorem (1.4.7) there exists an open set U in X such that some quasicomponent Q of U is not open.

i.e. there exist a point $q \in Q$ which is not an interior point of Q.

i.e. every neighbourhood of q contains points outside Q.

Since X is rim compact and regular, there exists an open neighbourhood U_2 of q such that $q \in U_2 \subset \overline{U}_2 \subset U$ and $\overline{U}_2 \setminus U_2$ is compact.

Using again the fact that X is rim compact and regular, there exists an open neighbourhood U_1 such that $q \in U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$ such that $\overline{U}_1 \setminus U_1$ is compact.

By theorem (1.1.21), we get every quasicomponent of q in $\overline{U}_1$ meets $\overline{U}_1 \setminus U_1$ and quasicomponent of q in $\overline{U}_2$ meets $\overline{U}_2 \setminus U_2$.

So by theorem (1.1.19) Q meets $\overline{U}_1 \setminus U_1$ and $\overline{U}_2 \setminus U_2$.

Clearly any quasicomponent of U which meets U_1 meets these boundaries.

Let $H = Q \cap (\overline{U_1} \setminus U_1)$.

Then H is compact, being a closed subset of the compact space $\overline{U_1} \setminus U_1$.

Let $U^* = U \setminus H$. Then U^* is open.

Since $X - (\overline{U_1} \setminus U_1) = U_1 \cup (X - \overline{U_1})$ we get the component C_q of q in U^* must be wholly contained U_1. Also $C_q \subset Q$.

To prove

The quasicomponent Q_q of q in U^* meets $\overline{U_2} \setminus U_2$.

Let S be any clopen set in U^* containing Q_q.

Since S is an open set containing q, S contains points outside Q. i.e. there exist a point $p \in U_1 \setminus Q$ with $p \in S$.

Let P be the quasicomponent of p in U, then P meets $\overline{U_2} \setminus U_2$.

Since $P \neq Q$, and $H \subset Q$ (also P and Q are disjoint) we get P and H are disjoint.

Hence P is also a quasicomponent p in $U^* = U \setminus H$.

Since $p \in S$, $P \subset S$ (because S is a clopen set in U^* containing p).

Since P meets $\overline{U_2} \setminus U_2$, S also meets $\overline{U_2} \setminus U_2$.

By theorem (1.1.20), since $C_q \subset U_1$, C_q does not intersect $\overline{U_2} \setminus U_2$.

Therefore we get Q_q meets $\overline{U_2} \setminus U_2$.

Therefore $C_q \neq Q_q$

Therefore we have constructed an open set U^* on which components and quasicomponents do not agree. This is a contradiction to our assumption.

Hence X is locally connected.

Theorem 1.5.5

Let X be a continuum, and let F be the set of all points of X at which X fails to be locally connected. Then either F is empty or F contains a continuum consisting of more than one point.

Proof

Let F be non-empty.

Therefore there exists a point p in F at which X is not locally conected.

Hence by theorem (1.4.7), X is not weakly locally connected at p.

By theorem (1.4.9), there is an open neighbourhood V of p, and a collection $\{C_\alpha\}$ of components of $\overline{V} = H$, none of which contain p, such that every neighbourhood of p meets infinitely many of the C_α.

Then obviously H is not connected.

Also as X is connected, H is not clopen.

Therefore boundary B of H is nonempty.

Since components and quasicomponents coincide on the compact set H by theorem (1.1.18), each C_α meets B, by theorem (1.1.21)

Consider the compact set $S = \overline{\bigcup_\alpha C_\alpha}$. Then $p \in S$.

Let C_p be the (quasi)component of p in $S \cup B$.

Since every clopen neighbourhood of p in $S \cup B$ contains a connected C_j, every such clopen set meets the compact set B.

By theorem (1.1.20), the component C_p meets B.

Hence C_p is a continuum containing more than one point, which, inturn, contains a sub continuum C containing p properly and lying in the interior of H {for example, the component of p in $\overline{U} \cap S$, where U is an open neighbourhood of p whose closure lies in the interior H).

By the definitions of S and C_p, every neighbourhood of any point in C meets infinitely many of the C_α.

Hence by theorem (1.4.9), X is not weakly locally connected, and therefore not locally connected at any point of C.

Chapter II

CHAPTER - 2

LOCAL CONNECTEDNESS IN BITOPOLOGICAL SPACES

This chapter is devoted to the study of local connectedness in bitopological spaces. Many authors have studied this concept and they have given various definitions of pairwise local connectedness. In this chapter we discuss the contributions of Dasgupta and Lahri [5], Birsan [3] and Lakshmi [11] towards the study of pairwise local connectedness. First section deals with the results due to Dasgupta and Lahri [5]. They have obtained some basic properties and have established that there are spaces which are locally connected with respect to the two topologies taken separately, but the space considered as a bitopological space, is not locally connected.

In the second section we have made a comparative study of the three notions of pairwise local connectedness and discussed examples to show that they are independent notions.

In the third section we have discussed DL - local connectedness and B-local connectedness of the special bitopological space (X, τ, τ^{α}).

Section 2.1

Pairwise Local Connectedness By Dasgupta And Lahri [5]

In this section we discuss the contributions of Dasgupta and Lahri [5].

Definition 2.1.1

A set X on which are defined two (arbitrary) topologies u and v is called a bitopogical space and denoted by (X, u, v)

Definition 2.1.2

A Bitopological space (X, u, v) is called **pairwise connected** if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that $[A \cap C_u(B)] \cup [C_v(A) \cap B] = \phi$, where C_u and C_v denote the closures with respect to the topologies u and v respectively. If X can be so expressed then, A and B are said to separate X . Also in this case A and B are called **Separated Sets.**

A Subset A of (X, u, v) is pairwise connected if the space (A, u_A, v_A) is pairwise connected.

According to pervin [19],Components are maximal connected subsets of (X, u, v).

Defnition 2.1.3

Let (X, u, v) be a bitopological space and $x \in X$. The Component $C(x)$ of x is the union of all pairwise connected subsets of X containing x.

Pervin [19] has shown that the union of any family of pairwise connected sets having a non-empty intersection is a pairwise connected set. Hence we get $C(x)$ is pairwise connected.

Definition 2.1.4

A function f mapping a bitopological space (X, u, v) into a bitopological space (X^*, u^*, v^*) is said to be **continuous** if and only if the induced mappings $f_1 : (X, u) \rightarrow (X^*, u^*)$ and $f_2 : (X, v) \rightarrow (X^*, v^*)$ of the topological spaces are continuous.

Definition 2.1.5

A mapping $f: (X, u, v) \rightarrow (X^*, u^*, v^*)$ is said to be open if and only if the induced mappings $f_1:(X, u) \rightarrow (X^*, u^*)$ and $f_2: (X, v) \rightarrow (X^*, v^*)$ of the topological spaces are open.

Definition 2.1.6

A bitopological space (X, u, v) is called **pairwise locally connected at a point** $x \in X$ if for every pair of u - openset P and v-openset Q each containing x, there exits a pairwise connected v-openset C and a pairwise connected u-openset D such that $x \in C \subset P$ and $x \in D \subset Q$.

(X, u, v) is called **locally connected** if and only if it is locally connected at every point of X..

Remark 2.1.7

From the above definition it follows that a bitopological space is pairwise locally connected if and only if the family of all pairwise connected v-opensets is a base for the u- topology and family of all pairwise connected u - opensets is a base for the v - topology.

Remark 2.1.8

Pairwise local connectedness for a bitopological space is not equivalent to local connectedness of the two topologies as is shown by the following example :

Example 2.1.9

Let (X, u, v) be a bitopological space where X = {a, b, c},

$u = \{\phi, X, \{a\}, \{a, b\}, \{a,c\}\}$ and $v = \{\phi, X, \{b\}, \{a, b\}, \{b,c\}\}$.

Clearly X is locally connected with respect to both the topologies u and v.

For the u - open set {a} containing a, there is no v - open set containing a and contained in $\{a\}$. Hence $(X, \mathcal{u}, \mathcal{v})$ is not pairwise locally connected at a.

As in the case of topological spaces, we get the following theorem for bitopological spaces.

Theorem 2.1.10

In a bitopological space $(X, \mathcal{u}, \mathcal{v})$,

(i) each component $C(x)$ is a maximal pairwise connected set in X.

(ii) the set of all distinct components of points of X form a partition of X.

(iii) each $C(x)$ satisfies the equation

$$C(x) = C_{u}(C(x)) \cap C_{v}(C(x))$$

(iv) Continuous image of a pairwise connected set is pairwise connected.

The following theorem gives a characterisation of pairwise locally connected spaces.

Theorem 2.1.11

A bitopological space $(X, \mathcal{u}, \mathcal{v})$ is pairwise locally connected if and only if the components of $\mathcal{u}$ - opensets are $\mathcal{v}$ - opensets and the components of $\mathcal{v}$ - opensets are $\mathcal{u}$ opensets.

Proof

Suppose that $(X, \mathcal{u}, \mathcal{v})$ is pairwise locally connected.

Let $G \subset X$ be $\mathcal{u}$ - open, C a component of G and $\{P_\alpha\}$ be a basis of u consisting of pairwise connected $\mathcal{v}$ - opensets.

If $y \in C$, then $y \in G$ and hence there exist P_α with $y \in P_\alpha \subset G$.

Since C is the component of y and P_α is pairwise connected, we have $y \in P_\alpha \subset C$. That is, C is $\mathcal{V}$ - open.

Similarly the components of $\mathcal{V}$ - open sets are U - opensets.

Conversely from the given condition we get that the family of all pairwise connected $\mathcal{V}$ (respectively $\mathcal{U}$) - opensets is a base for the $\mathcal{U}$ (respectively $\mathcal{V}$) - topology and hence by remark 2.1.7 $(X, \mathcal{U}, \mathcal{V})$ is pairwise locally connected.

Theorem 2.1.12

Let $(X, \mathcal{U}, \mathcal{V})$ and $(X^*, \mathcal{U}^*, \mathcal{V}^*)$ be two bitopological spaces and $f : (X, \mathcal{U}, \mathcal{V}) \rightarrow (X^*, \mathcal{U}^*, \mathcal{V}^*)$ be a mapping which is continuous, open and surjective. Then if X is pairwise locally connected X^* is also pairwise locally connected.

Proof

Let P be any $\mathcal{U}^*$ open subset of X^*. And C be any component of P.

As f is continuous, f^{-1} (P) is $\mathcal{U}$ - open in $(X, \mathcal{U})$.

Let A be any component of f^{-1} (P).

Since X is pairwise locally connected, by theorem (2.1.11), we get A is $\mathcal{V}$ - open.

Since continuous image of a pairwise connected space is pairwise connected, we get f (A) is pairwise connected.

Since C is a component of P, either $f(A) \subset C$ or $f(A) \cap C = \phi$.

Hence f^{-1} (C) is the union of collection of components of f^{-1} (P) and so f^{-1} (C) is $\mathcal{V}$ - open.

As f is open and surjective, $C = f(f^{-1}(C))$ is v^* - open.

Hence any component of u^* - open set is a v^* - open set.

Similarly we can prove that, any component of v^* - openset is a u^* - open set.

Hence by theorem (2.1.11) we get that X^* is pairwise locally connected.

Theorem 2.1.13

A bitopological space (X, u, v) is pairwise locally connected if and only if given any point $x \in X$ and a pair of u - openset P and v - openset Q each containing x, there are v - openset C and u - openset D each containing x, and such that C is contained in a single component of P and D is contained in a single component of Q.

Proof

Let X be pairwise locally connected, $x \in X$.

Let P be a u - openset containing x. Let A be a component of P that contains x. Since X is pairwise locally connected, there exists a pairwise connected v - openset C such that $x \in C \subset P$.

Since A is a maximal pairwise connected set containing x by theorem (2.1.10), $x \in C \subset A \subset P$.

Since components are disjoint sets, C is not contained in any other component of P, similar proof holds, if we start with a v- openset Q containing x.

Conversely, assume the given condition. Take any point $x \in X$ and any u - openset P containing x. By assumption, there is a v - openset C containing x, which is contained in a single component F of P.

Then $x \in C \subset F \subset P$.

Let $y \in F$, then $y \in P$.

Thus there is a v - openset O such that $y \in O$ and O is contained in a single component of P.

Since components are disjoints set and $y \in F$, the component containing O must be F.

Therefore $y \in O \subset F$.

Hence F is v - open.

For every $x \in X$, for every u - openset P containing x, there is a pairwise connected v - openset F such that $x \in F \subset P$.

Similarly, we can prove that for every v - openset Q containing x, there is a pairwise connected u - openset G such that $x \in G \subset Q$.

Hence (X, u, v) is pairwise locally connected at x. Since x is arbitrary, (X, u, v) is pairwise locally connected.

Section 2·2

Comparison of different notions of pairwise local connectedness.

In this section we discuss three definitions of pairwise local connectedness due to Birsan [3], Dasgupta and Lahri [5] and Lakshmi [11]. Interesting examples are discussed to illustrate that all these three definitions - B - Local connectedness, DL - Local connectedness and L - Local connectedness are independent and a comparative study of these definitions is made here.

Definition [Birsan, 3] 2.2.1

In a space (X, u, v), **u is B - locally connected with respect to v** if for each point $x \in$ X and each u - openset U containing x, there is a pairwise connected u - openset G such that $x \in$ G $\subset$ U. The space (X, u, v) is **B - pairwise locally connected** if u is B - locally connected with respect to v and v is B - locally connected with respect to u.

The concept of pairwise local connected as defined by Dasgupta and Lahri [5] which we have discussed in section 2.1 is referred to as DL - local connectedness. Their notion can be split into its two constituent parts as follows.

Definition 2.2.2

In a space (X, u, v), u is DL - locally connected with respect to v if for each point $x \in$X and each u openset U containing x, there is a pairwise connected v openset G such that $x \in$G $\subset$ U. The space (X, u, v) is **DL - pairwise locally connected** if u is DL - locally connected with respect to v and v is DL - locally connected with respect to u.

Definition 2.2.3

Lakshmi [11] has defined local connectedness for bitopological spaces in terms of partially open sets.

In (X, u, v) the intersection U $\cap$ V of a u openset U and a v - openset V is called a **partially openset**.

The space (X, u, v) is **L - locally connected** if for every point $x \in X$ and every partially openset $U \cap V$ containing x, there is a pairwise connected partially openset $G \cap H$ such that $x \in G \cap H \subset U \cap V$.

Remark 2.2.4

A direct comparision of definition 2.2.1 and 2.2.2 reveals that if $x \in U \in u$, then the required pairwise connected set G such that $x \in G \subset U$, is u open in Birsan's definition, but v open in Dasgupta and Lahri's definition.This defference determines the contrasting behaviour of the two notions of bitopological local connectedness causing the DL - pairwise locally connected bitopological space to collapse to the corresponding topological case.

Theorem 2.2.5

If in a bitopological space (X, u, v) the topology u is DL - locally connected with respect to v, then $u \subset v$.

Proof

Follows easily from definition 2.1.3

Corollary 2.2.6

If (X, u, v) is DL - pairwise locally connected then $u = v$.

In view of the above arguments we conclude that (X, u, v) is DL - pairwise locally connected $\Rightarrow$ X is locally connected $\Rightarrow$ X is B - pairwise locally connected.

Though DL - pairwise local connectedness causes a bitopological space to collapse to the topological case there are many examples of proper bitopological spaces in which only one topology is DL - locally connected with respect to the other.

Example 2.2.7

Consider a bitopological space (X, u, v) where the topology u is indiscrete and v is any topology on X. Then u is DL - and B - locally conected with respect to v.

Example 2.2.8

If u is any topology on X and v is the discrete topology then u is DL - locally connected with respect to v, while v is B - locally connected with respect to u.

Example 2.2.9

The space (R, L, ℝ) of the real numbers with the left hand and the right hand topologies is B - pairwise locally connected but neither L is DL - locally connected with respect to ℝ nor ℝ is DL - locally connected with respect to L.

Proof

Take any point $x \in$ X and an L - openset U containing x.

Since L has as basis $\{[a,\infty) / a \in \overline{R}\}$, there exists $[a, \infty) \in L$

such that $x \in [a, \infty) \subset$ U.

Since $[a, \infty)$ is L open and connected, we get L is B - locally connected with respect to $\mathbb{R}$.

Similarly, it can be shown that $\mathbb{R}$ is B locally connected with respect to L. Now, consider the L - openset $[x, \infty)$ containing x.

Suppose there is a $\mathbb{R}$ - openset V such that $x \in V \subset [x, \infty)$. Then V must contain $(-\infty, x]$.

Therefore $x \in (-\infty, x] \subset V \subset [x, \infty)$. That is $(-\infty, x] \subset [x, \infty)$, which is not possible. Hence L is not DL - locally connected with respect to $\mathbb{R}$. Similarly it can be shown that $\mathbb{R}$ is not DL - locally connected with respect to L.

Example 2.2.10

If X = R, u is the usual topology on R and v is the discrete topology on R, then (X, u, v) the topology u is DL - locally connected with respect to v. However v is not DL - locally connected with respect to u.

Proof

Since $\{x\}$ is open and connected we get that u is DL - locally connected with respect to v.

To prove

v is not DL - locally connected with respect to u.

Let V = $\{x\}$. Then V is v - open

Take $x \in V = \{x\}$

If there exist an u open connected set such that $x \in U \subset V = \{x\}$

Then U = $\{x\}$, but U is not u - open.

Therefore v is not DL - locally connected with respect to u.

For a bitopological space (X, u, v), partially opensets form a basis for the supremum topology $u \vee v$ on X. The next two examples show that

(i) L - local connectedness is not equivalent to local connectedness of the supremum topology.

(ii) Local connectedness of the supremum topology and B local connectedness are independent properties.

(iii) L - local connectedness does not imply B - or DL - local connectedness.

Example 2.2.11

Consider the real line (R, u) and let v be the countable complement topology on the reals.

The space (R, $u \vee v$) is not locally connected, while the bitopological space (R, u, v) is L - locally connected and B - pairwise locally connected. It does not satisfy any DL - local connectedness property. The spaces (R, u) and (R, v) are not locally connected.

Example 2.2.12.

Let u and v be the left and the right half - open interval topology on the reals.

The supremum topology is discrete, thus the space (R, $u \vee v$) is locally connected. Hence the space (R, u, v) is L - locally connected, but neither topology is either B - or DL - locally connected with respect to the other.

Further more, the spaces (R, u) and (R, v) are not locally connected. For example, that v is not B - locally connected with respect to u follows from the fact that the sets $[x, y)$ and $[y, x + h)$ form a separation in (R, u, v) of the subset $[x, x + h) \in v$, where $x < y < x + h$.

Analogous to the case of ordinary topological spaces we get the following result for bitopological spaces regarding B - local connectedness and DL - local connected ness.

Theorem [Birsan, 3] 2.2.13

In a space (X, u, v) the topology u is B locally connected with respect to v if and only if the components of each u openset are u open.

Theorem 2.2.14

In a space (X, u, v) the topology u is DL - locally connected with respect to v if and only if the components of each u openset are v open.

Theorem 2.2.14 is nothing else but theorem 2.1.11,

By modifying the proof of theorem 2.1.12, we can prove the following result :

Theorem 2.2.15

If f : $(X, u, v) \rightarrow (Y, S, T)$ is pairwise continuous and surjective, and f : $(X, v) \rightarrow (Y, T)$ is open, then DL - local connectedness of u with respect to v implies DL - local connectedness of S with respect to T.

By following the standard arguments it can be shown that product and quotient theorems hold good for DL - local connectedness and B - local connectedness.

Theorem (Product theorem) 2.2.16

In the product space (X, u, v) of a family $\{ (X_j, u_j, v_j) / j \in J \}$, u is DL - locally connected (B - locally connected) with respect to v if and only if u_j is DL - locally connected (B - locally connected) with respect to v_j for every $j \in J$ and the spaces (X_j, u_j, v_j) are pairwise connected except finitely many of them.

Theorem (Quotient theorem) (2.2.17)

Let (X, u, v) be a topological space and ρ an equivalence relation on X. If u is DL - locally connected (B locally connected) with respect to v, then in the quotient space $(X/\rho, u/\rho, v/\rho)$ the topology u/ρ is DL - locally connected (B - locally connected) with respect to v / ρ.

Remark 2.2.18

Consider the special case of a bitopological space (X, u, v) where one topology is finer than the other, say $u \subset v$ holds.

In this case the supremum topology $u \vee v$ on X is v, as well as the family of all partially open sets.

We conclude this section by stating the following relationship among the three notions of local - connectedness :

Theorem 2.2.19

Consider the space (X, u, v) the inclusion $u \subset v$ holds, then (X, v) is locally connected $\Rightarrow$ (X, u, v) is L - locally connected $\Leftrightarrow$ v is B - locally connected with respect to $u \Rightarrow u$ is DL - locally connected with respect to v.

Section 2.3

DL - local connectedness and B -local connectedness of the space (X, τ, τ^{α})

DL - local connectedness and B - local connectedness of the particular bitopological space (X, τ, τ^{α}) is discussed in this section.

Definition 2.3.1.

Let (X, τ) be a topological space. A subset B is called an **α- set** if $B \subset \tau$ int (τ Cl (τ int B)).

The family of all α - sets is denoted by τ^{α}.

Theorem [Njastad, 18] 2.3.2

τ^{α} is a topology on X larger than τ.

Theorem [Mrsevic and Reilly, 12] 2.3.3.

Let (X, τ) be a topological space (X, τ) and let $G = U \setminus N$, where $U \in \tau$ and N is nowhere dense. Then τ^{α} cl G $= \tau$ cl G $= \tau$ cl U $= \tau^{\alpha}$ cl U holds.

Theorem [Mrsevic and Reilly, 13]2.3.4

(i) Let W be an α - set in the space (X, τ). Then W is τ^{α} connected $\Leftrightarrow$ W is pairwise connected $\Leftrightarrow$ W is τ connected.

(ii) Let $W = U / N$, where $U \in \tau$ and N is nowhere dense. Then W is τ^{α} connected (= τ connected = pairwise connected) $\Rightarrow$ U is τ^{α} connected (= τ connected = pairwise connected).

Theorem [Mrsevic and Reilly, 13] 2.3.5

(X, τ^{α}) is locally connected $\Leftrightarrow$ τ^{α} is B - locally connected with respect to τ $\Rightarrow$ τ is B - locally connected with respect to τ^{α} $\Leftrightarrow$ (X, τ) is locally connected.

The following two theorems give characterisations of the spaces (X, τ) in which the topology (1) τ^{α} is DL - locally connected with respect to τ and

(2) τ is DL - locally connecteed with respect to τ^{α}.

Theorem 2.3.6

(X, τ, τ^{α}) is DL - pairwise locally connected $\Leftrightarrow$ τ^{α} is DL - locally connected with respect to τ $\Leftrightarrow \tau = \tau^{\alpha}$ and (X, τ) (= (X, τ^{α})) is locally connected.

Proof

Follows from theorem 2.2.5 and Corollary 2.2.6

Theorem 2.3.7

τ is DL - locally connected with respect to τ^{α} $\Leftrightarrow$ (X, τ) is localy connected.

Remark 2.3.8

DL - local connectedness of τ^{α} with respect to τ is not equivalent to local connectedness of the space (X, τ^{α}) as seen from the following example :

Example 2.3.9

Let X be an infinite set, p, q $\in$ X and p $\neq$ q.

Let $\tau = \{\phi, X\} \cup \{G \subset X \setminus \{p, q\} \setminus X \setminus G \text{ is finite}\}$. The space (X, τ^{α}) is an infinite set with the cofinite topology and hence locally connected. But τ^{α} is not DL - locally connected with respect to τ since $\tau \neq \tau^{\alpha}$.

Remark 2.3.10

DL - local connectedness of τ with respect to τ^{α} need not imply DL - local connectedness τ^{α} with respect to τ as the following example illustrates :

Example 2.3.11

Let (X, τ) be the unit interval $I = [0, 1]$ of the real line with the usual topology. Then τ is DL - locally connected with respect to τ^{α}, but τ^{α} is not DL - locally connected with respect to τ. It is enough to consider the point O and its τ^{α} neighbourhood $[0, 1] \setminus \left\{\frac{1}{n} \backslash n \in \mathbb{N}\right\}$.

Thus DL - local connectedness of τ with respect to τ^{α} does not imply DL - local connectedness of τ^{α} with respect to τ.

The chapter is concluded by giving the following relations between the notion of local connectedness of the spaces (X, τ), (X, τ^{α}) and (X, τ, τ^{α}).

Theorem 2.3.12

(X, τ^{α}) is locally connected $\Leftrightarrow$ (X, τ, τ^{α}) is L - locally connected $\Leftrightarrow$ τ^{α} is B - locally connceted with respect to $\tau \Rightarrow \tau$ is DL - locally connected with respect to $\tau^{\alpha} \Leftrightarrow \tau$ is B - locally connected with respect to $\tau^{\alpha} \Leftrightarrow (X, \tau)$ is locally connected.

Proof

Follows from theorems 2.2.19, 2.3.7 and 2.3.5 and using the fact that $\tau \subset \tau^{\alpha}$.

Chapter III

CHAPTER - 3

LOCAL CONNECTEDNESS IN FUZZY TOPOLOGICAL SPACES

In this chapter we study the concept of local connectedness in fuzzy topological spaces. Ajmal and Kohli [1] have introduced four definitions of C_i- local connectedness (i = 1, 2, 3, 4) using the concept of C_i -connectedness (i = 1, 2, 3, 4) is fuzzy topological spaces. The concepts C_2 - local connectedness and C_4 - local connectedness are proved to be "good extensions" of local connectedness, while the concept C_3 - local connectedness is not a "good extension" is established by means of an example. They [1] have proved certain results analogous to the classical situation. Moreover the authors [1] have obtained some interesting results on C_2 - components and C_4 - quasi components. Before discussing these results, let us give the preliminary definitions and results on fuzzy sets and fuzzy connectedness which are needed for our study.

Section 3.1

Preliminary definitions and results

Definition 3.1.1

Let X be a non - empty set. A **fuzzy set** in X is a function μ from X into [0, 1]

Remark 3.1.2

Let X be any set and $A \subset X$. Then every subset A of X can be identified with the fuzzy set $\psi_A : X \rightarrow [0, 1]$ defined as

$\psi_A(x) = 1$ if $x \in A$

$= 0$ otherwise

Definition 3.1.3

For $\alpha \in [0, 1]$, the **constant fuzzy set** α on X is defined as $\alpha(x) = \alpha$ for every $x \in X$.

Definition 3.1.4

Let $\{\mu_\lambda\}_{\lambda \in J}$ be a collection of fuzzy sets in X. Then

(i) $(\vee\mu_\lambda)(x) = \sup\{\mu_\lambda(x)/\lambda \in J\}$ for all x in X.

(ii) $(\wedge\mu_\lambda)(x) = \text{Inf}\{\mu_\lambda(x)/\lambda \in J\}$ for all x in X.

Definition 3.1.5

Two fuzzy sets μ and γ are said to **intersect** if $\mu \wedge \gamma \neq 0$.

Definition 3.1.6

Let μ be a fuzzy set in X. Then the **complement** of μ denoted by μ^c is defined to be fuzzy set $1 - \mu$, where 1 is the constant fuzzy set

ie., $\mu^c(x) = (1 - \mu)(x)$

$= 1 - \mu(x)$ for all x in X.

Definition 3.1.7

If μ and γ are fuzzy sets in X, then $\mu \leq \gamma$ means $\mu(x) \leq \gamma(x)$ for all x in X.

Definition 3.1.8

Let $\mu : X \rightarrow [0, 1]$ be a fuzzy set in X. Then the **support** of μ is $\{x \,/\, \mu(x) > 0\}$ and is written shortly as **supp** μ.

Definition 3.1.9

Let X and Y be any two non - empty sets. Let f be a function from X to Y and γ be a fuzzy set in Y. Then the **inverse of** γ **under f** written as $\mathbf{f^{-1}}(\gamma)$ is the fuzzy set in X defined by

$f^{-1}(\gamma)\,(x) = \gamma\,(f(x))$ for all $x \in X$. For every fuzzy set μ in X, the **image of** μ **under f**, written as $f(\mu)$ is the fuzzy set defined by

$$\begin{aligned} f(\mu)(y) \quad &= \quad \sup_{z \in f^{-1}(y)} \mu(z) \text{ if } f^{1}(y) \text{ is not empty.} \\ &= \quad 0 \qquad \text{otherwise.} \end{aligned}$$

Definition 3.1.10

A **fuzzy point** x_α, $\alpha \in [0, 1]$ is a fuzzy set on X defined by $x_\alpha(x) = \alpha$, $x_\alpha(y) = 0$ for all $y \neq x$. x is called the support of x_α and α is referred to as the **value** of x_α. A fuzzy point $x_\alpha \in \mu$ iff $\alpha \leq \mu(x)$.

Definition 3.1.11

A fuzzy topology on X is a family δ of fuzzy sets on X such that

i) all constant functions $\alpha \in \delta$

ii) δ is closed for arbitrary union.

iii) δ is closed for finite intersection.

The pair (X, δ) is called a **fuzzy topological space** (abbreviated as fts). The members of δ are called **fuzzy open sets**.

Definition 3.1.12

A fuzzy set μ in X is said to be **fuzzy closed** if its complement is fuzzy open.

Definition 3.1.13

Let (X, δ) be a fuzzy topological space. Let μ be a fuzzy set on X. Then **closure** of μ and **interior** of μ are defined respectively as follows :

$\mathrm{cl}\ \mu = \wedge\ \{\gamma : \gamma^c \in \delta \text{ and } \gamma \geq \mu\}$

$\mathrm{int}\ \mu = \vee\ \{\gamma : \gamma \in \delta \text{ and } \gamma \leq \mu\}$

Definiton 3.1.14

Let (X, δ) be a fts. A fuzzy set μ is a **neighbourhood of a fuzzy point** x_α iff there exists an open fuzzy set λ such that $x_\alpha \in \lambda \leq \mu$.

Theorem 3.1.15

A fuzzy set μ in a fts X is a fuzzy open if and only if μ is a neighbourhood of each of the fuzzy points contained in μ.

Definition 3.1.16

Let (X, δ) be a fuzzy topological space. Suppose $A \subset X$ and let

$\delta_A = \{\mu/A : \mu \in \delta$. Then (A, δ_A) is called a **fuzzy subspace** of the fts X. In short, we shall denote (A, δ_A) by A. (Here μ / A denotes the restriction map).

Definition 3.1.17

A fuzzy subspace A is said to be a **fuzzy open subspace** if its characteristic function Ψ_A is fuzzy open in δ.

Notation 3.1.18

Let μ be a fuzzy set in X and let $A \subset X$. Then we denote μ / A by μ^a. In particular if μ is a fuzzy point x_α is X, then we denote the fuzzy set x_α / A by x_α^a. We remark that x_α^a may not be a fuzzy point in A unless $x \in A$. On the contrary for each fuzzy point in A, there is a fuzzy point in X with the same support and value α.

Definition 3.1.19

Let (X, δ_1) and (Y, δ_2) be two fuzzy topological spaces. A mapping $f : (X, \delta_1) \rightarrow (Y, \delta_2)$ is said to be **fuzzy continuous** if for every $\gamma \in \delta_2$, $f^1(\gamma) \in \delta_1$

Definition 3.1.20

Let (X, δ_1) and (Y, δ_2) be two fuzzy topological spaces.

A mapping $f : (X, \delta_1) \rightarrow (Y, \delta_2)$ is said to be **fuzzy open** if for every $\gamma \in \delta_1, f(\gamma) \in \delta_2$.

Definition 3.1.21

Let $f : X \rightarrow Y$ be a surjective map. The map f is said to be a **fuzzy quotient map**, provided a fuzzy set γ in Y is open iff $f^{-1}(\gamma)$ is open in X.

Definition 3.1.22

If (X, δ) is a fts and Y a non empty set and if $f : X \rightarrow Y$ is a surjective map, then there exists exactly one fuzzy topology relative to which f is a quotient map, it is called the **fuzzy quotient topology** induced by f.

Definition 3.1.23

Let $(X_\alpha, \delta_\alpha)_{\alpha \in I}$ be a family of fts, then the product **fuzzy topology** on πX_α is defined as the coarsest fuzzy topology making all projections fuzzy continuous.

Fuzzy connectedness

Definition 3.1.24

Two fuzzy sets μ and η in X are said to be

(1) **non - overlapping** if $\mu \leq 1 - \eta$,

(2) **Overlapping (quasi - coincident)** if $\mu(x) + \eta(x) > 1$ for some $x \in X$.

In case (2) μ and η are **said to overlap** at x. It is clear that overlapping fuzzy sets are always intersecting but not conversely.

Definition 3.1.25

A fuzzy set η is a fts X is said to be **C_i - connected** (i = 1, 2, 3, 4) if for proper fuzzy open sets μ_1 and μ_2 in X the following hold :

C_1 : $\eta \leq \mu_1 \vee \mu_2,\ \mu_1 \Lambda \mu_2 \leq 1-\eta \Rightarrow \mu_1 \Lambda \eta = 0 \; or \; \mu_2 \Lambda \eta = 0$

C_2: $\eta \leq \mu_1 \vee \mu_2,\ \mu_1 \Lambda \mu_2 \Lambda \eta = 0 \Rightarrow \mu_1 \Lambda \eta = 0 \; or \; \mu_2 \Lambda \eta = 0$

C_3: $\eta \leq \mu_1 \vee \mu_2,\ \mu_1 \Lambda \mu_2 \leq 1-\eta \Rightarrow \mu_1 \leq 1-\eta \; or \; \mu_2 \leq 1-\eta$

C_4: $\eta \leq \mu_1 \vee \mu_2,\ \mu_1 \Lambda \mu_2 \Lambda \eta = 0 \Rightarrow \mu_1 \leq 1-\eta \; or \; \mu_2 \leq 1-\eta$

Analogous to classical situation, the following results are true for C_2-connectedness and C_4 - connectedness.

Theorem : 3.1.26

1) Let $\{\eta i\}_{i \in I}$ be a family of C_2 - connected pairwise intersecting fuzzy sets in a fts X. Then $V_{\eta i}$ is a C_2 - connected fuzzy set in X.

2) Let $\{\eta i\}_{i \in I}$ be a family of pairwise overlapping C_4 - connected fuzzy sets in a fts X. Then $V_{\eta i}$ is a C_4 - connected fuzzy set in X.

3) C_2 - connectedness is invariant under fuzzy continuity.

4) C_4 - Connectedness is preserved under fuzzy continuous surjections.

Section 3.2

Local Connectedness in fuzzy topological space.

Definition 3.2.1

A fts X is said to be **C_i - locally connected** $(i = 1, 2, 3, 4)$ at a fuzzy point x_α in X if for every fuzzy open set μ in X containing x_α, there exists a C_i - connected fuzzy open set η in X such that $x_\alpha \leq \eta \leq \mu$. The fts X is said to be **C_i - locally connected $(i = 1, 2, 3, 4)$** if it is C_i - locally connected at every fuzzy point in X.

Theorem 3.2.2

A fts X is C_i - locally connected $(i = 1, 2, 3, 4)$ if and only if every fuzzy open subspace of X is C_i - locally connected.

Proof

Let A be a C fuzzy open subspace of the fts X..

Let x_α^a be a fuzzy point in A and let η^a be a fuzzy open set in A containing x_α^a. Then $\eta^a = \eta / A$ for some fuzzy open set η in X.

Clearly, the fuzzy point x_α in X is contained in η.

Since X is C_4 - locally connected and $\eta \wedge \chi_A$ is a fuzzy open set in X containing x_α there exists a C_4 - connected fuzzy open set μ in X containing x_α such that $\mu \leq \eta \Lambda \Psi_A$.

It is easy to see that $x_\alpha^a \leq \mu^a \leq \eta^a$.

Claim

μ^a is C_4 - connected in A. For if not, then there exist fuzzy open sets μ_1 and μ_2 in X such that $\mu^a \leq \mu_1^a \vee \mu_2^a$, $\mu_1^a \wedge \mu_2^a \wedge \mu^a = 0$, $\mu_1^a \nleq 1 - \mu^a$ and $\mu_2^a \nleq 1 - \mu^a$.

Let $\mu_1 * = \mu_1 \wedge \psi_A$ and $\mu_2 * = \mu_2 \wedge \psi_A$.

Then $\mu_1 *$ and $\mu_2 *$ form C_4 - disconnection of μ is X., since $\mu_1 *$ and $\mu_2 *$ clearly overlap with μ.

A similar proof holds good for i = 1, 2, or 3.

"Good extension"

Definition 3.2.3

For a topological space (X, τ), let $\omega(\tau)$ denote the collection of all lower semicontinuous functions from X into the closed unit interval [0, 1]. Then $\omega(\tau)$ is a fuzzy topology on X.

Definition 3.2.4

A property P_f in (X, $\omega(\tau)$) is called a **good extension** of a property P in (X, τ) provided (X, $\omega(\tau)$) has P_f iff (X, τ) has P.

Theorem 3.2.5

A topological space (X, τ) is locally connected if and only if the fts (X, $\omega(\tau)$) is C_2 - locally connected.

Proof

Necessity : Let μ be a fuzzy open set in ($\omega(\tau)$ containing the fuzzy point x_α. Now, x is in the support of μ and support of μ is open in τ, since μ is a lower semi continuous function and supp $\mu = \mu^{-1}(0, 1)$. By local connectedness of (X, τ), there is an open connected set U in X containing x and contained in the support of μ. Now, ψ_U, being the characteristic function of an open set in τ, is lower semi - continuous, and so $\psi_\mu \wedge \mu$ is a fuzzy open set in $\omega(\tau)$.

Claim

$\psi_U \wedge \mu$ is a C_2 - connected fuzzy set containing x_α

For, if not, then there exist lower semi - continuous functions μ_1 and μ_2 Such that

$$\sigma = \psi_U \wedge \mu \leq \mu_1 \vee \mu_2, \ \mu_1 \wedge \mu_2 \wedge \sigma = 0,$$

$$\mu_1 \wedge \sigma_2 \neq 0 \neq \mu_2 \wedge \sigma$$

It is clear that supp σ = U and supp μ_1 and supp μ_2 are open sets in τ.

Such that U $\subset$ supp $\mu_1 \cup$supp μ_2 ,supp μ_1 ,$\cap$ supp $\mu_2 \cap U = \phi$,

supp $\mu_1 \cap U \neq \phi \neq$, supp $\mu_2 \cap$ U.

This implies U is not connected, a contradiction.

Sufficiency

Let $x \in X$ and let U be an open set in τ containing x.

Then the characteristic function Ψ_U contains the fuzzy point x_1.

Now since ψ_u is fuzzy open in $\omega(\tau)$, there exits a C_2 - connected fuzzy open set μ in $\omega(\tau)$ containing the fuzzy point x_1 and contained in ψ_U.

Claim

Supp μ is an open connected set in τ containing x and contained in U.

For, if not, then there exist open sets V and W in τ such that

supp $\mu \subset V \cup W$, $V \cap W \cap$ supp $\mu = \phi$, $V \cap$ supp

$\mu \neq \phi \neq . W \cap$ supp μ

It is clear that ψ_v *and* ψ_w give a C_2 - disconnection of μ in $\omega(\tau)$ which is a contradiction.

Hence the proof.

Theorem 3.2.6.

A topological space (X, τ) is locally connected if and only if the fts $(X,\omega(\tau))$ is C_4 - locally connected.

Proof

The proof is similar as that of them 3.2.5

The above two theorems show that C_2 and C_4 - local connectedness are good extensions of the corresponding notions of local - connectedness.

However the following example shows that **C_3 - local connectedness is not a "good extension".**

Example 3.2.7

Let (X, τ) be the closed unit interval $[0,1]$ equipped with the usual topology. Let σ be any lower semi continuous function in X containing the fuzzy point x_α where $x = \alpha = 1/2$ and $\sigma \leq 1/2$.

Claim

Every fuzzy open set μ contained in σ and containing the fuzzy point x_α has a C_3 - disconnection.

Let $\varepsilon = 1/4$. Since μ is lower semi continuous, there exists a $\delta > 0$ such that $\mu(x) > 1/4$. whenever $1/2 - 2\delta < x < 1/2 + 2\delta$.

We define lower semi continuous functions μ_1 and μ_2 as follows.

$$\mu_1(x) = \left\{ \begin{array}{ll} \frac{1}{2} + \frac{1}{2(1-2\delta)}x, & if\ 0 \leq x \leq \frac{1}{2} - \delta, \\ \frac{3}{4} - \frac{1}{4\delta}(x + \delta - \frac{1}{2}), & if\ \frac{1}{2} - \delta < x \leq \frac{1}{2}, \\ \frac{1}{2} & if\ \frac{1}{2} \leq x \leq 1 \end{array} \right\}$$

$$\text{and } \mu_2(x) = \left\{ \begin{array}{ll} \frac{1}{2}, & if\ 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{4\delta}(x - \frac{1}{2}), & if\ \frac{1}{2} < x \leq \frac{1}{2} + \delta, \\ \frac{3}{4} - \frac{1}{2(1-2\delta)}(x - \frac{1}{2} - \delta), & if\ \frac{1}{2} + \delta < x \leq 1 \end{array} \right\}$$

It is easily verified that μ_1 and μ_2 form a C_3 - disconnection of μ which overlaps with μ_1 and μ_2 at x_1 and x_2 respectively, where $x_1 = 1/2 - \delta$ and $x_2 = 1/2 + \delta$.

Section 3.3

C_2 - local connectedness

Definition 3.3.1

Let X be a fts and let μ be a fuzzy set in X. We introduce an equivalence relation $\mathbf{E}_{(2,\mu)}$ on the set $\mathbb{B}_\mu$ of all fuzzy points contained in μ as follows:

For x_α, y_β in $\mathbb{B}_\mu$ define $x_\alpha \mathrm{E}_{(2,\mu)} y_\beta$ if and only if there exists a C_2-connected fuzzy set η containing the fuzzy points x_α and y_β and contained in μ.

Let $\mathbf{E}_{(2,\mu)}(x_\alpha)$ denote the equivalence class containing x_α. We call the fuzzy set $\mathbf{C}_{(2,\mu)}(x_\alpha) = \mathrm{V}\mathrm{E}_{(2,\mu)}(x_\alpha)$ the **C_2 - component** of μ containing the fuzzy point x_α.

Analogous to the classical situation we get the following two theorems regarding C_2 - components:

Theorem 3.3.2

For each fuzzy point $x_\alpha \in B_\mu$, the component $C_{(2,\mu)}(x_\alpha)$ is the maximal C_2 - connected fuzzy set containing x_α and contained in μ.

Theorem 3.3.3

Every fuzzy set μ in a fts X is the disjoint union of its C_2 - components.

Theorem 3.3.4

A fts X is C_2 - locally connected if and only if every C_2 - component of every fuzzy open set in X is fuzzy open.

Proof

Suppose μ is a fuzzy open set in X. Let $C_{(2,\mu)}(x_\alpha)$ be a C_2 - component of μ containing the fuzzy point x_α and let $y_\beta \leq C_{(2,\mu)}(x_\alpha)$ be any fuzzy point. Since X is C_2 - locally connected, there exists a C_2 - connected fuzzy open set η containing y_β and contained in μ.

Now $\eta \wedge C_{(2,\mu)}(x_\alpha) \neq 0$.

So, by theorem 3.1.26(1), $\eta \vee C_{(2,\mu)}(x_\alpha)$ is a C_2 - connected fuzzy set containing x_α and contained in μ.

Therefore η is contained in $C_{(2,\mu)}(x_\alpha)$. Hence $C_{(2,\mu)}(x_\alpha)$ is a neighbourhood of x_α. Thus $C_{(2,\mu)}(x_\alpha)$, being the neighbourhood of each of the fuzzy points it contains, is a fuzzy open set by theorem 3.1.15.

Conversely, let μ be a fuzzy open set in X containing a fuzzy point x_α. Then $C_{(2,\mu)}(x_\alpha)$ is a C_2 - connected fuzzy open set containing x_α and contained in μ.

Quotients and products

Theorem 3.3.5

Let $f : X \rightarrow Y$ be a fuzzy quotient map of a C_2 - locally connected space onto a fts Y. Then Y is C_2 - locally connected.

Proof

Let y_α be a fuzzy point in Y and let μ be a fuzzy open set in Y containing y_α. Let $C_{(2,\mu)}(y_\alpha)$ be the C_2 - Component of μ containing y_α.

By fuzzy continuity of f,$f^{-1}(\mu)$ is fuzzy open in X.

Let x_β be a fuzzy point contained in $f^{-1}(C_{(2,\mu)}y_\alpha))$

Since X is C_2- locally connected, there is a C_2 - connected fuzzy open set σ containing x_β and contained in $f^{-1}(\mu)$.

Now $f(x_\beta) \leq f(\sigma) \wedge (.C_{(2,\mu)}(y_\alpha)$

Since f is fuzzy continuous, $f(\sigma)$ is C_2 - connected.

Moreover, $f(\sigma) \leq \mu$, and hence by theorem 3.1.26(1),

$f(\sigma) \vee C_{(2,\mu)}(y_\alpha)$ is C_2 connected. Consequently, $f(\sigma) \leq C_{(2,\mu)}(y_\alpha)$

Therefore, $\sigma \leq f^{-1}(C_{(2,\mu)}(y_\alpha))$.

Hence $f^{-1}(C_{(2,\mu)}(y_\alpha)$ is a neighbourhood of each of its fuzzy points.

Therefore by theorem 3.1.15 $f^{-1}(C_{(2,\mu)}(y_\alpha))$ is fuzzy open. Since f is a fuzzy quotient map, $C_{(2,\mu)}(y_\alpha)$ is a fuzzy open set in Y. Hence Y is C_2 - locally connected.

Corollary 3.3.6

Let $f: X \rightarrow Y$ be a fuzzy continuous, fuzzy open (fuzzy closed) map of a C_2 - locally connected space onto Y. Then Y is C_2 - locally connected.

Proof

Since every fuzzy continuous fuzzy open (fuzzy closed) surjection is fuzzy quotient, the result follows by above theorem.

Theorem 3.3.7

If the fuzzy product πX_α of a family fts is C_2 - locally connected, then each co - ordinate space X_α is C_2 - locally connected.

Proof

Follows easily from the fact that projections are fuzzy open maps.

Section 3.4

C_4 - Quasi Local Connectedness

Let X be a fts and let μ be a fuzzy set in X. Let $\mathbb{B}_\mu$ denote the set of all fuzzy points contained in μ. In analogy with the definition of C_2-components, it may not be possible to define an equivalence relation on $\mathbb{B}_\mu$. similar to $E_{(2,\mu)}$ in case of C_4 - connectedness. It turns out that such a relation may fail to be transitive. Transitivity breaks, since the union of two

C_4 - connected intersecting fuzzy sets may not be C_4 - connected. The following example clearly illustrates this phenomenon, where σ_1 and σ_2 are intersecting C_4 - connected fuzzy sets whose union σ is not C_4 - connected.

Example 3.4.1

Let X = [0, 1] and let the fuzzy sets μ, η and σ in X be defined as follows :

$$\mu(x) = \begin{cases} \frac{2}{3}, & if\ 0 \le x \le \frac{1}{3}, \\ \frac{5}{6}, & if\ \frac{1}{3} < x \le \frac{2}{3}, \\ 0, & if\ \frac{2}{3} < x \le 1, \end{cases}$$

$$\eta(x) = \begin{cases} \frac{5}{6}, & if\ 0 \le x \le \frac{1}{3}, \\ 0, & if\ \frac{1}{3} < x \le \frac{2}{3}, \\ \frac{5}{6}, & if\ \frac{2}{3} < x \le 1, \end{cases}$$

$$\sigma(x) = \begin{cases} 0, & if\ 0 \le x \le \frac{1}{3}, \\ \frac{1}{3}, & if\ \frac{1}{3} < x \le 1.. \end{cases}$$

Let δ denote the fuzzy topology on X generated by taking the subbase S consisting of μ, η, σ and all the constant fuzzy sets in X. Then the fuzzy sets σ_1 and σ_2 in X defined by

$$\sigma_1(x) = \left\{ \begin{array}{l} 0, \quad \text{if } 0 \leq x \leq \frac{1}{3}, \\[2ex] \frac{1}{3}, \quad \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\[2ex] \frac{1}{6}, \quad \text{if } \frac{2}{3} < x \leq 1, \end{array} \right\}$$

and

$$\sigma_2(x) = \left\{ \begin{array}{l} 0, \quad \text{if } 0 \leq x \leq \frac{1}{3}, \\[2ex] \frac{1}{6}, \quad \text{if } \frac{1}{3} < x \leq \frac{2}{3}, \\[2ex] \frac{1}{3}, \quad \text{if } \frac{2}{3} < x \leq 1. \end{array} \right\}$$

are maximal C_4 - connected fuzzy sets contained in the fuzzy open set σ. The fuzzy set $\sigma = \sigma_1 \vee \sigma_2$ is C_4 - disconnected and $\sigma_1 \vee \sigma_2 \neq 0$.

Further, if we define C_4 - components as maximal C_4 - connected fuzzy sets, we do not get a characterisation of C_4 - local connectedness and similar to theorem 3.3.4 as shown in the above example.

However the situation is corrected as follows :

Definition 3.4.2

Define an equivalence relation $E^*_{(4,\mu)}$ on $\mathbb{B}_{(1/2,\mu)}$, where $\mathbb{B}_{(1/2,\mu)}$ denotes the set of all fuzzy points contained in μ whose value exceeds 1/2, for a fuzzy set $\mu > 1/2$. By $\mu > 1/2$, we mean that $\mu(x) > 1/2$ for every x in X.

For fuzzy points x_α, y_β in $\mathbb{B}_{(1/2,\mu)}$ define $x_\alpha E^*_{(4,\mu)} y_\beta$ if and only if there exists a C_4 - connected fuzzy set η containing x_α and y_β and contained in μ.

We point out that the transitivity of the relation $E^*_{(4,\mu)}$ follows from the fact that the union of two C_4- connected overlapping fuzzy sets is C_4-connected by theorem 3.1.26(2)

Let $E^*_{(4,\mu)}(x_\alpha)$ denote the equivalence class containing the fuzzy point $x_\alpha \in \mathbb{B}_{(1/2,\mu)}$ and let $\{\eta_i\}$ be the family of all C_4-connected fuzzy sets containing x_α and contained in μ. We define a **C_4 - quasi component** of μ as the union $V\eta_i$ and denote it by $C^*_{(4,\mu)}(x_\alpha)$.

Theorem 3.4.3

A C_4 - quasi component $C^*_{(4,\mu)}(x_\alpha)$ is a C_4 - connected fuzzy set contained in μ and containing x_α such that $VE^*_{(4,\mu)}(x_\alpha) \leq C^*_{(4,\mu)}(x_\alpha)$

Theorem 3.4.4

A fuzzy set $\mu > 1/2$ is a fts X is the non - overlapping union of its C_4-quasi - components.

Definition 3.4.5

A fts X is said to be **C_4 - quasi - locally connected** if for each fuzzy open set $\mu > 1/2$ and every fuzzy point x_α(with $\alpha>1/2$) contained in μ there exists a C_4 - connected fuzzy open set η in X such that $x_\alpha \leq \eta \leq \mu$.

Remark 3.4.6

Every connected as well as every locally connected topological space considered as a fts is always C_4 - quasi - locally connected.

Theorem 3.4.7

If a fts X is C_4 - quasi locally connected, then the C_4-quasi-component $C^*_{(4,\mu)}(x_{\mu(x)})$ of each fuzzy open set $\mu > 1/2$ is a C_4 - connected neighbourhood of every fuzzy point x_α, $0 < \alpha \leq \mu(x)$.

Proof

Let X be C_4 - quasi locally connected and let x_α be a fuzzy point in $C^*_{(4,\mu)}(x_{\mu(x)})$ where $0 < \alpha \leq \mu(x)$clearly $x_\alpha \leq x_{\mu(x)} \leq \mu$.

By quasi - local - connectedness of X, there exists a C_4 connected fuzzy open set η such that $x_{\mu(x)} \leq \eta \leq \mu$. Since $\eta$and $C^*_{(4,\mu)}(x_{\mu(x)})$are overlapping fuzzy sets, their union is C_4-connected and consequently $\eta \leq C^*_{(4,\mu)}(x_{\mu(x)})$.

Thus $C^*_{(4,\mu)}(x_{\mu(x)})$ is a neighbourhood of the fuzzy point x_α.

Corollary 3.4.8

If a fts X is C_4 - locally connected, then the C_4 - quasi - component $C^*_{(4,\mu)}(x_{\mu(x)})$ of each fuzzy open set $\mu > 1/2$ is a neighbourhood of every fuzzy point x_α, $0 < \alpha \leq \mu(x)$.

Theorem 3.4.9

If a fts X is C_4 - quasi locally connected, then the C_4-quasi component $C^*_{(4,\mu)}(x_{\mu(x)})$ of each fuzzy open set $\mu > 1/2$ is a C_4-connected neighbourhood of the fuzzy set V $E^*_\mu(x_\alpha)$.

Proof

Follows easily from the theorem 3.4.7 and theorem 3.4.4

Corollary 3.4.10

If a fts X is C_4 - locally connected, then the C_4 - quasi - component $C^*{}_{(4,\mu)}(x_{\mu(x)})$ of each fuzzy open set $\mu > 1/2$ is a connected neighbourhood of the fuzzy set $\vee E^*{}_{\mu}(x_\alpha)$.

Summary and Conclusion

SUMMARY AND CONCLUSION

In this thesis we have collected some of the important results regarding local connectedness in topological spaces, bitopological spaces and fuzzy topological spaces.

The concepts connectedness and local connectedness are not related to one another. A space may possess one or both of these properties or neither. There are spaces which are locally connected at all points except at only one point. Examples are given to illustrate these facts. Apart from these illustrations, some nice properties and characterisations of local connected spaces are cited in chapter 1.

Groot and McDowell[6] have established that a systematic and simple approach to the theory of locally connected spaces can be obtained by the use of quasi components. Many interesting results on local connectedness using quasi components are discussed. moreover, the quasi components approach has led to the important result that in a connected Hausdorff space (i.e. in a continuum) the property "components coincide with quasi components on every open subset" is equivalent to local connectedness.

The three independent notions of pairwise local connectedness namely, DL - local connectedness, B- local connectedness and L - local connectedness are discussed in the second chapter. Though DL- Pairwise local connectedness causes a bitopological space to collapse to the topological case, examples of proper bitopological spaces are given in which only one topology is DL- locally connected with respect to the other. Relation among the three notions are given in the following two theorems:

(1) If in a space $(X, \mathcal{U}, \mathcal{V})$ the inclusion $\mathcal{U} \subset \mathcal{V}$ holds, then ,

$(X, \mathcal{V})$ is locally connected .

$\Rightarrow$ $(X, \mathcal{U}, \mathcal{V})$ is L-locally connected .

$\Leftrightarrow$ $\mathcal{V}$ is B- locally connected with respect to $\mathcal{U}$.

$\Rightarrow$ $\mathcal{U}$ is DL- locally connected with respect to $\mathcal{V}$.

(2) (X, τ^{α}) is locally connected

$\Leftrightarrow$ (X, τ, τ^{α}) is L-locally connected

$\Leftrightarrow$ τ^{α} is B- locally connected with respect to τ

$\Rightarrow$ τ is DL - locally connected with respect to τ^{α}

$\Leftrightarrow$ τ is B-locally connected with respect to τ^{α}

$\Leftrightarrow$ (X, τ) is locally connected.

Ajmal and Kohli [1] notions of local connectedness of fuzzy topological spaces referred to as C_i-local connectedness. Concentration is made on C_4 - local connnectedness and C_4- local connectedness which are proved to be "good extensions " of local connectedness.

Analogous to the classical situation , they [1] have defined C_2 - Components by introducing an equivalence relation among fuzzy points. But a similar relation when defined regarding C_4 - local connectedness does not satisfy the transitivity relation. Hence they [1] have modified the relation which had led to the introductionof C_4 - quasi components using which the class of C_4 - quasi locally connected fuzzy topological spaces is defined and interesting results are obtained

The authors [1] have not settled the problem whether product of C_i-locally connected spaces is C_i - locally connected. It would be worthy if one could find an answer to the above problem and generalise the results on pairwise local connectedness to the fuzzy situation.

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