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β^{**} Generalized Connectedness in Intuitionistic Fuzzy Topological Spaces

6.1 Introduction

Raja Sethupathy and Lakshmivarahan (1977) have introduced the concept of connectedness in fuzzy topological spaces. Connectedness in intuitionistic fuzzy special topological spaces was introduced by Ozcag and Coker (1998). In this chapter we have introduced intuitionistic fuzzy β^{**} generalized connected space, intuitionistici fuzzy β^{**} generalized super connected space and intuitionistici fuzzy β^{**} generalized extremely disconnected space. Also we have provided some characterizations of intuitionistic fuzzy β^{**} generalized super connected spaces.

6.2 Intuitionistic Fuzzy β^{**} Generalized Connected Spaces

In this section we have introduced intuitionistic fuzzy β^{**} generalized connected space and intuitionistic fuzzy β^{**} generalized super connected space. We have investigated some of their properties. Also we have provided a characterization theorem for an intuitionistic fuzzy β^{**} generalized super connected space.

Definition 6.2.1 : An IFTS (X, τ) is said to be an **intuitionistic fuzzy β^{**} generalized** (IFβ^{**}G) **connected space** if the only IFSs which are both an IFβ^{**}GOS and an IFβ^{**}GCS are $0_\sim$ and $1_\sim$.

Proposition 6.2.2 : Every IFβ^{**}G connected space is an IFC_5-connected space but not conversely in general.

Proof : Let (X, τ) be an IFβ^{**}G connected space. Suppose (X, τ) is not an IFC_5-connected space, then there exists a proper IFS B which is both an IFOS and an IFCS in (X, τ). That is B is both an IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ). This implies that (X, τ) is not an IFβ^{**}G connected space. This is a contradiction to our hypothesis. Therefore (X, τ) must be an IFC_5-connected space.

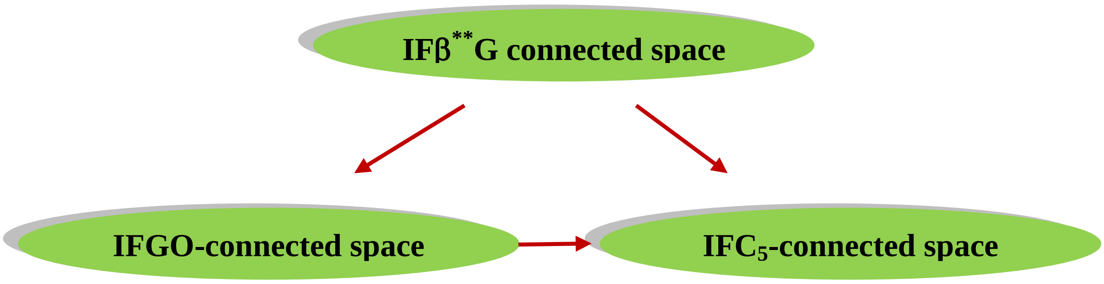
Example 6.2.3 : Let X = {a, b}, $G_1 = \langle x, (0.3_a, 0.3_b), (0.7_a, 0.7_b)\rangle$ and $G_2 = \langle x, (0.7_a, 0.6_b), (0.3_a, 0.4_b)\rangle$. Then $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X. Here (X, τ) is an IFC_5-connected space but not an IFβ^{**}G connected space, since the IFS $A = \langle x, (0.6_a, 0.5_b), (0.4_a, 0.5_b)\rangle$ is both an IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ).

Proposition 6.2.4 : Every IFβ^{**}G connected space is an IFGO-connected space but not conversely in general.

Proof : Let (X, τ) be an IFβ^{**}G connected space. Suppose (X, τ) is not an IFGO-connected space, then there exists a proper IFS B which is both an IFGOS and an IFGCS in (X, τ). That is B is both an IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ). This implies that (X, τ) is not an IFβ^{**}G connected space. This is a contradiction to our hypothesis. Therefore (X, τ) is an IFGO-connected space.

Example 6.2.5 : Let X = {a, b}, $G_1 = \langle x, (0.3_a, 0.3_b), (0.7_a, 0.7_b)\rangle$ and $G_2 = \langle x, (0.7_a, 0.6_b), (0.3_a, 0.4_b)\rangle$. Then $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X. Here (X, τ) is an IFGO-connected space but not an IFβ^{**}G connected space, since the IFS $A = \langle x, (0.6_a, 0.5_b), (0.4_a, 0.5_b)\rangle$ is both an IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ).

The relation between various types of intuitionistic fuzzy connectedness is given in the following diagram.



In the above diagram the reverse implications are not true in general.

Proposition 6.2.6 : An IFTS (X, τ) is an IFβ^{**}G connected space if and only if there exists no non-zero IFβ^{**}GOSs A and B in (X, τ) such that $A = B^c$.

Proof : Necessity : Let A and B be two IFβ^{**}GOSs in (X, τ) such that $A \neq 0_\sim$, $B \neq 0_\sim$ and $A = B^c$. Therefore B^c is an IFβ^{**}GCS. Since $B \neq 0_\sim$, $A = B^c \neq 1_\sim$. This implies A is a proper IFS which is both IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ). Hence (X, τ) is not an IFβ^{**}G connected space. But this is a contradiction to our hypothesis. Thus there exist no non-zero IFβ^{**}GOS A and B in (X, τ) such that $A = B^c$.

Sufficiency : Let A be both an IFβ^{**}GOS and IFβ^{**}GCS in (X, τ) such that $1_\sim \neq A \neq 0_\sim$. Now let $B = A^c$. Then B is an IFβ^{**}GOS and $B \neq 1_\sim$. This implies $B^c = A \neq 0_\sim$, which is a contradiction to our hypothesis. Therefore (X, τ) is an IFβ^{**}G connected space.

Proposition 6.2.7 : An IFTS (X, τ) is an IFβ^{**}G connected space if and only if there exists no non-zero IFβ^{**}GOSs A and B in (X, τ) such that $B = A^c$, $B = (\beta cl(A))^c$ and $A = (\beta cl(B))^c$.

Proof : Necessity : Assume that there exists IFSs A and B such that $A \neq 0_\sim$, $B \neq 0_\sim$, $B = A^c$, $B = (\beta cl(A))^c$ and $A = (\beta cl(B))^c$. Since $(\beta cl(A))^c$ and $(\beta cl(B))^c$ are IFβ^{**}GOSs in (X, τ), A and B are IFβ^{**}GOSs in (X, τ). This implies (X, τ) is not an IFβ^{**}G connected space, which is a contradiction. Therefore there exists no non-zero IFβ^{**}GOSs A and B in (X, τ) such that $B = A^c$, $B = (\beta cl(A))^c$ and $A = (\beta cl(B))^c$.

Sufficiency : Let A be both an IFβ^{**}GOS and IFβ^{**}GCS in (X, τ) such that $1_\sim \neq A \neq 0_\sim$. Now by taking B = A^c, we obtain a contradiction to our hypothesis. Hence (X, τ) is an IFβ^{**}G connected space.

Proposition 6.2.8 : If f : (X, τ) $\rightarrow$ (Y, σ) is an IFβ^{**}G continuous mapping and (X, τ) is an IFβ^{**}G connected space, then (Y, σ) is an IFC_5-connected space.

Proof : Let (X, τ) be an IFβ^{**}G connected space. Suppose (Y, σ) is not an IFC_5-connected space, then there exists a proper IFS A which is both an IFOS and an IFCS in (Y, σ). Since f is an IFβ^{**}G continuous mapping, f^{-1}(A) is both an IFβ^{**}GOS and an IFβ^{**}GCS in (X, τ). But this is a contradiction to our hypothesis. Hence (Y, σ) is an IFC_5-connected space.

Proposition 6.2.9 : If f : (X, τ) $\rightarrow$ (Y, σ) is an IFβ^{**}G irresolute surjection mapping and (X, τ) is an IFβ^{**}G connected space, then (Y, σ) is also an IFβ^{**}G connected space.

Proof : Suppose (Y, σ) is not an IFβ^{**}G connected space, then there exists a proper IFS B which is both IFβ^{**}GOS and IFβ^{**}GCS in (Y, σ). Since f is an IFβ^{**}G irresolute mapping, f^{-1}(B) is both IFβ^{**}GOS and IFβ^{**}GCS in (X, τ). But this is a contradiction to hypothesis. Hence (Y, σ) is an IFβ^{**}G connected space.

Definition 6.2.10 : An IFTS (X, τ) is called IFβ^{**}G connected between two IFSs A and B if there is no IFβ^{**}GOS E in (X, τ) such that A $\subseteq$ E and E_{q^c} B.

Example 6.2.11: Let X = {a, b} and τ = {$0_\sim$, G, $1_\sim$} be an IFT on X, where G = $\langle$x, (0.6_a, 0.7_b), (0.4_a, 0.3_b)$\rangle$. Then, the IFTS (X, τ) is an IFβ^{**}G connected between the two IFSs A = $\langle$x, (0.7_a, 0.6_b), (0.3_a, 0.4_b)$\rangle$ and B = $\langle$x, (0.3_a, 0.4_b), (0.6_a, 0.6_b)$\rangle$ as there exists no IFβ^{**}GOS E such that A $\subseteq$ E and E_{q^c} B.

Proposition 6.2.12 : If an IFTS (X, τ) is IFβ^{**}G connected between two IFSs A and B, then it is IFC_5-connected between two IFSs A and B but the converse may not be true in general.

Proof : Suppose (X, τ) is not IFC_5-connected between A and B, then there exists an IFOS E in (X, τ) such that $A \subseteq E$ and $E_{q^c} B$. Since every IFOS is an IFβ^{**}GOS, there exists an IFβ^{**}GOS E in (X, τ) such that $A \subseteq E$ and $E_{q^c} B$. This implies (X, τ) is not IFβ^{**}G connected between A and B, a contradiction to our hypothesis. Therefore (X, τ) is IFC_5-connected between A and B.

Example 6.2.13 : Let $X = \{a, b\}$ and $\tau = \{0_\sim, G, 1_\sim\}$ be an IFT on X, where $G = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$. Then (X, τ) is IFC_5-connected between the IFSs $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b)\rangle$ and $B = \langle x, (0.6_a, 0.6_b), (0.4_a, 0.4_b)\rangle$. But (X, τ) is not IFβ^{**}G connected between A and B, since the IFS $E = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b)\rangle$ is an IFβ^{**}GOS in X such that $A \subseteq E$ and $E \subseteq B^c$.

Proposition 6.2.14 : An IFTS (X, τ) is IFβ^{**}G connected between two IFSs A and B if and only if there is no IFβ^{**}GOS and IFβ^{**}GCS E in (X, τ) such that $A \subseteq E \subseteq B^c$.

Proof : Necessity : Let (X, τ) be IFβ^{**}G connected between two IFSs A and B. Suppose that there exists an IFβ^{**}GOS and IFβ^{**}GCS E in (X, τ) such that $A \subseteq E \subseteq B^c$, then $E_{q^c} B$ and $A \subseteq E$. This implies (X, τ) is not IFβ^{**}G connected between A and B, by Definition 6.2.10. A contradiction to our hypothesis. Therefore there is no IFβ^{**}GOS and IFβ^{**}GCS E in (X, τ) such that $A \subseteq E \subseteq B^c$.

Sufficiency : Suppose that (X, τ) is not IFβ^{**}G connected between A and B. Then there exists an IFβ^{**}GOS E in (X, τ) such that $A \subseteq E$ and $E_{q^c} B$. This implies that there exists IFβ^{**}GOS E in (X, τ) such that $A \subseteq E \subseteq B^c$. But this is a contradiction to our hypothesis. Hence (X, τ) is IFβ^{**}G connected between A and B.

Proposition 6.2.15 : If an IFTS (X, τ) is IFβ^{**}G connected between two IFSs A and B, $A \subseteq A_1$ and $B \subseteq B_1$, then (X, τ) is IFβ^{**}G connected between A_1 and B_1.

Proof : Suppose that (X, τ) is not IFβ^{**}G connected between A_1 and B_1, then by Definition 6.2.10, there exists an IFβ^{**}GOS E in (X, τ) such that $A_1 \subseteq E$ and $E_{q^c}\, B_1$. This implies $E \subseteq B_1^c$ and $A_1 \subseteq E$. That is $A \subseteq A_1 \subseteq E$. Hence $A \subseteq E$. Since $E \subseteq B_1{}^c$, $B_1 \subseteq E^c$. That is $B \subseteq B_1 \subseteq E^c$. Hence $E \subseteq B^c$. Therefore (X, τ) is not IFβ^{**}G connected between A and B, which is a contradiction to our hypothesis. Thus (X, τ) is an IFβ^{**}G connected between A_1 and B_1.

Proposition 6.2.16 : Let (X, τ) be an IFTS and A and B be IFSs in (X, τ). If $A_q B$, then (X, τ) is IFβ^{**}G connected between A and B.

Proof : Suppose (X, τ) is not IFβ^{**}G connected between A and B. Then there exists an IFβ^{**}GOS E in (X, τ) such that $A \subseteq E$ and $E \subseteq B^c$. This implies that $A \subseteq B^c$. That is $A_{q^c}\, B$. But this is a contradiction to our hypothesis. Therefore (X, τ) is IFβ^{**}G connected between A and B.

Definition 6.2.17 : An IFS A is called an **intuitionistic fuzzy regular β^{**} generalized open set** (IFRβ^{**}GOS) if $A = \beta^{**}\mathrm{gint}(\beta^{**}\mathrm{gcl}(A))$. The complement of an IFRβ^{**}GOS is called an **intuitionistic fuzzy regular β^{**} generalized closed set** (IFRβ^{**}GCS) in (X, τ).

Definition 6.2.18 : An IFTS (X, τ) is called an **intuitionistic fuzzy β^{**} generalized super** (IFβ^{**}G super) **connected space** if there exists no proper intuitionistic fuzzy regular β^{**} generalized open set in (X, τ).

Proposition 6.2.19 : Let (X, τ) be an IFTS, then the following are equivalent :

(a) (X, τ) is an IFβ^{**}G super connected space,

(b) For every non-zero IFRβ^{**}GOS A, β^{**}gcl(A) = $1_\sim$,

(c) For every IFRβ^{**}GCS A with A $\neq 1_\sim$, β^{**}gint(A) = $0_\sim$,

(d) There exists no IFRβ^{**}GOSs A and B in (X, τ) such that A $\neq 0_\sim \neq$ B, A $\subseteq B^c$,

(e) There exists no IFRβ^{**}GOSs A and B in (X, τ) such that A $\neq 0_\sim \neq$ B, B = $(\beta^{**}gcl(A))^c$, A = $(\beta^{**}gcl(B))^c$,

(f) There exists no IFRβ^{**}GCSs A and B in (X, τ) such that A $\neq 1_\sim \neq$ B, B = $(\beta^{**}gint(A))^c$, A = $(\beta^{**}gint(B))^c$.

Proof : (a) $\Rightarrow$ (b) Assume that there exists an IFRβ^{**}GOS A in (X, τ) such that A $\neq 0_\sim$ and β^{**}gcl(A) $\neq 1_\sim$. Now let B = $\beta^{**}gint(\beta^{**}gcl(A))^c$. Then B is a proper IFR$\beta^{**}$GOS in (X, τ). But this is a contradiction to the fact that (X, τ) is an IFβ^{**}G super connected space. Therefore β^{**}gcl(A) = $1_\sim$.

(b) $\Rightarrow$ (c) Let A $\neq 1_\sim$ be an IFRβ^{**}GCS in (X, τ). If B = A^c, then B is an IFRβ^{**}GOS in (X, τ) with B $\neq 0_\sim$. Hence β^{**}gcl(B) = $1_\sim$, by hypothesis. This implies $(\beta^{**}gcl(B))^c = 0_\sim$. That is $\beta^{**}gint(B^c) = 0_\sim$. Hence β^{**}gint(A) = $0_\sim$.

(c) $\Rightarrow$ (d) Suppose A and B be two IFRβ^{**}GOSs in (X, τ) such that A $\neq 0_\sim \neq$ B, A $\subseteq B^c$. Since B^c is an IFRβ^{**}GCS in (X, τ) and B $\neq 0_\sim$ implies $B^c \neq 1_\sim$, $B^c = \beta^{**}gcl(\beta^{**}gint(B^c))$ and we have $\beta^{**}gint(B^c) = 0_\sim$. But A $\subseteq B^c$. Therefore $0_\sim \neq A = \beta^{**}gint(\beta^{**}gcl(A)) \subseteq \beta^{**}gint(\beta^{**}gcl(B^c)) = \beta^{**}gint(\beta^{**}gcl(\beta^{**}gcl(\beta^{**}gint(B^c)))) = \beta^{**}gint(\beta^{**}gcl(\beta^{**}gint(B^c))) = \beta^{**}gint(B^c) = 0_\sim$. A contradiction arises. Therefore (d) is true.

(d) $\Rightarrow$ (a) Suppose $0_\sim \neq A \neq 1_\sim$ be an IFRβ^{**}GOS in (X, τ). If we take B = $(\beta^{**}gcl(A))^c$, then B is an IFRβ^{**}GOS, since $\beta^{**}gint(\beta^{**}gcl(B)) = \beta^{**}gint(\beta^{**}gcl(\beta^{**}gcl(A))^c) = \beta^{**}gint(\beta^{**}gint(\beta^{**}gcl(A)))^c = \beta^{**}gint(A^c) = (\beta^{**}gcl(A))^c$

= B. Also we get $B \neq 0_{\sim}$, since otherwise, if $B = 0_{\sim}$, this implies $(\beta^{**}gcl(A))^c = 0_{\sim}$. That is $\beta^{**}gcl(A) = 1_{\sim}$. Hence $A = \beta^{**}gint(\beta^{**}gcl(A)) = \beta^{**}gint(1_{\sim}) = 1_{\sim}$, which is a contradiction. Therefore $B \neq 0_{\sim}$ and $A \subseteq B^c$. But this is a contradiction to (d). Therefore (X, τ) is an IFβ^{**}G super connected space.

(a) $\Rightarrow$ (e) Suppose A and B are any two IFRβ^{**}GOSs in (X, τ) such that $A \neq 0_{\sim} \neq B$, $B = (\beta^{**}gcl(A))^c$ and $A = (\beta^{**}gcl(B))^c$. Now we have $\beta^{**}gint(\beta^{**}gcl(A)) = \beta^{**}gint(B^c) = (\beta^{**}gcl(B))^c = A$, $A \neq 0_{\sim}$ and $A \neq 1_{\sim}$, since if $A = 1_{\sim}$, then $1_{\sim} = (\beta^{**}gcl(B))^c \Rightarrow (\beta^{**}gcl(B)) = 0_{\sim} \Rightarrow B = 0_{\sim}$. But $B \neq 0_{\sim}$. Therefore $A \neq 1_{\sim} \Rightarrow$ A is a proper IFRβ^{**}GOS in (X, τ) , which is a contradiction to (a). Hence (e) is true.

(e) $\Rightarrow$ (a) Suppose A is an IFRβ^{**}GOS in (X, τ) such that $0_{\sim} \neq A \neq 1_{\sim}$. Now take $B = (\beta^{**}gcl(A))^c$. In this case we get $B \neq 0_{\sim}$ and B is an IFRβ^{**}GOS in (X, τ), $B = (\beta^{**}gcl(A))^c$ and $(\beta^{**}gcl(B))^c = (\beta^{**}gcl(\beta^{**}gcl(A))^c)^c = \beta^{**}gint(\beta^{**}gcl(A)^c)^c = \beta^{**}gint(\beta^{**}gcl(A)) = A$. But this is a contradiction to (e). Therefore (X, τ) is an IFβ^{**}G super connected space.

(e) $\Rightarrow$ (f) Suppose A and B be two IFRβ^{**}GCSs in (X, τ) such that $A \neq 1_{\sim} \neq B$, $B = (\beta^{**}gint(A))^c$ and $A = (\beta^{**}gint(B))^c$. Taking $C = A^c$ and $D = B^c$, C and D become IFRβ^{**}GOSs in (X, τ) with $C \neq 0_{\sim} \neq D$, $D = (\beta^{**}gcl(C))^c = (\beta^{**}gcl(D))^c$, which is a contradiction to (e). Hence (f) is true.

(f) $\Rightarrow$ (e) It can be proved easily by the similar way as in (e) $\Rightarrow$ (f).