SOME IMPORTANT RESULTS ON THE RADICAL AND SEMI-SIMPLICITY FOR ARBITRARY RINGS

By

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INTRODUCTION

INTRODUCTION

The radical of an algebra is a very important concept. It is defined by different authors in different ways. The first definition was given by N. Jacobson as follows:

> "The radical R of a ring U is the intersection of all maximal left ideals of R".

Later the radical of a ring U that satisfies the descending chain condition was defined to be the join of the nilpotent right(left) ideals of U.

Other possibilities for defining a radical are afforded by an important characterization of the radical R of an algebra U with a finite basis. The definition due to Perlis, makes use of the notion of quasi regularity as follows:

> "The radical of a ring is the join of all quasi regular right ideals of the ring". [14]

In his paper "The Radical and Semi Simplicity for arbitrary rings" by N. Jacobson, he has considered the above definition of the radical by Perlis as the definition and proved the other definitions as theorems.

The aim of this thesis is to discuss the definition of radical by Perlis, the properties of radical and some important results on Radicals and semi simplicity from the above paper.

The first chapter deals with an exhaustive discussion of the definition of radical by Perlis.

In the Second chapter several properties of the radical are discussed. The important theorems in connection with these are as follows:

i) If $U^* = U + (1)$ where (1) is isomorphic to the ring of integers, then $R(U^*)=R(U)$

ii) If R is the radical of U, then $\overline{U} = U-R$ is semi-simple.

iii) Any regular ring is semi-simple.

iv) If U is a ring that satisfies the descending chain condition for right(left) ideals then the radical of U is nilpotent.

v) If U_n is the ring of $n \times n$ matrices with elements in U, then $R(U_n) = R_n$, R the radical of U.

In the third chapter, using the properties of the radicals discussed in Chapter II, it is proved that the definitions of radical by Perlis is the same as the definition of radical by N. Jacobson.

DEFINITIONS

DEFINITIONS

Definition: 1 Right quasi inverse of z:

An element z of an arbitrary ring U is right quasi regular if there exists an element z' in U such that $z+z'+zz' = 0$. The element z' satisfying this equation is called a right quasi inverse of z.

Definition: 2 Right quasi regular ring:

An element z of a ring U is right quasi regular if the totality of elements $\{x+zx\} = U$.

Definition: 3 Radical

The Radical of a ring is the join of all the quasi regular right ideals of the ring.

Definition:4 Nilpotent element:

An element $z \in U$ is a nipotent element of index n, then $z^n = 0$ but $z^{n-1} \neq 0$.

Definition: 5 Regular element:

U is regular if every element a of U has a relative inverse u such that $aua = a$.

Definition: 6 Descending chain condition:

If $N_1 \supseteq N_2 \supseteq \ldots\ldots.$ is a descending sequence of submodules, then there exists an integer n such that $N_n = N_{n+1} = \ldots\ldots\ldots$

Definition: 7 Minimum condition:

In any non-vacuous collection $\{N\}$ of submodules, there exists a minimum submodule, that is a submodule that does not contain properly any submodule of the collection.

Definition:8 Maximum condition:

Any non-vacuous condition of submodules contains a maximum submodule (one not contained properly in any other module of the collection).

Definition:9 Transfinite Powers:

If U is an arrbitrarry ring we define the transfinite powers U^{α} by the condition,

i) $U' = U$

ii) $U^{\alpha+1} = U^{\alpha}U$.

iii) If α is the limit ordinal U^{α} is the join of all U^{β} with $\beta < \alpha$.

Definition: 10 Algebra:

An associated ring A is called an algebra over F if A is a vector space over F such that for all a, b $\in$ A and $\alpha \in$ F,

$$\alpha (ab) = (\alpha a)b = a(\alpha b).$$

Definition: 11 Idempotent element:

An element $e \in R$ is called an idempotent if $e^2 = e$.

Definition: 12 Lattice:

A lattice (structure) is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound.

Definition: 13 Algebraic Algebra:

An element 'a' of an algebra U is algebraic if it satisfies a non-trival algebraic equation with coefficients in the underlying field ϕ. An equivalent condition is that 'a' generates a subalgebra A with a finite basis.

Definition: 14 Transcendentalelement:

As in the special case of a field an element which is not algebraic will be called transcendental

Definition: 15 **Normed ring:**

Suppose U is a normed ring, that is U is an algebra over the field of complex numbers and for each a in U there is defined a non-negative real norm $||a||$ such that

i) $\|a\| > 0$ if $a \neq 0$ $\|0\| = 0$

ii) $\|a + b\| \leq \|a\| + \|b\|$

iii) $\|a \alpha\| = \|a\| \, |\alpha|$ if $\alpha \in \phi$

iv) $\|ab\| \leq \|a\| \, \|b\|$

v) U has an identity and $\|1\| = 1$.

vi) U is completely relative to the metric $D(a,b) = \|a - b\|$

Definition: 16 **Semi-simple ring:**

If the radical of the ring is zero then the ring is called semi-simple.

Definition: 17 **Zorn's Maximum Principle:**

Let S be a non-empty inductively ordered set. Then there exists a maximal element in S.

CHAPTER: 1

CHAPTER : I

Let U be a ring with an identity and let z be an element of U such that $1+z$ has a right inverse u. We write $u = 1+z'$.

Then $(1+z)(1+z') = 1$

$\Rightarrow \quad 1+z'+z+zz' = 1$

$\Rightarrow \quad z + z' + zz' = 0 \qquad ..(1)$

Conversely if z is an element for which there exists an elements z' satisfying (1) then $1+z$ has the right inverse $1+z'$. This leads to the following definition.

Definition of a Right Quasi Inverse of z:

An element z of an arbitrary ring U is right quasi regular if there exists an element z' in U such that $z+z'+zz' = 0$. The element z' satisfying this equation is called a right quasi-inverse of z.

We have noted that if U has an identity then z is right quasi regular with right quasi inverse z' if and only if, $1+z$ has the right inverse $1+z'$.

Lemma: 1.1

An element z of a ring U is right quasi-regular if and only if the totality of elements $\{ x+zx \} = U$[2].

Proof:($\Rightarrow$)

If x is any element of U the totality $x+zx$ of elements $\{x+zx\}$ where x ranges over U is a right ideal. If z is right quasi-regular with right quasi inverse z' then $z+z'+zz' = 0$.

$\Rightarrow \quad -z = z' + zz' \in \{x + zx\}$ (Since $z' \in U$)

that is $-z \in \{x + zx\}$

$\Rightarrow \quad z \in \{x + zx\}$

$\Rightarrow \quad zx \in \{x + zx\}$ and $x \in \{x + zx\}$

Then $\{x + zx\} = U$

(Since $x \in \{x+zx\}$ and $zx \in \{x+zx\}$ and the totality $\{x+zx\}$ of elements $x+zx$ where x ranges over U is right ideal).

$(\Leftarrow)$

On the other rhand if $x+zx = U$ then

$-z = z' + zz'$ forr suitable z'.

$\Rightarrow \quad z+z'+zz' = 0$

$\Rightarrow \quad z$ is right quasi regular.

Hence the lemma.

The definition of quasi regularity due to Bear is definition[2].

A right ideal τ will be called quasi-regular if all the elements of τ are rightt quasi regular. Let τ_1 and τ_2 be quasi-regular right ideals and let $z_1 \in \tau_1$ and $z_2 \in \tau_2$

Lemma: 1.2

$\tau_1 + \tau_2$ is quasi regular.

Proof:

Let τ_1 and τ_2 be quasi regular right ideals and let $z_1 \in \tau_1$, $z_2 \in \tau_2$ we have to prroduce a right quasi invese to $z_1 + z_2$.

By the definition there exists an element z_1' such that $z_1 + z_1' + z_1 z_1' = 0$.

Since $z_2 + z_2 z_1' \in \tau_2$ this element has a right quasi-inverrse w' such that

$$(z_2+z_2z_1') + w' + (z_2+z_2z_1')w' = 0 \qquad ..(2)$$

(by the definition)

Hence consider

$$(z_1+z_2) + (z_1'+w'+z_1'w') + (z_1+z_2)(z_1'+w'+z_1'w')$$

$$\Rightarrow \quad z_1+z_2+z_1'+w'+z_1'w'+z_1z_1'+z_1w'+z_1z_1'w' + z_2z_1'+z_2w'+z_2z_1'w'$$

and collecting the terms as follows.

$$= \quad (z_1+z_1'+z_1z_1') + [(z_2+z_2z_1') + w' + (z_2+z_2z_1')w'] + [(z_1+z_1'+z_1z_1')w']$$

$$= \quad 0+0+0$$

$$= \quad 0.$$

(since $z_1+z_1'+z_1z_1' = 0$ and
$[(z_1+z_2z_1') + w' + (z_2+z_2z_1')w'] = 0.$
by (2)].

$$\therefore \quad (z_1+z_2) + (z_1'+w'+z_1'w') + (z_1+z_2)(z_1'+w'+z_1'w') = 0.$$

Thus $z_1'+w'+z_1'w$; is a right quasi inverse of z_1+z_2.

Hence $T_1 + T_2$ is quasi-regular.

Theorem: 1.1

If U is an arbitrary ring the join R of all the quasi-regular right ideals of U is a (right) quasi-regular two-sided ideal.

Proof:

Let R be the join of all the quasi-regular right ideals of U. Since the right ideal generated by an element z is the totality of elements $zi + za$ where i is an integer and $a \in U$, it is clear that R is the totality of elements z such hat $zi + za$ is right quasi-regular for all integral i and all $a \in U$. The above result shows that R is a right ideal. Let $z \in R$ and $b \in U$. Then $zb \in R$, then there exists an element w' such that

$$zb + w' + (zb)w' = 0 \quad \text{(by definition)}$$

Then considerr

$$
\begin{aligned}
& bz + (-bz-bw'z) + bz\,(-bz-bw'z) \\
= \quad & bz - bz - bw'z - bzbz - bzbw'z \\
= \quad & -b\,(w' + zb + zbw')z \\
= \quad & -b(0)z \\
= \quad & 0
\end{aligned}
$$

(Since $w' + zb + zbw' = 0$).

and so $-(bz + bw'z)$ is a right quasi-inverse for bz. Similarly if i is an integar and $a \in U$,

$(bz)i + (bz)a = b\,(zi + za)$ is right quasi-regular. Hence $bz \in R$.

We have proved that if U is a arbitrary ring the join R of all the quasi-regular right ideals of U is a (right) quasi-regular two-sided ideal. Hence the theorem.

Now we shall define the radical of a ring as follows.

The radical of a ring is the join of all the quasi-regular right ideals of the ring.

WE know that R is the set of elements z such that zi + za is right quasi-regular for all integers i and all a ϵ U. If U is a ring with an identity the connection between quasi-regularity and regularity shows that R is the totality of elements z such that 1+za has a right inverse for every a$\in$U.

We define the left quasi-regularity left quasi-inverse, quasi-regular left ideal and left radical R' in a manner analogous to the above. As for ordinary inverses we say that an element is quasi-regular if it is both right and left quasi-regular.

<u>Lemma: 1.3</u>

If z is quasi-regular any right(left) quasi-inverse of z is a left(right) quasi-inverse of it and is uniquely determined and commutes with z.

To prove: $z'' = z'$

Consider $z'' = z'' + 0 + z''(0)$

$= z'' + (z+z'+zz') + z''(z+z'+zz')$

$= z' + (z+z''+z''z) + (z+z''+z''z)z'$

$= z' + 0 + 0.$

$= z'$ (Since $z + z'' + z''z = 0$).

This proves that any right quasi-inverse of a quasi-regular element is also a left quasi-inverse.

Since $z+z'+zz' = 0 = z+z''+z''z = z+z'+z'z$

(Since $z' = z''$)

$\Rightarrow z+z'+zz' = z + z' + z'z$

$\Rightarrow zz' = z'z$

z' commutes with z. We call z' the quasi-inverse of z.

Hence the lemma.

Theorem: 1.2

The radical of a ring U is the join of all the quasi-regular left ideal of U.

Proof:

Now let $z \in R$. Then z has a right quasi-inverse $z' = -z-zz'$. Since R is an ideal $z' \in R$. Hence z' has a right quasi-inverse. Since z is a left quasi-inverse of z' it follows by lemma (1.3) that z' is quasi-regular and z is its quasi-inverse. Hence z is quasi-regular, since R is a left ideal $R \subseteq$ the left radical R'. By symmetry $R' \subseteq R$. Therefore this proves the radical of a ring U is the join of all the quasi-regular left ideals of U.

CHAPTE: 2

CHAPTER : II

PROPERTIES OF RADICALS:

If U is an arbitrary ring, we know that we can imbed U in a ring with an identity U^* such that $U^* = U+(1)$, $U \cap (1) = 0$, where (1), the ring generated by 1, is isomorphic to the ring of integers[1].

Theorem: 2.1

Let U be an arbitrary ring and let U^* be a ring with an identity containing U such that $U^* = U+(1)$. Then the radical $R(U) = R(U^*) \cap U$. If in addition $U \cap (1) = 0$ and (1) is isomorphic to the ring of integers then $R(U) = R(U^*)$.

Proof:

First we assume that $U^* = U+(1)$. Let $R(U^*)$ be the radical of U^*.

Claim: 1

$$R((U^*) \cap U = R(U)$$

Let z be an element of $R(U^*) \cap U$. Since $z \in R(U^*)$ the radical of U^*, z has a quasi-inverse z' in U^* (by the definition of radical)

that is $z+z'+zz' = 0$

$\Rightarrow \quad z' = -z - zz'$

$\Rightarrow \quad z' \in U$ (Since $z \in U$)

Hence z has a quasi-regular inverse z' in U, that is z in quasi regular in U.

that is $z \in R(U)$.

Therefore $R(U^*) \cap U \subseteq R(U)$ ..(1)

We know that $U^* = U+(1)$. Any right ideal of U is also a right ideal of U*.

Therefore $R(U) \subseteq R(U^*\_)$

$\Rightarrow$ $R(U) \subset R(U^*) \cap U$ ..(2)

From (1) and (2)

$R(U^*) \cap U = R(U)$

Claim: 2

$$R(U) = R(U^*) \text{ if } U \cap (1) = 0.$$

We assume that $U \cap (1) = 0$ and (1) is isomorphic to the ring of integers. Let $z^* \in R(U^*)$, then the coset $\bar{z}^*$ of z^* in the difference ring U*-U is in the radical of the difference ring.

But the radical of ring of integers is zero for let $n \neq 0$ and $n \in R(1)$. If there exists n' (1), such that

$n + n' + nn' = 0.$

$\Rightarrow$ $n + n'(1+n) = 0$

$\Rightarrow$ $n' = \dfrac{-n}{1+n} \notin (1)$ a contracdiction.

Therefore $R(1) = 0$

That is R(1) is 0 and hence $\bar{z}^* = 0$ and therefore $z^* \in U$. (By the definition of cosets).

Hence $U \subseteq R(U^*)$

Hence from the claim (1)

$R(U) = R(U^*) \cap U$. **We obtained that $R(\dot{U}) = (U^*)$.**

Hence the theorem.

Theorem: 2.2

If R is a radical of U, $\bar{U} =$ U-R is semi-simple.

Proof:

Let $\bar{z}$ be an element of the radical of $\bar{U}$, that is $R(\bar{U})$ where $\bar{U} = U - R$, let z be an element in the coset $\bar{z}R(\bar{U})$. Then there exists an element z' such that $z+z'+zz' = u \in R$.

Since $u \in R$ there exists u' such that $u + u' + uu' = 0$.

Hence $0 = (z+z'+zz')+u+(z+z'+zz')u'$

$= z+(z'+u'+z'u') + z(z'+u'+z'u')$

$\therefore$ z is a right quasi-regular.

Now consider all such z elements in the cosets $\bar{z}$ of $R(\bar{U})$ that will form an ideal which is quasi-regular.

Hence z R and therefore $\bar{z} = 0$. Hence $R(\bar{U})=0$. Hence $\bar{U} = U - R$ is semi-simple (By the definition of semi-simple ring).

Theorem: 2.3

The radical of a ring contains every nil right (left) ideal of the ring.

Proof:

Assume that z is a nil potent element of index n of a ring of a ring U. Consider the element

$$
\begin{aligned}
z' &= \sum_{i=1}^{n-1} (-i)^i z^i \\
&= (-1)z+(-1)^2 z^2+\ldots\ldots\ldots+(-1)^{n-1} z^{n-1} \\
&= -z+z^2-z^3+\ldots\ldots\ldots+(-1)^{n-1} z^{n-1}
\end{aligned}
$$

Claim: z' is a quasi-inverse of z.

$$
\begin{aligned}
z+z'+zz' &= z+(-z+z^2-z^3+\ldots\ldots+(-1)^{n-1}z^{n-1}) \\
&\quad + z(-z+z^2-z^3+\ldots\ldots+(-1)^{n-1}z^{n-1}) \\
&= z-z+z^2-z^3+z^4+\ldots\ldots\ldots+(-1)^{n-1}z^{n-1}-z^2 \\
&\quad + z^3-z^4+\ldots\ldots\ldots+(-1)^{n-1}z^n \\
&= (-1)^{n-1}z^n \\
&= 0 \text{ (Since } z^n=0 \text{ by the definition of nil potent elements of index } n\text{)}
\end{aligned}
$$

z' is a right quasi-inverse of z.
Similary z' can be proved to be the left quasi-inverseof z. Hence the claim.

Hence every nil right(left) ideal of the ring is contained in the radical of the ring.

COROLLARY:

If z is an element such that $U_zU \subseteq R$ then $z \in R$.

Proof:

For if τ is the right ideal generated by z, $\tau \subseteq U_zU \subseteq R$.

Cosider $\bar{\tau} = (\tau + R) - R$

It is nil potent in the semi-simple ring $\bar{u} = u-R$.
By the definition semi-simplicity $R(U) = (0)$.

By theorem 2.3 $\bar{\tau} \subseteq R(U) = 0$.

$\Rightarrow \bar{\tau} = 0$

$\Rightarrow \tau \subseteq R$

Hence $z \in \tau \subseteq R$

$\Rightarrow z \in R$

Hence the corollary.

This corollary implies that $z \in R$ if and only if $za(az)$ is right(left) quasi-regular for all a in U.

By theorem 2.3 we know that the nil potent elements are the elements of Radical. But the converse need not be true. The elements of the radical need not be nil potent. This can be seen by the example.

U be the ring of P-adic integers. The radical of this ring is the intersection of the maximum right ideals of U. Hence R = PU. PU has no nil potent element.

Theorem: 2.4

If N is a subring of R and $z \in N$ then for any positive integer h either $z^{h-1}N \supset z^hN$ or $z^h = 0$.

Proof:

Evidently $z^{h-1} N \supseteq z^hN$. Suppose that $z^{h-1}N = z^hN$ then $z^h = z^hy$ forr some y in N. Let y' be quasi-inverse of -y. Then by the definition of quasi-inverse we know that

$$(-y) + y' + (-y)y' = 0$$

By our assumption $z^h = z^hy$

$\Rightarrow \quad z^h - z^h y = 0$ ..(1)

that is $z^h y' = z^h yy'$

$\Rightarrow \quad z^h y' - z^h yy' = 0$ ..(2)

(1) + (2) we get

$$z^h - z^h y + z^h y' - z^h yy' = 0$$

$$\Rightarrow \quad z^h + z^h(-y + y' + (-y)y') = 0$$

$$\Rightarrow \quad z^h + z^h(0) = 0 \quad (\text{Since } -y+y'+(-y)y'=0)$$

$$\Rightarrow \quad z^h = 0$$

Hence the theorem.

Theorem: 2.5

If U is a ring with an identity whose lattice of right(left) ideals is completely reducible, then U is semi-simple.

Proof:

Theorem 2.4 implies that the radical contains no idempotent element $\neq 0$. It is known that if U is a ring with an identity whose lattice of right ideals is completely reducible then every right ideal τ in U has the form eU, where e is an idempotent element in τ [ ]. Hence if U is a ring with an identity whose lattice of right (left) ideals is completely reducible, then U is semi-simple, because a semi-simple rings radical is 0.

Hence the theorem.

Theorem: 2.6

Any regular ring is semi-simple.

Proof:

By the definition, "U is regular if every element a of U has a relative inverse u such that aua = a".

Suppose that $a \in R$, the radical then $-ua$ has a quasi-inverse ϑ such that $-ua + \vartheta + (-ua) = 0$ ..(1)

$\Rightarrow \quad -aua + a\vartheta - aua\vartheta = 0$

$\Rightarrow \quad -a + a\vartheta - a\vartheta = 0$ (Since $aua = a$)

$\Rightarrow \quad -a = 0$

$\Rightarrow \quad -a = 0$

$\Rightarrow \quad a = 0$

$\Rightarrow \quad R = 0$

$\therefore$ Any regular ring is semi-simple.

The Radical of of a Ring Satisfying the Descending Chain Condition:

Theorem: 2.7

If U is a ring that satisfies the descending chain condition for right(left) ideals, then the radical of U is nil potent.

Proof:

We suppose now that U is a ring for which the descending chain condition for right(left) ideals holds. Let N be a two-sided ideal contained in R, the radical of U, and suppose that $N^2 = N$. Then $N \neq 0$ there exists a minimum right ideal α of U with the properties.

i) $\tau \subseteq N$ (ii) $\tau N \neq 0$.

Let b be an element of τ. Such that $bN \neq 0$.

Then

$$(bN)\, N = bN^2$$
$$= bN \quad 0(\text{Since } N^2 = N)$$

and since $bN \subseteq \Upsilon$, by the definition of minimality of τ. $\tau \subseteq bN$

We have $bn = \tau$

Since $b \in \tau$ there exists an element $y \in N$ such that $by = b$.

$$\Rightarrow \quad by - b = 0$$
$$\Rightarrow \quad b(y-1) = 0.$$

But $y \neq 1$. Since if $y = 1$, it is idempotent $y = 1 = 1^2$ and theorem 2.4 the radical cannot have non-zero idempotent element implies y cannot be in R and hence cannot be in N.

Hence $y \neq 1$. $\therefore b = 0$.

Contrary to $bN \neq 0$. Thus $N = 0$. Now the positive integral power R^k of R are two-sided ideals and $R \supseteq R^2 \supseteq \ldots\ldots..$

Hence there is an ρ such that $R^{\rho} = R^{\rho+1}$

(Because of the minimality condition)

Then $N = R^{\rho}$

$$\Rightarrow \quad N^2 = N.N = R^{\rho}.R^{\rho}$$

$$= R^{2\rho}$$

$$= R^{\rho} = N$$

Hence $N = R^{\rho} = 0$

This proves the theorem.

Note:

Since any nil ideal is contained in the radical this proves that any nil ideal in a ring that satisfies the descending chain condition for right (left) ideals is nil potent. It is clear also that R coincides with the usual radical defined as the join of all nil potent ideals.

Finitely Generated Ideals Contained in R:

Theorem: 2.8

If N is a right ideal with a finite basis contained in the radical R then either $NR \subset N$ or $N = 0$.

Proof:

Since N is a right ideal either NR⊂N or NR = N. Assume that NR = N.

Let $y_1, y_2 \ldots\ldots\ldots\ldots y_n$ be a basis for N. Then every element of N has the form $\Sigma y_i a_i + \Sigma y_i j_i$ where $a_i \in U$ and the j_i are integers. Since NR=N every element of N also has the form $\Sigma y_i z_i$, z_i in R.

In particular $y_1 = \Sigma y_i z_i$. Let z_1' be the quasi-inverse of $-z_1$ then $-z + z_1' + (-z_1)z_1' = 0$.

Then consider

$$y_1 = y_1 + (-z_1 + z_1' + (-z_1)z_1')$$

(Since $-z_1 + z_1' + (-z_1{}')z_1' = 0$).

$$= (y_1 - y_1 z_1) + (y_1 - y_1 z_1) z_1'$$

$$= (\Sigma y_i z_i - \Sigma y_i z_i z_1) + (\Sigma y_i z_i - \Sigma y_i z_i z_1)z_1'$$ (Since put $y_1 = \Sigma y_i z_i$)

$$= (\Sigma y_i z_i + \Sigma y_i z_i z_1') - (\Sigma y_i z_i z_1 + \Sigma y_i z_i z_1 z_1')$$

$$= [(y_1 z_1 + y_1 z_1 z_1') + (y_2 z_2 + y_2 z_2 z_1') + \ldots\ldots\ldots + (y_n z_n + y_n z_n z_1')] - (y_1 z_1 + y_1 z_1 z_1')$$ (Put $\Sigma y_i z_i = y_1$)

$$= (y_2 z_2 + y_2 z_2 z_1') + \ldots\ldots + (y_n z_n + y_n z_n z_1')$$

$$= y_2 (z_2 + z_2 z_1') + \ldots\ldots\ldots + y_n (z_n + z_n z_1')$$

Hence y_1 can be eleminated from the basis. Similarly every y_i can be eliminated and so $N=0$.

Definition:

If U is an arbitrary ring we define the transfinite powers U^α by the conditions.

i) $U' = U$ ii) $U^{\alpha+1} = U^\alpha U$

iii) If α is the limit ordinal U^α is the join of all U^β with $\beta < \alpha$.

There exists a least ordinal ρ such that $U^\rho = U^{\rho+1}$. We shall call ρ the index of U and we shall say that U is transfinite nil potent if $U^\rho = 0$.

Theorem: 2.9

The radical of a ring that satisfies the ascending chain condition for right ideals is transfinite nil potent.

Proof:

Now suppose that U is a ring that satisfies the ascending chain condition for right ideals. We recall that this condition is equivalent to the requirement that every ideal has a finite basis.

Let $N = R^{\rho}$ where ρ is the index of the radical N of U. Then $NR = N$ and so by theorem 2.8, $N = R^{\rho} = 0$.

The Radical of a Matrix Ring:

If U is an arbitrary ring we denote as usual the ring of n x n matrices with elements in U by U_n. If B is a subring (ideal) in U then B_n is a subring (ideal) in U_n. Let R be the radical of U. Then we wish to show that R_n is the radical of U_n denoted by $R(U_n)$.

Lemma: 2.1

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Any matrix $z\ (z_{ij})$ of U_n in which z_{11} is right quasi-regular and the $z_{ij} = 0$ for $i > 1$ is right quasi regular in U_n.

Proof:

Consider
$$z = \begin{bmatrix} z_1 & z_{12} & \cdots & z_{1n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

z_{11} is given to be a right quasi regular element. Then by lemma 1.1 the ideal, $\{x + z_{11}x\} = 0$. Hence there exists z_{1i} such that $z_{1i} + (z_{1i}' + z_{11} + z_{1i}') = 0$

$$\Rightarrow \quad z_{1i}' + z_{11} z_{1i}' = - z_{1i}$$

Consider $z_{ij}' = \forall\ i > 1$

let $z' = \begin{bmatrix} z_{11}' & z_{12}' & \ldots & z_{1n}' \\ 0 & 0 & \ldots & 0 \\ \cdot & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$

Then we can prove z' is the right quasi inverse of z.

that is, $z + z' + zz'$

$$= \begin{bmatrix} z_{11} & z_{12} & \ldots & z_{1n} \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} z'_{11} & z'_{12} & \ldots & z'_{1n} \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} z_{11} & z_{12} & \ldots & z_{1n} \\ 0 & 0 & & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} z'_{11} & z'_{12} & \ldots & z'_{1n} \\ 0 & 0 & & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_{11} + z'_{11} + z_{11}z'_{11} & z_{12}+z'_{12}+z_{11}z'_{12} \cdots & z_{1n}+z'_{1n}+z_{11}z'_{1n} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$\therefore$ z' is the right quasi inverse of z.

Theorem: 2.10

If U is an arbitrary ring and U_n is the ring of n x n matrices with elements in U, then the radical $R(U_n) = Rn$, R the radical of U.

Proof:

Now consider the totality τ_1 of matrices with first row consisting of elements in R and other rows 0. Then τ_1 is a right ideal. Hence by the above lemma $\tau_1 \subseteq R(U_n)$.

Similarly the totality τ_j of matrices with jth row consisting of elements of R and other rows 0 is contained in $R(U_n)$.

Since $R_n = \tau_1 + \tau_2 + \ldots\ldots. + \tau_n$.

$R_n \subseteq R(U_n)$

Conversely let $y = (y_{ij}) \in R(U_n)$. If 'a' is any element of U we denote the matrix that has a in the (i,j) position and 0 elsewhere by A_{ij}. Let a and b be arbitrary and form the matrix $D = \Sigma_k A_{kp} Y B_{qk}$. Then D is the diagonal matrix$\{ ay_{pq}b, \ldots\ldots. ay_{pq}b\}$ and $D \in R(U_n)$. If $D' = (d'_{ij})$ is a right quasi inverse of D it is easy to see that $d' - d'_{11}$ is a right quasi-inverse of $d - ay_{pq}b$. Evidently this implies for arbitrary C in U and arbitrary integral i, dc+di is right quasi regular. Hence $d \in R$. Since a and b are arbitrary the corollary to theorem 2.3, "If z is an element such that $U_z U \subseteq R$ then $z \in R$" shows that $y_{pq} \in R$ and $Y \in R_n$.

The Radical of an Algebra:

Let U be an algebra of poissibly infinite order over a field ϕ. By an ideal in the algebra U we mean, of course, an ideal of the ring U that is invarient under the scalar multiplications. $x \to x\alpha$, α in ϕ. If U has an identity,

$$x\alpha = (x)(1\alpha) = (1\alpha)(x)$$

and so any ideal of the ring U is an ideal of the algebra U.

The above discussion is valid without change for an arbitrary algebra U. Thus the radical R of U can be defined to be join of all the quasi-regular right ideals of U. R is also the maximum quasi-regular left ideal of U. All of our theorems hold for algebras.

Theorem: 2.11

If U is an algebra over a field the elements of the radical R of U are either nil-potent or transcendental over ϕ .

Proof:

An element 'a' of an algebra U is algebraic if it satisfies a non-trivial algebraic equation with coefficients in the underlying field ϕ . An equivalent condition is that 'a' generates a subalgebra. A with a finite basis. As in the special case of field, an element which is not algebraic will be called transcendental. If every element of U is algebraic, then U is algebraic. If 'a' is an algebraic element and A is the subalgebra generated by 'a' then there exists a positive integer h such that $a^{h-1}A = a^hA$. Hence if 'a' is in the radiccal, by theorem 2.4 "if N is a subring of R and $z \in N$, then for any positive integer h either $z^{h-1}N$ z^hN or $z^h = 0$". 'a' is nilpotent

Hence the theorem.

Theorem: 2.12

The radical of an algebraic algebra U is the join of all nil right(left) ideals of U.

Proof:

If U is algebraic every z in R is nilpotent. Hence since R cocntains every nil ideal, we have that the radical of an algebraic algebra U is the join of all nil right(left) ideals of U.

Corollary:

If U is commutative algebraic algebra, the radical of U is the totality of its nilpotent elements.

Proof:

If U is commutative an element z generates a nil ideal if and only if it is nilpotent.

Hence the corollary.

The Radical of a Normed Ring:

We suppose now that U is a normed ring, that is U is an algebra over the field ϕ of complex numbers and for each a in U there is defined a non negative real norm $\|a\|$

such that

i) $\|a\| > 0$ if $a \neq 0$ $\|0\| = 0$.

ii) $\|a+b\| \leq \|a\| + \|b\|$

iii) $\|a\alpha\| = \|a\| \, |\alpha|$ if $\alpha \in \phi$

iv) $\|ab\| \leq \|a\| \, \|b\|$.

v) U has an identity and $\|1\| = 1$.

vi) U is completely relative to the metric $D(a,b) = \|a-b\|$

Following Gelfand [5] we call an element of U a generalized nilpotent element if $\lim \|z^n\|^{1/n} = 0$ [5]. For commutative normed rings Gelfands has defined the radical to be the totality of generalized nilpotent element. We shall show that this set coincides with the radical as defined here we shall obtain a similar characterization of the radical for non-commutative normed rings.

Theorem: 2.13

The radical of a normed ring U is the totality of elements such that $(za)^n \to 0$ $[(az)^n \to 0]$ for every a in U.

Proof:

Since U has an identity R is the totality of elements z such that 1+za has an inverse for every a in U. Now suppose that z is an element such that $(za)^n \to 0$ for every a. Then for any α in ϕ, $\|(z\alpha)^n\| < 1$ for n sufficiently large.

Thus $\| z^n \| \beta^n$ where $\beta = \frac{1}{|\alpha|}$. We choose α so that $| \alpha | > 1$. The series $1-z+z^2-\ldots\ldots.$ is ultimately dominated by the convergent series $1+\beta+\beta^2+\ldots.$ Hence $1-z+z^2-\ldots\ldots.$ exists in U and this elemennt is the inverse of $1+z$.

that is,

$$(1+z)(1-z+z^2-z^3+\ldots\ldots)$$

$$= 1-z+z^2-z^3+\ldots\ldots+z-z^2+z^3+\ldots\ldots$$

$$= 1+0+0+\ldots\ldots = 1.$$

Similarly we can show that if $z' = za$, then $1+z'$ has an inverse. Hence $z \varepsilon R$.

Conversely suppose that $z \varepsilon R$ then $1+z\alpha$ has an inverse $(1+z\alpha)^{-1}$ for every α in ϕ. Using the fact that $(1+z\alpha)^{-1}$ is an analytic function of α, we may prove exactly as Gelfand has done in the commutative case, that $(z\alpha)^n \to 0$[5]. In particular $z^n \to 0$ and since R is an ideal $(za)^n \to 0$ for every a.

Theorem: 2.14

The radical of a normed ring is a generalized nil ideal that contains every generalized nil right (left) ideal of the ring.

Proof:

We shall call an ideal a generalized nil ideal if all of its elements are generalized nilpotent elements.

If $z \in R$, $(z\alpha)^n \to 0$ then for n sufficiently large.

$\| (z\alpha)^n \| < 1$ and $\| z^h \| < \beta^n$ for $\beta = \frac{1}{|\alpha|}$

Hence $0 \leq \| z^n \|^{1/n} < \beta$. Since β is arbitrary $\lim \| z^n \|^{1/n} = 0$. Thus z^n is a generalized nilpotent element and R is a generalized nil ideal. Then if $y \in \tau$ $\| y^n \|^{1/n} \to 0$. Hence $y^n \to 0$. Since τ is a right ideal, $y' = ya \in \tau$ and $(y')^n \to 0$ by theorem 2.13, $y \in R$ and so $\subseteq$ R. Hence the theorem

Let U be commutative. Then if z and $a \in U$.

$$\begin{aligned} \| (za)^n \| &= \| z^n a^n \| \\ &\leq \| z^n \| \; \| a^n \| \\ &\leq \| z^n \| \; \| a \|^n \end{aligned}$$

Hence $\| (za)^n \|^{1/n} \leq \| z^n \|^{1/n} \; \| a \|$

and if z is a generalized nilpotent element then za is a generalized nilpotent element. Thus any generalized nilpotent element generates a generalized nil ideal and N is the totality of generalized nilpotent elements.

Quotient ideals:

We return to the consideration of an arbitrary ring U. The results that we obtain are also valid for algebras but we shall not state them explicitly for these.

Let τ be a right ideal in U. Then if $a \in U$ the right multiplication $x \to xa$ determined by a induces an endomorphism $\bar{a}$ in the difference group $M = U - \tau$. The mapping $\bar{a}$ sends the coset $x + \tau$ into $xa + \tau$. The totality of elements $\bar{a}$ is a subring $\bar{U}$ of the ring of endomorphisms of M, and the correspondence $a \to \bar{a}$ is a homomorphism between U and $\bar{U}$. The Kernal of this homomorphism is a two-sided ideal $\tau : U$ which we shall call the quotient of τ relative to U. Evidently $U(\tau : U) \subseteq \tau$, and if U has an identity, $(\tau : U)$ is the largest two sided ideal of U contained in τ. By the fundamental theorem on the homomorphisms $\bar{U} \cong U - (\tau : U)$

Let τ be maximal, that is $U \supset \tau$ and there is no right ideal τ' such that $U \supset \tau' \supset \tau$.

Then $M = U - \tau$ is irreducible relative to $\bar{U}$. As usual we call a ring of endomorphisms $\bar{U}$ ireducible if the group M in which U acts is irreducible. Let $\bar{U} \neq U$, have this property. Then the totality B of elements z in M such that $Z\bar{U} = 0$ is a subgroup of M invarient under U. Hence either $B=0$ or $B=M$. Since $\hat{U} \neq 0$, $B \neq M$ and so $B=0$.

It follows that if x is any element $\neq 0$ of M then $\overline{U} \neq 0$. Since xU is a subgroup invarient under $\overline{U}$, $x\overline{U} = M$.

Theorem: 2.15

Any irreducible ring of endomorphisms is semi-simple.

Proof:

We can use the above facts. Let $\bar{z}$ be an element of the radical U and let $x \neq 0$ be arbitrary in M. If $x\bar{z} \neq 0$, $(x\bar{z})\overline{U} = M$.

Hence there is an a in U such that $x\bar{z}\bar{a} = x$. the element $-\bar{z}\bar{a}$ has a quasi-inverse $\bar{z}'$.

Hence

$$\begin{aligned} x &= x - (x\bar{z}\bar{a} - x\bar{z}' + x\bar{z}\bar{a}\bar{z}') \\ &= x - x\bar{z}\bar{a} + (x - x\bar{z}\bar{a})\bar{z}' \\ &= 0. \end{aligned}$$

This contradiction show that $x\bar{z} = 0$ for all x. Thus $\bar{z} = 0$ and $\overline{U}$ is semi-simple.

Corollary:

If τ is a maximal right ideal $(\tau : U)$ contains the radical R of U.

If $(\tau : U) = U$ there is nothing to prove. Hence suppose that $(\tau : U) \neq U$. Then $\bar{U} \cong U-(\tau : U)$ is an irreducible ring of endomorphisms $\neq 0$. If $z \in R$ the coset $\bar{z} = z + (\tau : U)$ is in the radical of $U - (\tau : U)$. Since $\bar{U}$ is semi-simple $z = 0$. Hence $\bar{z} \in (\tau : U)$ and $R \subseteq (\tau : U)$.

Hence the corollary.

CHAPTER : 3

CHAPPTER: 3

The radical as intersection of maximal right ideals:

Lemma: 3.1

If a is an element of U that is not right quasi-regular the right ideal $\{x + ax\}$ can be imbedded in a maximal right ideal.

Proof:

Let U be any ring that is not a radical ring. Then U contains an element a which is not right quasi regular. Hence the right ideal $\{x + ax\}$ does not contain a . Moreover if is a right ideal that contains a and contains the ideal $\{x + ax\}$ then $\tau = U$. By using Zorn's maximum principle derive the result. Hence the lemma.

Theorem: 3.1

If U is a ring that contains maximal right ideals and $\pi\tau$ is the intersection of these maximal right ideals, then $\subseteq R$ and $UR \subseteq \pi\tau$.

Proof:

The lemma 3.1 shows that any ring that is not a radical ring contains maximal right ideals. We assume now that U is any ring that contains maximal right ideals. Consider the intersection of maximal right ideals $\Pi\tau$ of U. Let $y \in \Pi\tau$. Then y is right quasi-regular. For otherwise $\{x + yx\}$ can be imbedded in the maximal right ideal τ. Then $y \in \tau$ and $\tau = U$ contary to the maximality of τ. Thus every element of $\Pi\tau$ is quasi-regular and since $\Pi\tau$ is a right ideal, $\Pi\tau \subseteq R$. On the other hand by the corollary to theorem 2.15, $R \subseteq (\tau : U)$. Hence $URR \subseteq U(\tau : U)$ and $UR \subseteq \Pi\tau$. Hence the theorem.

Corollary: 1

If U is not a radical ring and is the intersection of the maximal right ideals of U then $\Pi\tau \subseteq R$ and $UR \subseteq \Pi\tau$.

Corollary: 2

If U is a ring with an identity the radical of U is the intersection of the maximal right ideals of U.

For UR = . Hence $R \subseteq \Pi\tau$ as well as $\Pi\tau \subseteq R$.

The following results also are consequences of Theorem: 3.1.

If U is a ring that contains maximal right ideals then is a two sided ideal. It is known that any maximal ideal is closed[5]. Hence $R = \Pi\tau$ is closed.

Proof:

If U is a semi-simple rig, $\Pi\tau = 0$. Suppose, in addition, that U satisfies the descending chain condition forr right ideals τ_j, j = 1,2 ......n such that $\Pi\tau_j = 0$.

We may suppose that the set $\{\tau_j\}$ is minimal in the sense that $\prod_{k \neq i} \tau_k = \tau'_k \neq \tau'_i \neq 0$ for every i. Then $\tau_i \cap \tau'_i = 0$ and $\tau'_i = \tau_i$. Since τ_i is maximal it follows that $U = \tau_i + \tau_i'$. Hence τ'_i is isomorphic to the difference U-goup $U - \tau_i$ add τ_i' is minimal. Using a simple lattice theoretic argument we can conclude that $U = \tau_i' \oplus \ldots\ldots \oplus \tau_n'$ [6].

Hence the theorem.

Theorem: 3.3

Let U be an arbitrary ring that contains maximal right ideals. Then the radical of U is the intersection $\Pi(\tau : U)$ where ranges over the maximal right ideals of U.

Proof:

Suppose again that U is any ring that contains maximal right ideals. Then, by the corollary to the theorem 2.15,

$R \subseteq \Pi(\tau : U)$ for all maximal τ.

Conversely let $y \in \Pi(\tau : U)$. Then $Uy \subseteq \Pi\tau \subseteq R$. Hence $UyU \subseteq R$ and this implies that $y \in R$. Hence the theorem.

Corollary:

If U is not a radical ring, $R = \Pi(\tau : U)$ where τ ranges over the maximal right ideals.

The results of this section hold also for left ideals. An interesting consequence of the second corollary to theorem 3.1 is that if U is a ring with an identity then the intersection of the maximal right ideals of U coincides with the intersection of the maximal left ideals of U.

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