



# *Chapter I*

# CHAPTER I

## PRELIMINARY DEFINITIONS AND RESULTS

### Definition: 1.1

Let  $X$  be a non-empty set. A relation  $\leq$  on  $X$  is said to be a **pre-order** or **quasi-order** if it is reflexive and transitive. An anti symmetric pre-order is called a **partial order**.

### Definition: 1.2

Every set can be considered as a partially ordered set equipped with the **discrete order**  $x \leq y$  if and only if  $x = y$ .

### Definition: 1.3

Let  $X$  be a non-empty set. A **fuzzy set** in  $X$  is a function with domain  $X$  and values in the closed unit interval  $I = [0, 1]$ .  $I^X$  denote the set of all fuzzy sets in  $X$ .

### Definition: 1.4

Let  $\mu \in I^X$ . The subset of  $X$  in which  $\mu$  assumes nonzero values is known as the **support of  $\mu$** .

### Definition: 1.5

Let  $\mu, \rho \in I^X$ . We define the following fuzzy sets.

- (1)  $\mu \leq \rho \Rightarrow \mu(x) \leq \rho(x) \quad \forall x \in X$
- (2)  $\mu \wedge \rho \in I^X$  by  $(\mu \wedge \rho)(x) = \min\{\mu(x), \rho(x)\}$  for every  $x \in X$ .
- (3)  $\mu \vee \rho \in I^X$  by  $(\mu \vee \rho)(x) = \max\{\mu(x), \rho(x)\}$  for every  $x \in X$ .

Let  $\Lambda$  be an indexing set and  $\{\mu_\lambda / \lambda \in \Lambda\}$  be a family of fuzzy sets in  $X$ . Then their **union** and **intersection** are defined as follows.

$$(\vee \mu_\lambda)(x) = \sup\{\mu_\lambda(x) / \lambda \in \Lambda\}.$$

$$(\wedge \mu_\lambda)(x) = \inf\{\mu_\lambda(x) / \lambda \in \Lambda\}.$$

(4)  $\mu^c \in I^X$  by  $\mu^c(x) = 1 - \mu(x)$  for every  $x \in X$ .  $\mu^c$  is called **complement of  $\mu$**  and is also denoted as  $A'$ .

(5) Let  $f : X \rightarrow Y$ ,  $\mu \in I^X$  and  $\rho \in I^Y$ . The  $f(A)$  is a fuzzy set in  $Y$  called **image of  $A$**  defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) ; x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

and  $f^{-1}(\rho)$  is a fuzzy set in  $X$  called **inverse image of  $\rho$**  defined by

$$f^{-1}(\rho)(x) = \rho(f(x)), \forall x \in X.$$

(6) By the constant function  $\alpha$  on  $X$  we mean the function  $\alpha : X \rightarrow I$  given by  $\alpha(x) = \alpha$ , for every  $x \in X$  and  $\alpha \in [0, 1]$ .

**Definition: 1.6 (Chang, C.L., [4])**

A **fuzzy topological space** is a pair  $(X, \delta)$ , where  $X$  is a set and  $\delta$  is a **fuzzy topology** on it, that is, a family of fuzzy sets  $(\delta \subseteq I^X)$  satisfying the following three axioms.

(1)  $\mathbf{0}, \mathbf{1} \in \delta$  where  $\mathbf{0}(x) = 0, \forall x \in X$

$$\mathbf{1}(x) = 1, \forall x \in X$$

(2) If  $\mu, \rho \in \delta$ ,  $\mu \wedge \rho \in \delta$ .

(3) If  $\{\mu_j; j \in J\} \subseteq \delta$ , then  $\vee\{\mu_j; j \in J\} \in \delta$ .

The elements of  $\delta$  are called **fuzzy open sets**. A fuzzy set  $K$  is called a **fuzzy closed set** if  $K^c \in \delta$ . We denote by  $\delta^c$  the collection of all fuzzy closed sets in this fuzzy topological space.

**Definition: 1.7 (Lowen, R., [17])**

A **fuzzy topology** on  $X$  is a subset  $\delta \subset I^X$  such that

- (i) for every  $\alpha$  constant,  $\alpha \in \delta$ .
- (ii) for every  $\mu, \gamma \in \delta$ ,  $\mu \wedge \gamma \in \delta$ .
- (iii) for every  $(\mu_j)_{j \in J} \in \delta$ ,  $\sup_{j \in J} \mu_j \in \delta$ .

**Definition: 1.8**

The **closure**  $cl(\mu)$  or  $\mu^-$  and the **interior**  $int(\mu)$  or  $\mu^0$  of a fuzzy set  $\mu$  of  $X$  is defined as

$$Cl(\mu) = \inf\{\gamma; \mu \leq \gamma, K^c \in \delta\},$$

$$Int(\mu) = \sup\{L : L \leq \mu, L \in \delta\} \text{ respectively.}$$

**Definition: 1.9**

A fuzzy set in  $X$  is called a **fuzzy point** if and only if it takes the value 0 for all  $y \in X$  except one, say  $x \in X$ . If its value at  $x$  is  $\lambda$  ( $0 < \lambda \leq 1$ ) we denote the fuzzy point  $X_\lambda$ , where the point  $x$  is called its **support**. When the support and the value of a fuzzy point are trivial, we use briefly the symbol  $e$  to denote the fuzzy point.

**Definition: 1.10**

A fuzzy set in  $X_\lambda$  is said to be **contained** in a fuzzy set  $\mu$  or **to belong to**  $\mu$ , denote by  $x_\lambda \in \mu$ , if and only if  $\lambda \leq \mu(x)$ . Evidently, every fuzzy set  $\mu$  can be expressed as the union of all the fuzzy points which belong to  $\mu$ .

**Definition: 1.11**

A fuzzy set  $\mu$  in a fuzzy topological  $(X, \delta)$  is called a **neighbourhood** of a fuzzy point  $X_\lambda$  iff there exists a  $v \in \delta$  such that  $x_\lambda \in v \subseteq \mu$ .

**Definition: 1.12**

A function  $f$  from a fuzzy topological space  $X$  into a fuzzy topological space  $Y$  is **fuzzy continuous** if and only if for every fuzzy point  $x_\lambda$  in  $X$  and every neighbourhood  $V$  of  $f(x_\lambda)$ , there exists a neighbourhood  $U$  of  $x_\lambda$  such that  $f(U) \leq V$ .

**Definition: 1.13**

A subset  $\sigma \subset \delta$  is a **base for  $\delta$**  iff for every  $\mu \in \delta$ , there exists  $(\mu_j)_{j \in J} \subset \sigma$  such that  $\mu = \sup_{j \in J} \mu_j$ .

**Definition: 1.14**

A subset  $\sigma' \subset \delta$  is a **subbase for  $\delta$**  iff the family of finite infima of members of  $\sigma'$  is a base for  $\delta$ .

**Definition: 1.15**

Any subset  $\mu$  of  $X$  can be identified with the fuzzy set  $\chi_\mu$ , the **characteristic function** of  $\mu$ . The function  $\chi_\mu : X \rightarrow I$  is defined by

$$\begin{aligned} \chi_\mu(x) &= 1 \quad \text{iff } x \in \mu \\ &= 0 \quad \text{iff } x \notin \mu \end{aligned}$$

The characteristic function  $\chi_\mu$  is also denoted by  $I_\mu$ .

**Definition: 1.16**

A fuzzy topological space  $X$  is called a **Hausdorff space** if  $x \neq y$  implies that there are neighbourhoods  $\mu, \rho$  of  $x, y$  respectively, with  $\mu \wedge \rho = 0$ .

**Definition: 1.17**

A fuzzy topological space  $X$  is a  **$T_1$ -space** if for any two distinct points  $x, y$  of  $X$  there are neighbourhoods  $\mu, \rho$  of  $x, y$  respectively, such that  $\mu(y) = 0$  and  $\rho(x) = 0$ .

**Definition: 1.18**

Let  $X$  and  $Y$  be topological space. Let  $P : X \rightarrow Y$  be a surjective map. The map  $P$  is said to be a **quotient map** provided a subset  $U$  of  $Y$  is open in  $Y$  if and only if  $P^{-1}(U)$  is open in  $X$ .

**Definition: 1.19**

Let  $\mathcal{S}_\beta$  denoted the collection  $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) / U_\beta \text{ open in } X_\beta\}$  and let  $\mathcal{S}$  denote the union of these collections.

$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$ . The topology generated by the subbasis  $\mathcal{S}$  is called the **product topology**.

In this topology  $\prod_{\alpha \in J} X_\alpha$  is called a **product space**.

**Definition: 1.20**

Let  $D$  be a non-empty set. Let  $\geq$  be a semi-order on  $D$ . The pair  $(D, \geq)$  is called a **directed set**, directed by  $\geq$  iff for every pair  $m, n \in D$  there exists a  $p \in D$  such that  $p \geq m$  and  $p \geq n$ .

**Definition: 1.21**

Let  $(X, \delta)$  be a fuzzy topological space. A **net**  $(X, \delta)$  is a function  $S : D \rightarrow M(I^X)$ , where  $D$  is a directed set.

**Definition: 1.22**

A net  $(x_\alpha)_{\alpha \in J}$  in a fuzzy topological space  $(X, \tau)$  is said to be **converge** to  $x \in X$ , written as  $x = \lim x_\alpha$  or  $x_\alpha \rightarrow x$ , if for each neighbourhood  $\mu$  of  $x$  there exists or index  $\alpha_0$  such that  $\mu$  is a neighbourhood of  $x_\alpha$  for each  $\alpha \geq \alpha_0$ .