

CHAPTER 1

CHAPTER 1

PRELIMINARIES ON FUZZY SETS AND INTUITIONISTIC FUZZY SETS

In this chapter we discuss preliminaries on fuzzy sets, fuzzy topological spaces, intuitionistic fuzzy sets and intuitionistic fuzzy topological spaces.

In section one of chapter 1, preliminary definitions on fuzzy sets, fuzzy topological spaces, closure and interior operators and the image and pre image of fuzzy sets that are needed for our study are collected.

In section two of chapter 1, the basic concept of intuitionistic fuzzy sets, basic properties of inclusion and complementation and some relations defined over the set of intuitionistic fuzzy sets are studied.

In section three of chapter 1, the concept of intuitionistic fuzzy topological spaces, the properties of image and pre image of intuitionistic fuzzy sets are discussed.

SECTION: 1.1

PRELIMINARIES ON FUZZY SETS AND FUZZY TOPOLOGY

Definition: 1.1.1

Let X be a non-empty set and I be the unit interval $[0,1]$. A *Fuzzy set* in X is a function with domain X and values in I , that is an element of I^X .

By 0^X and 1^X , we denote respectively the constant fuzzy set on X taking the value 0 and 1.

Definition: 1.1.2

Let $A, B \in I^X$. We define the following fuzzy sets,

(i) A include B (i.e $B \subset A$) by $B(x) \leq A(x)$ for every $x \in X$.

(ii) $A \cap B \in I^X$ by $(A \cap B)(x) = \min \{A(x), B(x)\}$ for every $x \in X$.

(iii) $A \cup B \in I^X$ by $(A \cup B)(x) = \max \{A(x), B(x)\}$ for every $x \in X$.

(iv) $A' \in I^X$ by $A'(x) = 1 - A(x)$ for every $x \in X$.

Let Δ be an indexing set and $\{A_\lambda(x)/\lambda \in \Delta\}$ be a family of fuzzy sets in X . Then their *Union and Intersection* are defined as follows:

$$(\cup A_\lambda)(x) = \cup \{A_\lambda(x)/\lambda \in \Delta\}$$

$$(\cap A_\lambda)(x) = \cap \{A_\lambda(x)/\lambda \in \Delta\}$$

Note: 1.1.3

The ordinary subset of X can be considered as fuzzy sets by identifying them with their characteristic functions.

Ordinary subsets are referred to as *Crisp sets* when they are considered as fuzzy sets. Ordinary topological space is referred to as *Crisp topological space*.

If $A \subset X$ and if we consider A as a fuzzy set then we mean

$$A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Definition: 1.1.4

Let f be a function from X to Y . Let B be a fuzzy set in Y . Then *inverse image of B* or *preimage of B* written as $f^{-1}(B)$ is a fuzzy set in X defined by $f^{-1}(B)(x) = B(f(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in Y defined by

$$f(A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is non-empty, where } f^{-1}(y) = \{x : f(x) = y\} \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in Y$.

Properties: 1.1.5

Let f be a function from X to Y . Then

- (a) $f^{-1}(B') = \{f^{-1}(B)\}'$ for any fuzzy set B in Y .
- (b) $f(A') \supset \{f(A)\}'$ for any fuzzy set A in X .
- (c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$ where B_1 and B_2 are fuzzy sets in Y .
- (d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$ where A_1 and A_2 are fuzzy sets in X .
- (e) $B \supset f\{f^{-1}(B)\}$ for any fuzzy set B in Y .
- (f) $A \subset f^{-1}\{f(A)\}$ for any set A in X .
- (g) Let f be a function from X to Y and g be a function from Y to Z . Then
 $(g \circ f)^{-1}\{c\} = f^{-1}\{g^{-1}(c)\}$ for any fuzzy set C in Z , where $(g \circ f)$ is the composition of g and f .
- (h) If f is onto then $f(f^{-1}(A)) = A$.

Definition: 1.1.6(Chang [8])

A fuzzy topology on a set X is a family T of fuzzy sets in X satisfying the following axioms.

- (i) $\Phi, X \in T$
- (ii) $A, B \in T \Rightarrow A \cap B \in T$
- (iii) $A_i \in T$ for each $i \in \Delta$ then $\bigcup_i A_i \in T$.

The pair (X, T) is referred to as a *Fuzzy topological space*. A fuzzy topological space is denoted by *fts* in short.

Definition: 1.1.7

If (X, T) is a fuzzy topological space, members of T are called *open fuzzy sets*. A fuzzy set is called a *closed fuzzy set* iff $A' \in T$.

Definition: 1.1.8

Let (X, T) be a fts. Then the clouser and interior of a fuzzy set $A \in I^X$ are defined respectively as

$$\bar{A} = \bigcap \{B/B \supset A, B \in T\}$$

$$A^\circ = \bigcup \{B/B \subset A, B \in T\}$$

It is easily seen that \bar{A} is the smallest closed fuzzy set larger than A and that A° is the largest open fuzzy set smaller than A .

Definition: 1.1.9

Let $(X, T), (Y, T')$ be a two fts's. A mapping $f : (X, T) \rightarrow (Y, T')$ is *fuzzy continuous* iff for each open fuzzy set V in T' the inverse image $f^{-1}(V)$ is open in T .

Definition: 1.1.10

Let $(X, T), (Y, T')$ be two fts's. A mapping $f : (X, T) \rightarrow (Y, T')$ is called *fuzzy open* iff for each open fuzzy set V in T the image $f(V)$ is open in T' .

Definition: 1.1.11

Given two fuzzy topologies T_1, T_2 on the same set X , we say T_1 is *finer* than T_2 (and that T_2 is *coarser* than T_1) if the identity mapping of (X, T_1) into (X, T_2) is fuzzy continuous.

SECTION: 1.2

INTUITIONISTIC FUZZY SETS

Definition: 1.2.1

Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}$$

where the functions $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each element $x \in X$ to the set A , respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition: 1.2.2

Let X be a non-empty set, and the IFSs A and B be in the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}, \quad B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle : x \in X \}.$$
 Then

(a) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

(b) $A = B$ iff $A \subset B$ and $B \subset A$.

$$(c) \bar{A} = \{ \langle x, \nu_A(x), \mu_A(x) \rangle : x \in X \}$$

$$(d) \Box A = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X \}$$

$$(e) \nabla A = \{ \langle x, 1 - \nu_A(x), \nu_A(x) \rangle : x \in X \}$$

$$(g) A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle : x \in X \}$$

$$(h) A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle : x \in X \}$$

$$(i) A + B = \{ \langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x) \rangle : x \in X \}$$

$$(j) A \cdot B = \{ \langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x) \rangle : x \in X \}$$

Proposition: 1.2.3

(a) The operations \cap and \cup are commutative, associative, distributive to the left and to the right among themselves, idempotent and satisfy the law of DeMorgan.

(b) The operations $+$ and \cdot are commutative, associative and satisfy a law, similar to the DeMorgan law.

(c) The operations $+$ and \cdot are distributive to the left and to the right with respect to the operations \cap and \cup .

Proposition: 1.2.4

For every IFS A :

- (a) $\square A = \overline{\nabla A}$;
- (b) $\nabla A = \overline{\square A}$;
- (c) $\square A \subset A \subset \nabla A$;
- (d) $\square \square A = \square A$;
- (e) $\square \nabla A = \nabla A$;
- (f) $\nabla \square A = \square A$;
- (g) $\nabla \nabla A = \nabla A$.

Proposition: 1.2.5

For every two IFSs A and B :

- (a) $\square(A \cup B) = \square A \cup \square B$;
- (b) $\nabla(A \cup B) = \nabla A \cup \nabla B$.

Proposition: 1.2.6

For every two IFSs A and B :

- (a) $\square(A \cap B) = \square A \cap \square B$;
- (b) $\nabla(A \cap B) = \nabla A \cap \nabla B$;

$$(c) \quad \square(A + B) = \square A + \square B;$$

$$(d) \quad \nabla(A + B) = \nabla A + \nabla B;$$

$$(e) \quad \square(A.B) = \square A.\square B;$$

$$(f) \quad \nabla(A.B) = \nabla A.\nabla B.$$

Definition: 1.2.7

Let X be a non-empty fixed set. Then,

$$A \subset_{\square} B \text{ iff } (\forall x \in X) (\mu_A(x) \leq \mu_B(x));$$

$$A \subset_{\nabla} B \text{ iff } (\forall x \in X) (\nu_A(x) \geq \nu_B(x)).$$

Proposition: 1.2.8

For every two IFSs A and B :

$$(a) A \subset_{\square} B \text{ iff } \square A \subset \square B;$$

$$(b) A \subset_{\nabla} B \text{ iff } \nabla A \subset \nabla B.$$

Definition: 1.2.9

Let X be a non-empty fixed set. Then,

$$A \sqsubset B \text{ iff } (\forall x \in X) (\Pi_A(x) \leq \Pi_B(x)).$$

Proposition: 1.2.10

For every two IFSs A and B :

$$(a) \text{ if } A \subset_{\square} B \text{ and } A \subset_{\nabla} B \text{ then } A \subset B;$$

$$(b) \text{ if } A \subset_{\square} B \text{ and } A \subset_{\sqsubset} B \text{ then } A \subset B;$$

$$(c) \text{ if } A \subset_{\nabla} B \text{ and } B \subset_{\sqsubset} A \text{ then } A \subset B.$$

Definition: 1.2.11

Let A be a given IFS. We determine for its the four numbers

$$\begin{aligned} K &= \max_{x \in X} \mu_A(x), & L &= \min_{x \in X} \nu_A(x), \\ k &= \min_{x \in X} \mu_A(x), & l &= \max_{x \in X} \nu_A(x) \end{aligned}$$

and the sets

$$\begin{aligned} C(A) &= \{ \langle x, K, L \rangle : x \in X \}, \\ I(A) &= \{ \langle x, k, l \rangle : x \in X \}. \end{aligned}$$

will be called *closure and interior* where $0 \leq K + L \leq 1$ and $0 \leq k + l \leq 1$.

Proposition: 1.2.12

For every two IFSs A and B :

- (a) $C(A)$ and $I(A)$ are IFSs;
- (b) $I(A) \subset A \subset C(A)$;
- (c) $C(A \cup B) = C(A) \cup C(B)$;
- (d) $C(C(A)) = C(A)$;
- (e) $C(\bar{0}) = \bar{0}$, where $\bar{0} = \{ \langle x, 0, 1 \rangle : x \in X \}$.

Proposition: 1.2.13

For every IFS A :

- (a) $\square A \subset C(A)$;
- (b) $I(A) \subset \nabla A$;
- (c) $\overline{I(\bar{A})} = C(A)$.

Definition: 1.2.14

Let X_1 and X_2 be two universums and let $A = \{ \langle x_1, \mu_A(x_1), \nu_A(x_1) \rangle : x_1 \in X_1 \}$ and $B = \{ \langle x_2, \mu_B(x_2), \nu_B(x_2) \rangle : x_2 \in X_2 \}$ be two IFSs. The cartesian product of these two IFSs are

$$A \times B = \{ \langle \langle x_1, x_2 \rangle, \mu_A(x_1) \cdot \mu_B(x_2), \nu_A(x_1) \cdot \nu_B(x_2) \rangle : x_1 \in X_1, x_2 \in X_2 \}.$$

Because $0 \leq \mu_A(x_1) \cdot \mu_B(x_2) + \nu_A(x_1) \cdot \nu_B(x_2) \leq \mu_A(x_1) + \nu_A(x_1) \leq 1$, it follows that $A \times B$ is an IFS, but now with universum $X_1 \times X_2$.

Proposition: 1.2.15

For every three universums X_1, X_2, X_3 and four IFSs A, B (over X_1), C (over X_2), D (over X_3).

- (a) $A \times C = C \times A$;
- (b) $(A \times C) \times D = A \times (C \times D)$;
- (c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- (d) $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- (e) $(A + B) \times C \subset (A \times C) + (B \times C)$;
- (f) $(A \cdot B) \times C \supset (A \times C) \cdot (B \times C)$.

Proposition: 1.2.16

For every two universums X_1, X_2 and two IFSs A, B over them:

- (a) $\Box(A \times B) \subset \Box A \times \Box B$;
- (b) $\nabla(A \times B) \supset \nabla A \times \nabla B$;
- (c) $C(A \times B) \subset C(A) \times C(B)$;
- (d) $I(A \times B) \supset I(A) \times I(B)$.

SECTION: 1.3**Corollary: 1.3.1**

Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}$;

$B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle : x \in X \}$;

$C = \{ \langle x, \mu_C(x), \nu_C(x) \rangle : x \in X \}$;

$D = \{ \langle x, \mu_D(x), \nu_D(x) \rangle : x \in X \}$ are IFSs in X. Then

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$,

(b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \cap C$,

(c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$,

(d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$,

(e) $\overline{A \cup B} = \overline{A} \cap \overline{B}$,

(f) $\overline{A \cap B} = \overline{A} \cup \overline{B}$,

(g) $A \subseteq B \Rightarrow \overline{B} \cup \overline{A}$,

(h) $\overline{\overline{A}} = A$.

Definition: 1.3.2

$\tilde{0} = \{ \langle x, 0, 1 \rangle : x \in X \}$ and

$\tilde{1} = \{ \langle x, 1, 0 \rangle : x \in X \}$.

Definition: 1.3.3

(a) If $B = \{ \langle y, \mu_B(y), \nu_B(y) \rangle : y \in Y \}$ is an IFS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{ \langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x) \rangle : x \in X \}$$

(b) If $A = \{ \langle x, \lambda_A(x), \vartheta_A(x) \rangle : x \in X \}$ is an IFS in X, then the image of A under f, denoted by $f(A)$, is the IFS in Y defined by

$$f(A) = \{ \langle y, f(\lambda_A)(y), (1 - f(1 - \vartheta_A))(y) \rangle : y \in Y \}$$

where

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

$$(1 - f(1 - \vartheta_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \vartheta_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Corollary: 1.3.4

Let $A, A_i (i \in J)$ be IFSs in X , $B, B_j (j \in K)$ IFSs in Y and $f : X \rightarrow Y$ a function.

Then

- (a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,
- (b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,
- (c) $A \subseteq f^{-1}(f(A))$ [if f is injective, then $A = f^{-1}(f(A))$],
- (d) $f(f^{-1}(B)) \subseteq B$ [if f is surjective, then $f(f^{-1}(B)) = B$],
- (e) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$,
- (f) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$,
- (g) $f(\cup A_i) = \cup f(A_i)$,
- (h) $f(\cap A_i) \subseteq \cap f(A_i)$, [if f is injective, then $f(\cap A_i) = \cap f(A_i)$],
- (i) $f^{-1}(\tilde{1}) = \tilde{1}$,
- (j) $f^{-1}(\tilde{0}) = \tilde{0}$,
- (k) $f(\tilde{1}) = \tilde{1}$, if f is surjective,
- (l) $f(\tilde{0}) = \tilde{0}$,
- (m) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective
- (n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition: 1.3.5

An intuitionistic fuzzy topology (IFT for short) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(T₁) $\tilde{0}, \tilde{1} \in \tau$,

(T₂) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(T₃) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in τ is known as an intuitionistic fuzzy open sets (IFOS for short) in X .

Proposition: 1.3.6

Let (X, τ) be an IFTS on X . Then, we can also construct several IFTS on X in the following way:

$$(a) \tau_{0,1} = \{ \{ G : G \in \tau \} \},$$

$$(b) \tau_{0,2} = \{ \langle \rangle G : G \in \tau \}$$

Remark: 1.3.7

Let (X, τ) be an IFTS.

(a) $\tau_1 = \{ \mu_G : G \in \tau \}$ is a fuzzy topological space on X in Chang's sense.

(b) $\tau_2' = \{ \nu_G : G \in \tau \}$ is a family of all fuzzy closed sets of the fuzzy topological space

$\tau_2 = \{ 1 - \nu_G : G \in \tau \}$ on X in Chang's sense.

(c) Since $0 \leq \mu_G(x) + \nu_G(x) \leq 1$ for each $x \in X$ and each $G \in \tau$, we obtain $\mu_G \leq 1 - \nu_G$.

(d) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bifuzzy topological space.

Definition: 1.3.8

Let $(X, \tau_1), (X, \tau_2)$ be two IFTS on x . Then τ_1 is said to be contained in τ_2 (in symbols, $\tau_1 \subseteq \tau_2$) if $G \in \tau_2$ for each $G \in \tau_1$. In this case, we also say that τ_1 is coarser than τ_2 .

Proposition: 1.3.9

Let $\{\tau_i : i \in j\}$ be a family of IFTS on X . Then $\bigcap \tau_i$ is an IFT on X . Furthermore, $\bigcap \tau_i$ is the coarsest IFT on X containing all τ_i 's.

Definition: 1.3.10

An intuitionistic fuzzy topological space in the sense of Lowen is a pair (X, τ) where (X, τ) is an IFTS and each IFS in the form $C_{\alpha, \beta} = \{ \langle x, \alpha, \beta \rangle : x \in X \}$, where $\alpha, \beta \in I$ are arbitrary and $\alpha + \beta \leq 1$, belongs to τ .

Definition: 1.3.11

The complement \bar{A} of an IFOS A in an IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS for short) in X .

Definition: 1.3.12

Let (X, τ) be an IFTS and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in X . Then the *fuzzy interior and fuzzy closure* of A are defined by

$$\begin{aligned} cl(A) &= \bigcap \{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\}, \\ int(A) &= \bigcup \{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}. \end{aligned}$$

Proposition: 1.3.13

For any IFS A in (X, τ)

$$(a) \quad cl(\bar{A}) = \overline{int(A)},$$

$$(b) \quad int(\bar{A}) = \overline{cl(A)}.$$

Proposition: 1.3.14

Let (X, τ) be an IFTS and A, B be IFSs in X . Then the following properties hold:

(a) $\text{int}(A) \subseteq A$,

(a') $A \subseteq \text{cl}(A)$,

(b) $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$,

(b') $A \subseteq B \Rightarrow \text{cl}(A) \subseteq \text{cl}(B)$,

(c) $\text{int}(\text{int}(A)) = \text{int}(A)$,

(c') $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,

(d) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$,

(d') $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$,

(e) $\text{int}(\tilde{1}) = \tilde{1}$,

(e') $\text{cl}(\tilde{0}) = \tilde{0}$.