

**PERFORMANCE ANALYSIS OF RETRIAL QUEUEING MODELS WITH  
MULTI STAGES OF SERVICES**

**By**

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**Supervisor**

**Dr. N. Sangeetha**

**Thesis submitted to**

**Avinashilingam Institute for Home Science and Higher Education for Women,**

**Coimbatore – 641 043**

**In Partial Fulfilment of the requirements for the Degree of**

**Master of Science in Mathematics**

**May 2022**

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19/05/2022

**Signature of the Supervisor**

## DECLARATION

I declare that the dissertation, entitled on "**PERFORMANCE ANALYSIS OF RETRIAL QUEUEING MODELS WITH MULTI STAGES OF SERVICES**" submitted by me for the Degree of Master of Science is a record of work carried out by me during the period from December 2021 to May 2022 under the guidance of **Dr. N. SANGEETHA** Teaching and Research Fellow, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and it has not formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or other similar title in this University or any other University or Institution of Higher Learning.

*S.T. Sowmya Natarajani.*  
**Signature of the Candidate**

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## ACKNOWLEDGEMENT

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## **CONTENTS**

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<b>CHAPTER</b>	<b>TITLE</b>	<b>PAGE NO.</b>
<b>I</b>	<b>INTRODUCTION</b>	<b>1</b>
<b>II</b>	<b>BATCH ARRIVAL MULTISTAGE RETRIAL QUEUEING SYSTEM WITH BERNOULLI VACATION, SERVER BREAKDOWN AND REPAIR</b>	<b>13</b>
<b>III</b>	<b>BATCH ARRIVAL MULTISTAGE RETRIAL QUEUEING SYSTEM WITH PRIORITY AND COLLISION</b>	<b>25</b>
	<b>SUMMARY AND CONCLUSION</b>	<b>36</b>
	<b>BIBLIOGRAPHY</b>	<b>37</b>

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## **CHAPTER 1**

# CHAPTER 1

## INTRODUCTION

Queueing theory is a branch of applied mathematics utilizing concepts from the field of stochastic processes. It has been developed for the purpose of better understanding of queueing systems and to take appropriate measures to maintain the system efficiency.

### Classical Queueing System

A classical queueing system can be described as the customers arriving for service, waiting for the service if it is not immediate, and if having waiting for service, leaving the system after being served. The general structure of a queueing system is presented in Fig: 1.1.

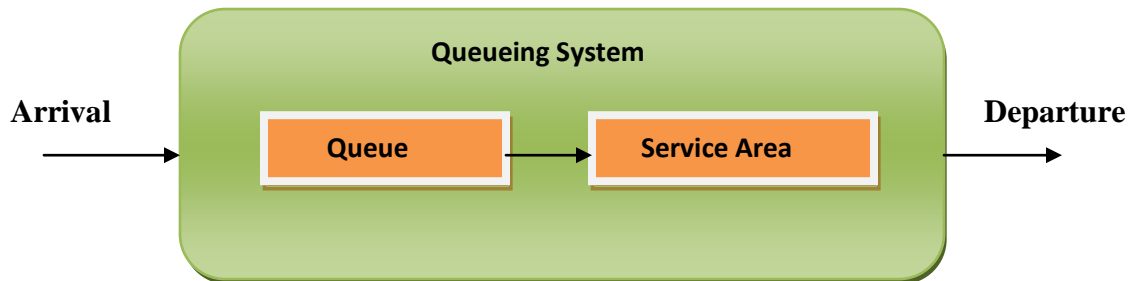


Fig: 1.1. General Structure of a Queueing system

### APPLICATION AREAS OF QUEUEING SYSTEMS

- Bank counters
- Tollgate
- Ration shop
- Mainframe computer centre
- Library
- Traffic signal
- Final inspection station of television assembly line
- Airport runways

- Telephone booth
- Maintenance shop

## **CHARACTERISTICS OF QUEUEING PROCESS**

The basic characteristics of a queueing system are arrival pattern of customers, service pattern of servers, queue discipline, system capacity and number of service channels.

### **Arrival pattern**

Arrival pattern describes the way how units arrive and join the system. The arrival may be either single or batches of variable or fixed size. The source of units may be finite or infinite. The arrival patterns are often measured in terms of average number of arrivals per unit time. The inter-arrival time is the interval between two consecutive arrivals. In case, the arrival times are known with certainty, the queueing problems are characterized as deterministic models. However, in usual queueing situations, the process of arrivals is stochastic and it is necessary to know the probability distribution associated with the successive arrivals. The most common stochastic queueing models assume that inter-arrival times follow an Exponential distribution. The arrival pattern also describes the behaviour of the customers as some customers may wait patiently in the queue and some may be impatient if it takes a long time to receive the desired service. If an arriving customer decides not to join the queue, the customer is said to have balked. If a customer leaves the queue after joining due to impatience, it is known as reneging. In case there are two or more parallel waiting lines and a customer moves from one queue to another, the customer is said to have jockeyed. An arrival process could be stationary or non-stationary according to the probability distribution describing the arrival pattern being time independent or dependent of time.

### **Service pattern**

Service pattern describes the manner in which the service is rendered. The customers may be served either in single or in batches of variable or fixed size. The time required for serving one customer or customers in batches is called the service time. The service pattern of customers may be stationary or non-stationary with respect to time and state dependent or

independent with respect to the number of customers waiting for service. The service time may be deterministic or probabilistic.

## **Queue discipline**

The service discipline indicates the manner in which the units are taken from the queue and allowed into service. Customers may also be served in groups or one by one. The most known disciplines are:

- **First In First Out(FIFO)**

The usual queue discipline in First Come First Served (FCFS), a customer that finds the service centre busy goes to the end of the queue.

- **Last In First Out(LIFO)**

Last Come First Served (LCFS), a customer that find the service Centre busy proceeds immediately to the head of the queue, this customer will be served next, given that no further customers arrive.

- **Service in Random Order(SIRO)**

Service in Random Order also called (SIRO), customers are selected for service at random, irrespectively of their arrivals in the service system. Every customer in the queue is equally likely to be selected. In the case the time of arrival of the customers is therefore, of no relevance.

- **Priority disciplines**

Under this discipline, the service offered is of two types pre-emptive priority and non pre-emptive priority. Under pre-emptive rule, high priority customers are given preference over low priority customers. As such, low priority customer's service is interrupted (pre-empted) to offer service to a priority customer. Under non pre-emptive rule, the high priority customers go ahead in the queue but his service starts only after the completion of service of the existing customer.

## **Number of servers**

A system may have single server or a group of servers providing service to the customers. Increasing the number of service channels helps to decrease the waiting time. Given a

number of service channels they may operate in parallel being able to serve customers simultaneously. It's generally assumed that the service mechanisms of the parallel channels operate independently of each other. An arrival who finds more than one free server may choose any one of them for receiving the service. If he finds all the servers busy, he joins a queue common to all the servers, the first customer in the common queue goes to the server who becomes free first.

## **System Capacity**

In some queueing processes there is a physical limitation to the amount of waiting room so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available, as a result of a service completion. These are referred to as finite queueing situations. A queue with limited waiting room can be viewed as one with forced balking.

## **Service Channels**

The number of servers in a queueing models may be finite or infinite. The number of servers may be arranged in series, parallel or a combination of both, depending upon the nature of the services required. In parallel channels, all the channels provide identical services so that the several customers may be served simultaneously. In series channels, a customer must pass through successively in several ordered channels before service is completed.

## **KENDALL'S NOTATION**

Kendall's notation describes the arrival process to the queue, the nature of service process, the number of servers, maximum number in the queue and some basic queue disciplines. The notation has been considerably extended to allow it to represent a wide variety of queues. A queue is represented by a sequence  $A/B/C/X/Y$  with the following meaning attached to the letters A, B, C, X, Y.

A: This symbolically represents the nature of arrival process to the queue. Special letters are used to symbolize the nature of inter-arrival time distribution as follows:

M - Exponentially distributed Markovian inter-arrival times

D - Deterministic inter-arrival time

$E_k$  - Erlang distribution of order k for the inter-arrival time

G - General distribution for the inter-arrival time

B: This symbolically represents the nature of service time distribution for the customers getting served in the queue. Here, the symbols are similarly to inter-arrival time distribution.

C: Number of servers in the queue

X: System capacity. The Default is infinity

Y: Queue discipline. Default is First Come First Served (FCFS)

## **Analysing of Queueing models**

Queueing models can be classified into Markovian and non-Markovian models. If the inter arrival time of customers and service times are exponentially distributed, then the queueing model is called Markovian queueing model. It can be analysed by difference differential equation method or Neuts matrix geometric method. Queueing models with inter arrival times and/or service times which are not exponentially distributed are called non-Markovian queueing models. The methods generally used in studying non-Markovian queues are embedded Markov chain method and supplementary variable technique.

## **Retrial Queueing system**

In classical queueing theory it is very often assumed that an arriving customer who cannot get service immediately either joins the waiting line to get the service or leaves the system. Sometimes impatient customers leave the queue, but it is also assumed that they are leaving the system forever. However, as a matter of fact, the assumption about loss of customers who elected to leave the system is just a first order approximation to a real situation. Usually such a customer after a random time returns to the system and tries to get the service again. We may find queues with returning customers in our daily activities. In retail shopping, a customer who finds a long waiting line may wish to do something else and return later on, with the hope

that the queue dissolves. The standard queueing models do not take the retrial phenomenon into account and therefore cannot be applied in solving a number of practically important problems. Retrial queues (or queues with returning customers, repeated attempts, etc.) have been introduced to solve this insufficiency. Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time.

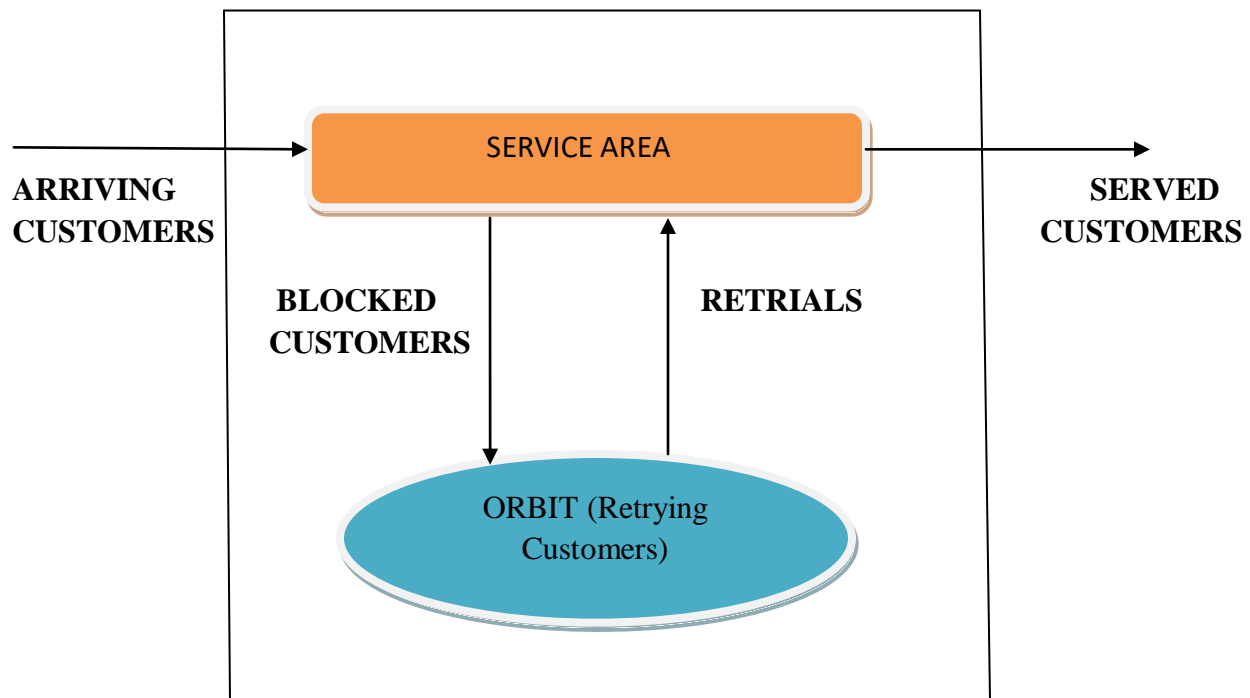


Fig: 1.2 General structure of a Retrial Queueing System

## Review of Literature

Queueing theory was developed to provide models to predict the behavior of systems that attempt to provide service for randomly arising demands. The Pioneer investigator of queueing theory was the Danish Mathematician A.K.Erlang (1909) who investigated the theory of probabilities and telephone conversations. Molina (1927) published his paper, Application of the theory of probability to telephone trunking problems. Fry (1928) extended much of Erlang's work in a book, Probability and its Engineering Uses. Kendall (1953) on imbedded chain, Takacs (1955) on waiting time, Cox (1955) on supplementary variables, Champernowne (1956) on the

use of random walks, Saaty (1960) on absorbing barriers and time dependent solutions, Little (1961) on deriving the queueing formula  $L = \lambda W$ , Conway and Maxwell (1962) on state dependent service and Neuts (1973) on discrete time queueing model are some remarkable work in the queueing literature. Review of queueing literature can be found in the survey papers of Prabhu (1987) and Shastrakar et al. (2016).

## Retrial Queues

Queues in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called retrial queues or queues with repeated attempts. Retrial queues have been widely used to provide stochastic modeling of many problems arising in computer systems, packet switching networks, cloud computing systems and in real life situations. The first results on M/G/1 retrial queues are due to Keilson et al. (1968), who used the method of supplementary variable to investigate the joint distribution of the channel state and the number of customers in the orbit in steady state. The first investigation with general retrial time was done by Kapyrin (1977), in which each customer in the orbit generates a stream of repeated attempts that are independent of the customers in the orbit and the state of the server. Review of retrial queueing literature can be found in the survey papers of Yang and Templeton (1987), Falin (1990), Kim and Kim (2016), Shekar et al. (2016) and Tuan Phung-Duc (2019), the bibliographies of Artalejo (1999a, 1999b and 2010) and the books by Falin and Templeton (1997) and Artalejo and Gomez Corral (2008). The applications of retrial queues in science and engineering are given by Kulkarni and Liang (1997).

Retrial queueing systems with **batch arrival** are common in many practical situations. In digital communication systems, messages which are transmitted could consist of a random number of packets. Falin (1976) introduced the single server batch arrival retrial queueing model. He used the embedded Markov chain technique to derive the joint distribution of the channel state and the queue length. Kulkarni (1986) and Falin (1988) examined multiple classes of customers with batch arrivals. Artalejo and Atencia (2004) presented a single server retrial queue with batch arrivals and carried out an extensive analysis of the system, including existence of the stationary regime, embedded Markov chain and stochastic decomposition. A detailed study on batch arrival queue under both classical and constant retrial policies was considered by Jain et al. (2008). Yamamuro (2012) analysed an M/G/1 retrial queue with batch arrivals and

obtained the expected queue length. Many authors including Florea and Nanau (2015), Kim (2017), Niranjana et al. (2019), Krishnamoorthy et al. (2021) and Sadhna Singh and Srivastava (2021) discussed retrial queueing situations with batch arrival.

## **Unreliable Queueing Models**

Breakdown is a remarkable and unavoidable phenomenon in the service facility of a queueing system. Queues with server subject to breakdowns and repairs are often encountered in many practical applications. The study of queueing models with repairable server can be found earlier in the articles of Avi-Itzhak and Naor (1962), Thiruvengdan (1963), Mitranjy and Avi-Itzhak (1968), Neuts and Lucantoni (1979) and Cao and Cheng (1982).

Retrial queueing systems with server breakdowns and repairs were considered by Aissani (1988) and Kulkarni and Choi (1990). Aissani and Artalejo (1998) studied a single server retrial queueing system subject to active breakdown. Wang et al. (2001) obtained explicit expressions of availability, failure frequency and reliability function of the server for M/G/1 retrial queue with server breakdown. Li et al. (2006) provided BMAP/G/1 retrial queue with server breakdowns and repairs considering both from queueing view point and reliability view point. Choudhury and Deka (2009) obtained the limiting distribution of the number of customers in the system at departure epoch and idle period completion epoch for an M/G/1 retrial queue with two types of heterogeneous service subject to random breakdown and repair under linear retrial policy. Falin (2010) investigated an M/G/1 retrial queue with an unreliable server and general retrial times with the help of embedded Markov chain.

In most of the papers with unreliable server it is assumed that whenever the system breaks down the repair process starts instantaneously. However, this is not the case in many real life situations. The system has to wait for repair to start. Prakash Rani et al. (2011) discussed the retrial queue with server breakdown and delayed repairs. Choudhury and Ke (2014) studied an unreliable retrial queue with delaying repair and general retrial times under Bernoulli vacation schedule and derived system size distribution at departure epoch. Gao et al. (2020) studied an M/G/1 retrial queue with two types of breakdowns and delayed repairs. Tzu-Hsin Liu, Fu-Min Chang and Jan-Chuan Ke (2022) studied unreliable servers subject to imperfect coverage and reboot delay.

## **Pre-emptive Resume Priority and Collision**

Priority queues, both pre-emptive and non pre-emptive have important uses in the modeling and analysis of computer systems, manufacturing models, operating systems and are widely used in production practice and transportation management. Today service priority is clearly a main feature of any manufacturing system. Falin et al. (1993) considered retrial queueing model with priority customers and carried out an extensive analysis of the system including stochastic decomposition. Krishna Kumar et al. (2002) developed a two phase M/G/1 retrial queue with pre-emptive resume service discipline and derived analytical expressions for various system measures. Jain and Bhargava (2008) suggested an unreliable batch arrival retrial model with two class of priority subscribers and established the condition of ergodicity, mean queue length and expected waiting time. Priority retrial queue with balking and feedback customers was studied by Chen and Zhu (2010). The authors analyzed ergodicity of the embedded Markov chain along with performance characteristics. Senthil Kumar et al. (2013) examined preemptive resume priority retrial queue with two classes of MAP arrivals. Devos et al. (2018) studied priority retrial queue under constant retrial policy with different service time distribution and obtained the expected orbit length, expected system length and Laplace Stieltje's transform of the stationary waiting times of both type of customers. Performance analysis of batch arrival feedback pre-emptive priority retrial queue with disaster under working breakdown services and balking was proposed by Rajadurai et al. (2020). Jia Xu et al. (2022) analysed an M/G/1 retrial queueing system with priority service and modified multiple vacations.

Retrial queues with collision arise from the medium access control protocols for wireless local area networks. In many situations involving data transmission from diverse sources there can be a conflict for a limited number of channels or other facilities. Uncoordinated attempts by several sources to use a single server facility can result in collision leading to the loss of the transmission and hence the need for retransmission. Choi et al. (1992) discussed a retrial queueing system with constant retrial rate and collision arising from the specific communication protocol CSMA-CD and derived the limiting distribution of the number of customers in the system at arbitrary time points using the theory of Markov regenerative processes. Kim (2010) considered a retrial queue with collision and impatience and using supplementary variable technique obtained the performance characteristics in steady state such

as blocking probability and the mean number of customers in the orbit. Nazarov et al. (2014) provided an asymptotic analysis of closed Markov retrial queueing system with collision and established formulas for computing preliminary distribution of the number of sources in waiting state. Li et al. (2019) derived the explicit expressions of performance measures along with the conditional stochastic decomposition for the queue length in the orbit when the server is busy for an M/M/1 retrial queue with working vacation and collision. Toth and Sztrik (2020) examined the waiting time distribution and sensitivity analysis for a finite source retrial queue with non-reliable server, collisions and impatient customers.

## **Server Vacation**

Retrial queueing systems with server vacations have been investigated extensively because of their applications in modelling the digital data distributed switching systems, wireless adhoc networks, production lines and inventory systems. A vacation in a queueing system is a period of time where the server may not be available. During the vacation period, the server may also perform other supplementary tasks like maintenance activity, working at other queues or simply taking break. Excellent surveys on earlier works on vacation models in the queueing literature may be found in Doshi (1986), books by Takagi (1991) and Tian and Zhang (2006). Aissani (2009) presented an  $M^x/G/1$  energetic retrial queue with vacations and derived recursive relation which can be used as a discrete event simulation algorithm. Stochastic inequality for M/G/1 retrial queues with vacations under constant retrial policy was considered by Boualem et al. (2009). The authors derived several comparison properties in the sense of strong stochastic ordering and convex ordering. Wang (2012) extended the continuous time retrial queue with Bernoulli vacation to discrete time. Lakshmi and Kasturi Ramanath (2013) examined a vacation queue under constant retrial policy and obtained the joint probability generating function of the steady state orbit size, system size at random epoch using supplementary variable technique. Ebenesar Anna Bagyam and Udaya Chandrika (2013) discussed a single server state dependent batch arrival retrial queue with Bernoulli vacation and obtained the explicit formula for the stationary distributions and performance measures of the system. Radha et al. (2017) considered the Bernoulli vacation in group arrival retrial queues with an unreliable server. A single server non-Markovian two phase retrial queue with Bernoulli vacation was given by Arivaudainambi and Gowsalya (2018). The authors investigated the

system size probabilities in steady state using supplementary variable method and established stochastic decomposition law. Gao and Wang (2019) suggested a single server retrial queue with server vacation, two waiting buffers based on ATM networks and carried out an extensive analysis to give the mean queue lengths in the original buffer, retrial buffer and in the system. Ke and Wang (2021) studied the single server constant retrial queue with Bernoulli vacation and obtained the threshold equilibrium strategies of blocked customers in different information levels.

### **Multi Phase Queueing Models**

Shahkar and Badamchi Zadeh (2006) analysed vacation queueing model with multiphases of services in succession. Salehirad and Badamchi Zadeh (2009) and Abdollahi and Salehirad (2012) studied the multi phase M/G/1 queueing system with feedbacks. Bagyam and Udaya Chandrika (2013a) analysed multistage retrial queueing system with breakdown and reserved time. Bagyam and Udaya Chandrika (2013b) obtained the joint distribution of the server state and orbit length of a multistage retrial queueing system with feedback at each stage. Radha et al. (2014) studied a batch arrival retrial queue with  $K$  – optional stages of service, Bernoulli feedback, single vacation and random breakdown. Using generating function technique, Bagyam and Udaya Chandrika (2015) obtained important performance measures of a bulk arrival retrial queueing model with multistages of service and Bernoulli vacation. Radha et al. (2017) considered a group arrival multistage unreliable retrial queueing system with vacation. Bagyam and Udaya Chandrika (2018) considered a bulk arrival retrial queueing system with multistages of heterogeneous services, active breakdown and delayed repair. Radha et al. (2020) analysed a batch arrival retrial queueing model with multi optional stages of service, orbital search, extended Bernoulli vacation and stand by server. Sangeetha et al. (2020) analysed a single server bulk arrival retrial queue with multistages of heterogeneous service and feedback.

## **Profile of work**

Introduction and review of literature are presented in chapter-1

Batch arrival retrial queueing system with multi-stages of services Bernoulli vacation, server breakdown and repair is analysed in chapter-2

Batch arrival multi stage retrial queueing system with priority and collision is derived in chapter-3

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## **CHAPTER 2**

## CHAPTER-2

### **Batch Arrival Multi Stage Retrial Queueing System with Bernoulli Vacation, Server Breakdown and Repair**

A single server batch arrival retrial queueing system with multi stages of services is considered. If the arriving batch of customers finds the server idle, one of the arriving customers receives the first stage service and the rest joins the orbit. After the completion of  $i^{th}$ , ( $i=1,2,\dots,M$ ) stage service, the customer may proceed to  $(i+1)^{th}$  stage with probability  $p_i$ , or leave the system with probability  $q_i = 1 - p_i$ , ( $i=1,2,3,\dots,M-1$ ) and  $q_M = 1$ . The server is subject to active breakdown when the server is busy at any of the stages. The repair of the failed server starts immediately and the interrupted customers remains in the service area until the service gets completed. The idle server may take a single vacation with certain probability or remain idle in the system with complementary probability. The joint distributions of the server state and the number of customers in the orbit are obtained using the supplementary variable technique. The impact of parameters in the system performance measures is examined. The reliability indices are also obtained.

#### **2.1 Model descriptions**

A single server batch arrival retrial queueing system with multi-stages of heterogeneous services is considered.

##### **Arrival process**

The customers arrive in batches according to the Poisson process with rate  $\lambda$ . At every arrival epoch, a batch of  $k$  customers arrives with probability  $C_k$ ,  $k=1, 2, 3,\dots$ . The generating function of sequence  $\{ C_k \}$  is  $C(z)$  with first two moments  $m_1$  and  $m_2$

##### **Retrial process**

If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit. If the server is free, the first stage service commences for one of the arriving

customers and others join the orbit. Inter-retrial times follow an arbitrary distribution with distribution function  $A(x)$ , Laplace-Stieltjes transform  $A^*(s)$  and the hazard rate function

$$\eta(x) = \frac{dA(x)}{1 - A(x)}$$

## Service process

The server provides  $M$  stages of heterogeneous service in succession. After the completion of the first stage service, the customer may proceed to second stage with probability  $p_1$  or depart the system with probability  $q_1 = 1 - p_1$ . After completing the second stage service, the customers may proceed to third stage with probability  $p_2$  or depart the system with probability  $q_2 = 1 - p_2$ . This process continues up to  $M$  stages. The service times are generally distributed with distribution function  $B_i(x)$ , Laplace-Stieltjes transform  $B_i^*(s)$  with first two moments  $\mu_i^{(1)}$  and

$$\mu_i^{(2)} \text{ and the hazard rate function } \mu_i(x) = \frac{dB_i(x)}{1 - B_i(x)}, \quad i = 1, 2, \dots, M.$$

## Breakdown and repair

The server is subject to active breakdown when the server is busy. The life time of the server is assumed to be exponentially distributed with rate  $\alpha_i$ . The repair of the failed server starts immediately and the interrupted customers waits in the service area until the remaining service gets completed. The repair time is generally distributed with distributed function  $R(x)$ , Laplace-Stieltjes transform  $R^*(S)$  with first two moments  $\beta^{(1)}$  and  $\beta^{(2)}$  and the hazard rate function

$$\beta_i(y) = \frac{dR_i(y)}{1 - R_i(y)}, \quad i = 1, 2, \dots, M.$$

## Server vacation

At the service completion epoch, the server may take a single vacation with probability  $\nu$  or wait for the customers in the system with probability  $\bar{\nu} = (1 - \nu)$ . The vacation time is

generally distributed with distributed function  $V(x)$  Laplace-Stieltjes transform  $V^*(S)$  with the first moments  $\gamma^{(1)}$  and  $\gamma^{(2)}$ . and the hazard rate function  $\gamma(x) = \frac{dV(x)}{1-V(x)}$ ,  $i = 1, 2, \dots, M$ .

## 2.2 Orbit size Distribution at Random Epoch

For the process,  $\{N(t) ; t \geq 0\}$  the probability densities are defined as follows.

$$I_0(t) = P\{C(t) = 0, N(t) = 0\}$$

$$I_n(x, t) = P\{C(t) = 0, N(t) = n, x < \xi_0(t) \leq x + dx\}, \quad n \geq 1$$

$$P_{i,n}(x, t) dx = P\{C(t) = 1, N(t) = n, x < \xi_1(t) \leq x + dx\}, \quad n \geq 0, \quad i = 1, 2, \dots, M$$

$$V_n(x, t) = P\{C(t) = 2, N(t) = n, x < \xi_2(t) \leq x + dx\}, \quad n \geq 1$$

$$R_{i,n}(x, t) = P\{C(t) = 3, N(t) = n, x < \xi_3(t) \leq x + dx\}, \quad n \geq 1$$

The steady state equations governing the model under consideration are given below.

$$\lambda I_0 = \int_0^\infty V_0(x) \gamma(x) dx + (1-\nu) \sum_{i=1}^M q_i \int_0^\infty P_{i,0}(x) \mu_i(x) dx \quad (2.1)$$

$$\frac{dI_n(x)}{dx} = -(\lambda + \eta(x)) I_n(x), \quad n \geq 1 \quad (2.2)$$

$$\frac{dP_{i,n}(x)}{dx} = -(\lambda + \alpha_i + \mu_i(x)) P_{i,n}(x) + \int_0^\infty R_{i,n}(x, y) \beta_i(y) dy + \lambda (1 - \delta_{on}) \sum_{k=1}^n C_k P_{i,n-k}(x), \quad n \geq 0 \quad (2.3)$$

$$\frac{dV_n(x)}{dx} = -(\lambda + \gamma(x)) V_n(x) + \lambda (1 - \delta_{on}) \sum_{k=1}^n C_k V_{n-k}(x), \quad n \geq 0 \quad (2.4)$$

$$\frac{dR_{i,n}(x, y)}{dy} = -(\lambda + \beta_i(y)) R_{i,n}(x, y) + \lambda (1 - \delta_{on}) \sum_{k=1}^n C_k R_{i,n-k}(x, y), \quad n \geq 0 \quad (2.5)$$

with boundary conditions at  $x=0$  and  $y=0$  are

$$I_n(0) = (1-\nu) \sum_{i=1}^M q_i \int_0^{\infty} P_{i,n}(x) \mu_i(x) dx + \int_0^{\infty} V_n(x) \gamma(x) dx, \quad n \geq 1 \quad (2.6)$$

$$P_{1,n}(0) = \int_0^{\infty} I_{n+1}(x) \eta(x) dx + \lambda C_{n+1} I_0 + \lambda \sum_{k=1}^n C_k \int_0^{\infty} I_{n-k+1}(x) dx, \quad n \geq 0 \quad (2.7)$$

$$P_{i,n}(0) = p_{i-1} \int_0^{\infty} P_{i-1,n}(x) \mu_{i-1}(x) dx, \quad n \geq 0, \quad i = 2, 3, \dots \quad (2.8)$$

$$V_n(0) = \nu \sum_{i=1}^M \int_0^{\infty} P_{i,n}(x) \mu_i(x) dx, \quad n \geq 0 \quad (2.9)$$

$$R_{i,n}(x, 0) = \alpha_i [P_{i,n}(x)], \quad n \geq 0, \quad i = 1, 2, \dots, M \quad (2.10)$$

### 2.3 Steady state Solutions

The following probability generating functions are defined to solve the equations that govern the model

$$\left. \begin{aligned} I(x, z) &= \sum_{n=1}^{\infty} I_n(x) z^n \quad ; \quad P_i(x, z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^n \\ V(x, z) &= \sum_{n=0}^{\infty} V_n(x) z^n \quad ; \quad R_i(x, y, z) = \sum_{n=0}^{\infty} R_{i,n}(x, y) z^n \end{aligned} \right\} \quad (2.11)$$

By using the definition of probability generating functions, the equation (2.2) to (2.10) give

$$\left[ \frac{\partial}{\partial x} + \lambda + \eta(x) \right] I(x, z) = 0 \quad (2.12)$$

$$\left[ \frac{\partial}{\partial x} + \lambda(1 - C(z)) + \alpha_i + \mu_i(x) \right] P_i(x, z) = \int_0^{\infty} R_i(x, y, z) \beta_i(y) dy \quad (2.13)$$

$$\left[ \frac{\partial}{\partial x} + \lambda(1 - C(z)) + \gamma(x) \right] V(x, z) = 0 \quad (2.14)$$

$$\left[ \frac{\partial}{\partial y} + \lambda(1 - C(z)) + \beta_i(y) \right] R_i(x, y, z) = 0 \quad (2.15)$$

$$I(0, z) = (1 - \nu) \sum_{i=1}^M q_i \int_0^{\infty} P_i(x, z) \mu_i(x) dx + \int_0^{\infty} V(x, z) \gamma(x) dx - \lambda I_0 \quad (2.16)$$

$$P_1(0, z) = \frac{\lambda}{z} C(z) I_0 + \frac{1}{z} I(0, z) [A^*(\lambda) + C(z)(1 - A^*(\lambda))] \quad (2.17)$$

$$P_i(0, z) = p_{i-1} \int_0^{\infty} P_{i-1}(x, z) \mu_{i-1}(x) dx \quad (2.18)$$

$$V(0, z) = \nu \sum_{i=1}^M q_i \int_0^{\infty} P_i(x, z) \mu_i(x) dx, \quad n \geq 0 \quad (2.19)$$

$$R_i(x, 0, z) = \alpha_i [P_i(x, z)], \quad n \geq 0, \quad i = 1, 2, \dots, M \quad (2.20)$$

Solving the partial differential equations (2.12) to (2.15), we get

$$I(x, z) = I(0, z) e^{-\lambda x} [1 - A(x)] \quad (2.21)$$

$$P_i(x, z) = P_i(0, z) e^{-\omega_i(z)x} [1 - B_i(x)] \quad i = 1, 2, \dots, M \quad (2.22)$$

$$V(x, z) = V(0, z) e^{-\lambda(1-C(z))x} [1 - V(x)] \quad (2.23)$$

$$R_i(x, y, z) = R_i(x, 0, z) e^{-\lambda(1-C(z))y} [1 - R_i(y)] \quad i = 1, 2, 3, \dots, M \quad (2.24)$$

where

$$\omega_i(z) = \lambda(1 - C(z)) + \alpha_i(1 - R_i^*(\lambda(1 - C(z))))$$

Using the expression (2.22) and (2.23) the equations (2.16), (2.19) and (2.20) give

$$I(0, z) = (1 - \nu) \sum_{i=1}^M q_i P_i(0, z) B^*(\omega_i(z)) + V(0, z) V^*(\lambda - \lambda C(z)) - \lambda I_0 \quad (2.25)$$

$$V(0, z) = \nu \sum_{i=1}^M q_i P_i(0, z) B^*(\omega_i(z)) \quad (2.26)$$

$$R_i(x, 0, z) = \alpha_i P_i(0, z) e^{-\omega_i(z)x} [1 - B_i(x)] \quad (2.27)$$

Using the expression  $P_i(0, z)$ , the equation (2.18) gives

$$\begin{aligned} P_i(0, z) &= p_{i-1} P_{i-1}(0, z) B_{i-1}^*(\omega_{i-1}(z)) \\ &= p_{i-1} p_{i-2} P_{i-2}(0, z) B_{i-2}^*(\omega_{i-2}(z)) B_{i-1}^*(\omega_{i-1}(z)) \\ &= p_{i-1} p_{i-2} \dots p_2 p_1 P_1(0, z) B_1^*(\omega_1(z)) B_2^*(\omega_2(z)) \dots B_{i-1}^*(\omega_{i-1}(z)) \\ &= \Theta_{i-1} S_{i-1}^* P_1(0, z) \end{aligned} \quad (2.28)$$

where

$$\Theta_{i-1} = p_1 p_2 \dots p_{i-1} \text{ and } S_i^* = B_1^*(\omega_1(z)) B_2^*(\omega_2(z)) \dots B_i^*(\omega_i(z))$$

Using the expression (2.28) and (2.26), the equations (2.25) to (2.27) give

$$I(0, z) = (1 - \nu) \sum_{i=1}^M q_i \Theta_{i-1} S_i^* P_1(0, z) + \nu \sum_{i=1}^M q_i \Theta_{i-1} S_i^* P_1(0, z) V^*(b(z)) - \lambda I_0 \quad (2.29)$$

$$V(0, z) = \nu \sum_{i=1}^{M1} q_i \Theta_{i-1} S_i^* P_1(0, z) \quad (2.30)$$

$$R_i(x, 0, z) = \alpha_i \Theta_{i-1} S_{i-1}^* P_1(0, z) \quad (2.31)$$

Substituting the expression of  $I(0, z)$ , the equation (2.17) yields

$$P_1(0, z) = \frac{\lambda I_0 A^*(\lambda)(C(z) - 1)}{D(z)} \quad (2.32)$$

where

$$D(z) = z - (A^*(\lambda) + C(z)(1 - A^*(\lambda))) \left[ (1 - \nu) \sum_{i=1}^M q_i \Theta_{i-1} S_i^* + \nu \sum_{i=1}^M q_i \Theta_{i-1} S_i^* V^*(\lambda(1 - C(z))) \right]$$

Using the equation (2.32), the equations (2.28), (2.29) to (2.31) give

$$I(0, z) = \frac{\lambda I_0 A^*(\lambda)(C(z) - 1) \left[ (1 - \nu) \sum_{i=1}^M q_i \Theta_{i-1} S_i^* + \nu \sum_{i=1}^M q_i \Theta_{i-1} S_i^* V^*(b(z)) \right]}{D(z)} - \lambda I_0 \quad (2.33)$$

$$P_i(0, z) = \frac{I_0 \lambda \Theta_{i-1} S_{i-1}^* A^*(\lambda)(C(z) - 1)}{D(z)} \quad (2.34)$$

$$V(0, z) = \frac{I_0 \lambda \nu \sum_{i=1}^M q_i \Theta_{i-1} S_i^* A^*(\lambda)(C(z) - 1)}{D(z)} \quad (2.35)$$

$$R_i(x, 0, z) = \frac{I_0 \lambda \alpha_i \Theta_{i-1} S_{i-1}^* A^*(\lambda)(C(z) - 1) e^{-\omega_i(z)x} [1 - B_i(x)]}{D(z)}, \quad n \geq 0, \quad i = 1, 2, \dots, M \quad (2.36)$$

The probability generating function of the orbit size when the server is idle in the non-empty system is given by

$$I(z) = \int_0^{\infty} I(x, z) dx$$

$$= \frac{I_0 (1 - A^*(\lambda)) \{ C(z) \left[ (1 - \nu) \sum_{i=1}^M q_i \Theta_{i-1} S_i^* + \nu \sum_{i=1}^M q_i \Theta_{i-1} S_i^* V^*(\lambda - \lambda C(z)) \right] - z \}}{D(z)} \quad (2.37)$$

The probability generating function of the orbit size when the server is busy in providing the  $i^{th}$  stage service is given by

$$\begin{aligned}
P_i(z) &= \int_0^{\infty} P_i(x, z) dx \\
&= \frac{\lambda I_0 (1 - B_i^*(\omega_i(z)))}{\omega_i(z)} \left[ \frac{\Theta_{i-1} A^*(\lambda) (C(z) - 1) S_{i-1}^*}{D(z)} \right]
\end{aligned} \tag{2.38}$$

The probability generating of the orbit size when the server is on vacation is given by

$$\begin{aligned}
V(z) &= \int_0^{\infty} V(x, z) dx \\
&= \frac{\nu I_0 A^*(\lambda) \sum_{i=1}^M q_i \Theta_{i-1} S_i^* (1 - V^*(b(z)))}{D(z)}
\end{aligned} \tag{2.39}$$

The probability generating of the orbit size when the server is under repair is given by

$$\begin{aligned}
R_i(z) &= \int_0^{\infty} \int_0^{\infty} R_i(x, y, z) dx \\
&= \frac{I_0 \alpha_i A^*(\lambda) \Theta_{i-1} (C(z) - 1) S_{i-1}^* (1 - B_i^*(\omega_i(z))) (R_i^*(\lambda(1 - C(z)) - 1)}{\omega_i(z) D(z)}
\end{aligned} \tag{2.40}$$

Using the normalizing condition

$$I_0 + \lim_{z \rightarrow 1} I(z) + \lim_{z \rightarrow 1} \sum_{i=1}^M P_i(z) + \lim_{z \rightarrow 1} V(z) + \lim_{z \rightarrow 1} \sum_{i=1}^M R_i(z) = 1$$

$$\text{we obtain } I_0 = \frac{\{1 - m_1(1 - A^*(\lambda)) + m_1 \nu \lambda \sum_{i=1}^M \Theta_{i-1} q_i \gamma^{(1)} - \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)}\}}{A^*(\lambda) \{1 - \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} + \lambda \sum_{i=1}^M \Theta_{i-1} M_i^{(1)} (1 - \alpha_i \beta^{(1)})\}} \tag{2.41}$$

where

$$M_i^{(1)} = \frac{\lim_{z \rightarrow 1} dS_i^*[\omega_i(z)]}{dz} = \sum_{j=1}^i \lambda m_1 \mu_j^{(1)} (1 - \alpha_i \beta^{(1)})$$

The probability generating function of the orbit size is given by

$$\begin{aligned} P_q(z) &= I_0 + I(z) + \sum_{i=1}^M P_i(z) + \sum_{i=1}^M R_i(z) + V(z) \\ &= \frac{I_0 A^*(\lambda) (z - \sum_{i=1}^M \Theta_{i-1} q_i S_i^* + \sum_{i=1}^M q_i \Theta_{i-1} S_i^* (1 - B_i^*(\omega_i(z))))}{D(z)} \end{aligned} \quad (2.42)$$

The probability generating function of the system size is given by

$$\begin{aligned} P_s(z) &= I_0 + I(z) + z \sum_{i=1}^M P_i(z) + z \sum_{i=1}^M R_i(z) + V(z) \\ &= \frac{I_0 A^*(\lambda) (z - \sum_{i=1}^M \Theta_{i-1} q_i S_i^* + z \sum_{i=1}^M \Theta_{i-1} S_i^* [1 - B_i^*(\omega_i(z))])}{D(z)} \end{aligned} \quad (2.43)$$

## 2.4 Stability condition

The necessary and sufficient condition for the system to be stable is

$$m_1 (1 - A^*(\lambda)) + \nu \lambda m_1 \gamma^{(1)} \sum_{i=1}^M \Theta_{i-1} q_i + \sum_{i=1}^M \Theta_{i-1} M_i^{(1)} < 1$$

## 2.5 Performance Measures

- The probability of the server is idle in the non-empty system is

$$I = \lim_{z \rightarrow 1} I(z)$$

$$= \frac{I_0(1-A^*(\lambda)) \left[ m_1 \left( 1 + \sum_{i=1}^M \Theta_{i-1} q_i v \lambda \gamma^{(1)} \right) + \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} - 1 \right]}{D'} \quad (2.44)$$

where

$$D' = \{1 - m_1(1 - A^*(\lambda)) + v m_1 \lambda \sum_{i=1}^m \Theta_{i-1} q_i \gamma^{(1)} - \sum_{i=1}^m \Theta_{i-1} q_i M_i^{(1)}\}$$

- The probability of the server is busy in providing  $i^{th}$  ( $i=1,2,3,\dots,M$ ) stage service is given by

$$P_i = \lim_{z \rightarrow 1} P_i(z)$$

$$= \frac{\lambda I_0 m_1 A^*(\lambda) \Theta_{i-1} M_i^{(1)}}{D'} \quad (2.45)$$

- The probability of the server is on vacation is given by

$$V = \lim_{z \rightarrow 1} V(z)$$

$$= \frac{\lambda I_0 m_1 A^*(\lambda) v \gamma^{(1)} \sum_{i=1}^M \Theta_{i-1} q_i}{D'} \quad (2.46)$$

- The probability of the server is under repair on  $i^{th}$  ( $i=1,2,\dots,M$ ) stage is given by

$$R_i = \lim_{z \rightarrow 1} R_i(z)$$

$$= \frac{\lambda I_0 m_1 \beta^{(1)} A^*(\lambda) \sum_{i=1}^M \alpha_i \Theta_{i-1} \mu_i^{(1)}}{D'} \quad (2.47)$$

Expected orbit size is given by

$$L_q = \lim_{z \rightarrow 1} \frac{d}{dz} P_q(z)$$

$$= \frac{D' N_q'' - N_q' D''}{2D'^2} \quad (2.48)$$

where

$$\begin{aligned}
N'_q &= I_0 A^*(\lambda) \left[ 1 - \sum_{i=1}^M q_i \Theta_{i-1} M_i^{(1)} - \sum_{i=1}^M \Theta_{i-1} \lambda m_1 (1 + \alpha_i \beta^{(1)}) \mu_i^{(1)} \right] \\
N''_q &= -I_0 A^*(\lambda) \left[ \sum_{i=1}^M q_i \Theta_{i-1} M_i^{(2)} + 2 \sum_{i=1}^M \Theta_{i-1} M_i^{(1)} \lambda m_1 (1 + \alpha_i \beta^{(1)}) \mu_i^{(1)} + \sum_{i=1}^M \Theta_{i-1} \lambda m_1 (1 + \alpha_i \beta^{(1)}) \mu_i^{(1)} \right] \\
D'' &= -[m_2 (1 - A^*(\lambda)) + m_1 (1 - A^*(\lambda))] \left[ (1 - \nu) \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} + \nu \sum_{i=1}^M \Theta_{i-1} q_i \lambda m_1 \nu \gamma^{(1)} \right] \\
&\quad + m_1 (1 - A^*(\lambda)) \left( \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(2)} + 2 M_i^{(1)} \lambda m_1 \gamma^{(1)} + \lambda^2 m_1^2 \gamma^{(2)} + \lambda m_2 \gamma^{(1)} \right) \\
M_i^{(2)} &= \sum_{j=1}^i [-M_i^{(1)} (\lambda m_2 + \alpha_i) [\lambda m_2 \beta^{(1)} - \alpha_i (\lambda m_1)^2 \beta^{(1)}] + (\lambda m_1 \mu_j^{(1)})^2 \mu_j^{(2)} (1 - \alpha_i \beta^{(1)})]
\end{aligned}$$

Expected system size is given by

$$\begin{aligned}
L_s &= \lim_{z \rightarrow 1} \frac{d}{dz} P_s(z) \\
&= L_q + \sum_{i=1}^M P_i
\end{aligned} \tag{2.49}$$

## 2.6. Reliability Indices

Let  $\mathcal{A}(t)$  be the availability of the server at time  $t$ , and let  $\mathcal{F}(t)$  be the failure frequency

### Theorem: 2.1

The steady state availability of the server is

$$\mathcal{A} = \frac{I_0 A^*(\lambda) \left[ 1 - \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} - \lambda \sum_{i=1}^M \Theta_{i-1} m_1 M_i^{(1)} \right]}{1 - m_1 \left( (1 - A^*(\lambda)) \left( 1 + \sum_{i=1}^M \Theta_{i-1} q_i \nu \lambda \gamma^{(1)} \right) - \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} \right)} \tag{2.50}$$

**Proof**

$$\begin{aligned}
\mathcal{A} &= I_0 + \lim_{z \rightarrow 1} \left[ \int_0^{\infty} I(x, z) dx + \sum_{i=1}^M \int_0^{\infty} P_i(x, z) dx + \int_0^{\infty} V(x, z) dx \right] \\
&= I_0 + \lim_{z \rightarrow 1} \left[ I(z) + \sum_{i=1}^M P_i(z) + V(z) \right] \\
&= I_0 + I + \sum_{i=1}^M P_i + V
\end{aligned} \tag{2.51}$$

Equation (2.50) is obtained by substituting the expression of  $I, I_0, P_i$  in equation (2.51).

**Theorem: 2.2**

The failure frequency of the server in the system is given by

$$\mathcal{F} = \frac{\sum_{i=1}^M \alpha_i \lambda I_0 m_1 A^*(\lambda) \Theta_{i-1} M_i^{(1)}}{D'} \tag{2.52}$$

**Proof**

$$\begin{aligned}
&= \sum_{i=1}^M \alpha_i \lim_{z \rightarrow 1} \int_0^{\infty} P_i(x, z) dx \\
&= \sum_{i=1}^M \alpha_i \lim_{z \rightarrow 1} P_i \\
&= \sum_{i=1}^M \alpha_i P_i
\end{aligned} \tag{2.53}$$

The result in equation (2.52) can be obtained by using equation (2.45).

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## **CHAPTER 3**

## CHAPTER 3

### Batch Arrival Multistage Retrial Queueing System with Priority and Collision

In this chapter, a single server batch arrival retrial queue with multi stages of the service is considered. Upon the arrival of a batch of customers, if the server is busy, the arriving batch either joins the orbit or collides with the customer in service resulting in all being shifted to the orbit or one of the customers in the batch pushes out the customer in service to the orbit to get his own service. The service time, retrial time and repair time are assumed to be arbitrarily distributed. A mathematical model is constructed and the steady state distributions of the server state including the average number of customers in the system are derived.

#### 3.1 Model Description

The basic assumptions of the model under study are described as follows.

##### Arrival Process

Customers arrive in batches according to the Poisson process with arrival rate  $\lambda$ . At every arrival epoch, a batch of  $k$  customers arrives with probability  $C_k$  where  $k=1, 2, 3, \dots$ . The generating function of the sequence  $\{C_k\}$  is  $C(z)$  with first two moments  $m_1$  and  $m_2$ . Upon the arrival of a batch of customers, if the server is free, then one of the customers from the batch enters the first stage service and others join the orbit. If the server is busy, then the arriving batch proceeds to the server with probability  $\delta$  or join the orbit with complementary probability  $\bar{\delta} = 1 - \delta$ .

##### Retrial process

If an arriving batch of customers finds the server busy, on vacation or down, the batch joins the orbit. If the server is free, the first stage commences for one of the arriving customers and other join the orbit. The inner-retrial times follow general distribution with distribution function  $A(x)$ , Laplace- Stieltjes transform  $A^*(s)$  and the hazard function  $\eta(x) = \frac{dA(x)}{1 - A(x)}$ .

## Service process

The server provides  $M$  stage of heterogeneous service in succession. After the completing the first stage service, the customer may proceed to the second stage with probability  $p_1$  or depart the system with probability  $q_1 = 1 - p_1$ . After completing the second stage service, the customers may proceed to third stage with probability  $p_2$  or depart the system with probability  $q_2 = 1 - p_2$ . This process continues up to  $M$  stages. The service times are generally distributed with distribution function  $B_i(x)$  and LST  $B_i^*(s)$  with first two moments  $\mu_i^{(1)}$  and  $\mu_i^{(2)}$  and the hazard rate function  $\mu_i(x) = \frac{dB_i(x)}{1 - B_i(x)}$ ,  $i = 1, 2, \dots, M$ .

Define  $S_i^* = B_1^* B_2^* B_3^* \dots B_i^*$  with  $S_0^* = 1$  and  $\Theta_{i-1} = \theta_1 \theta_2 \theta_3 \dots \theta_{i-1}$  with  $\Theta_0 = 1$ .

The state of the system at time  $t$  can be described by the Markov process  $\{ X(t), t \geq 0 \} = \{ C(t), N(t), \xi_1(t), \xi_2(t) \}$ , where  $C(t)$  denotes the server state 0, 1, or 2, according as the server being idle in the empty system, idle in the non-empty system and busy in  $i^{\text{th}}$  stage of the service.  $N(t)$  corresponds to the customers in the orbit at time  $t$ . Define the supplementary variables  $\xi_1(t)$  and  $\xi_2(t)$  as follows

$\xi_1(t) =$  elapsed retrial time, if  $C(t) = 0$

$\xi_2(t) =$  elapsed service time, if  $C(t) = 2$  and 1

### 3.2 Orbit Size Distribution at Random Epoch

For the process,  $\{N(t); t \geq 0\}$  the probability densities are defined as follows.

$$I_0(t) = P\{C(t) = 0, N(t) = 0\}$$

$$I_n(x, t) = P\{C(t) = 1, N(t) = n, x < \xi_1(t) \leq x + dx\}, n \geq 1$$

$$P_{i,n}(x, t) dx = P\{C(t) = 2, N(t) = n, x < \xi_2(t) \leq x + dx\}, n \geq 0, i = 1, 2, \dots, M$$

The steady state equations governing the model under consideration are given below

$$\lambda I_0 = \sum_{i=1}^M q_i \int_0^{\infty} P_{i,0}(x) \mu_i(x) dx \quad (3.1)$$

$$\frac{d}{dx} I_n(x) = -[\lambda + \eta(x)] I_n(x), n \geq 1 \quad (3.2)$$

$$\frac{d}{dx} P_{i,n}(x) = -[\lambda + \mu_i(x)] P_{i,n}(x) + \lambda \bar{\delta} (1 - \delta_{0,n}) \sum_{k=1}^n C_k P_{i,n+k}(x), n \geq 0 \quad (3.3)$$

with boundary condition

$$I_1(0) = \sum_{i=1}^M q_i \int_0^{\infty} P_{i,1}(x) \mu_i(x) dx \quad (3.4)$$

$$I_n(0) = \sum_{i=1}^M \int_0^{\infty} P_{i,n}(x) \mu_i(x) dx + \lambda \delta \bar{\alpha} \sum_{k=1}^{n-1} C_k \int_0^{\infty} \sum_{i=1}^M P_{i,n-(k+1)}(x) dx, n \geq 2 \quad (3.5)$$

$$P_{1,0}(0) = \lambda C_1 I_0 + \int_0^{\infty} I_1(x) \eta(x) dx \quad (3.6)$$

$$P_{1,n}(0) = \lambda C_{n+1} I_0 + \lambda \sum_{k=1}^n C_k \int_0^{\infty} I_{n-k+1}(x) dx + \int_0^{\infty} I_{n+1}(x) \eta(x) dx + \lambda \delta \alpha \sum_{k=1}^n C_k \sum_{i=1}^M \int_0^{\infty} P_{i,n-k}(x) dx, n \geq 1 \quad (3.7)$$

$$P_{i,n}(0) = p_{i-1} \int_0^{\infty} \sum_{i=1}^M P_{i-1,n}(x) \mu_{i-1}(x) dx, n \geq 0, i = 1, 2, \dots, M \quad (3.8)$$

The following probability generating functions are defined to solve the equations that govern the model.

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n \quad \text{and} \quad P_i(x, z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^n \quad (3.9)$$

Using the definition of the generating functions, the equation (3.1) to (3.8) give

$$\left(\frac{\partial}{\partial x} + \lambda + \eta(x)\right)I(x, z) = 0 \quad (3.10)$$

$$\left(\frac{\partial}{\partial x} + \lambda(1 - \bar{\delta}C(z)) + \mu_i(x)\right)P_i(x, z) = 0 \quad (3.11)$$

$$I(0, z) = \sum_{i=1}^M q_i \int_0^{\infty} P_i(x, z) \mu_i(x) dx + \lambda \delta \bar{\alpha} z C(z) \sum_{i=1}^M \int_0^{\infty} P_i(x, z) dx - \lambda I_0 \quad (3.12)$$

$$P_1(0, z) = \frac{1}{z} \lambda C(z) I_0 + \frac{\lambda}{z} C(z) \int_0^{\infty} I(x, z) dx + \frac{1}{z} \int_0^{\infty} I(x, z) \eta(x) dx + \lambda \delta \alpha C(z) \int_0^{\infty} P_1(x, z) dx \quad (3.13)$$

$$P_i(0, z) = p_{i-1} \int_0^{\infty} \sum_{i=1}^M P_{i-1}(x, z) \mu_{i-1}(x) dx \quad (3.14)$$

Solving the partial differential equations (3.10) and (3.11), we obtain

$$I(x, z) = I(0, z) e^{-\lambda x} (1 - A(x)) \quad (3.15)$$

$$P_i(x, z) = P_i(0, z) e^{-(\lambda - \lambda \bar{\delta} C(z))x} (1 - B_i(x)) \quad (3.16)$$

Using the expressions in equations (3.13) and (3.14) equation (3.12) yields

$$\begin{aligned} I(0, z) &= \sum_{i=1}^M q_i \int_0^{\infty} P_i(0, z) e^{-(\lambda - \lambda \bar{\delta} C(z))x} (1 - B_i(x)) \mu_i(x) dx \\ &\quad + \lambda \delta \bar{\alpha} z C(z) \sum_{i=1}^M \int_0^{\infty} P_i(0, z) e^{-(\lambda - \lambda \bar{\delta} C(z))x} (1 - B_i(x)) dx - \lambda I_0 \\ &= \sum_{i=1}^M q_i P_i(0, z) B_i^*(g(z)) + \lambda \bar{\alpha} \delta z C(z) \sum_{i=1}^M P_i(0, z) (1 - B_i^*(g(z))) - \lambda I_0 \end{aligned} \quad (3.17)$$

Substituting the expression  $I(x, z)$  and  $P_i(x, z)$  from the equations (3.15) and (3.16) in the equation (3.13), we have

$$\begin{aligned}
 z P_1(0, z) &= \lambda C(z) I_0 + (A^*(\lambda) + C(z)(1 - A^*(\lambda))) I(0, z) \\
 &+ \lambda \alpha \delta z C(z) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) P_1(0, z) (1 - B_i^*(g(z)) / g(z))
 \end{aligned} \tag{3.18}$$

Using the expression of  $P_i(x, z)$ , the equation (3.14) gives

$$\begin{aligned}
 P_i(0, z) &= p_{i-1} \int_0^\infty \sum_{i=1}^M P_{i-1}(x, z) \mu_i(x) dx \\
 &= p_{i-1} \sum_{i=1}^M P_{i-1}(0, z) B_{i-1}^*(g(z)) \\
 &= p_{i-1} p_{i-2} \sum_{i=1}^M P_{i-2}(0, z) B_{i-2}^*(g(z)) B_{i-1}^*(g(z)) \\
 &= p_{i-1} p_{i-2} \dots p_1 P_1(0, z) B_{i-2}^*(g(z)) B_{i-1}^*(g(z)) B_2^*(g(z)) \dots B_{i-1}^*(g(z))
 \end{aligned}$$

$$P_i(0, z) = \Theta_{i-1} S_{i-1}^*(g(z)) P_1(0, z) \tag{3.19}$$

where

$$\Theta_i = p_1, p_2, \dots, p_i \text{ and } S_i^* = B_1^*(g(z)) B_2^*(g(z)) \dots B_i^*(g(z))$$

Substituting the equation (3.19) in equation (3.17) and on simplifying, we get

$$I(0, z) = T(z) P_1(0, z) - \lambda I_0 \tag{3.20}$$

where

$$T(z) = \sum_{i=1}^M q_i \Theta_{i-1} S_i^*(g(z)) + \lambda \bar{\alpha} \delta z C(z) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) (1 - B_i^*(g(z)) / g(z))$$

Inserting the equation (3.20) in equation (3.18), we obtain

$$P_1(0, z) = \lambda I_0 A^*(\lambda) (C(z) - 1) / D(z) \quad (3.21)$$

where

$$D(z) = z - T(z) [A^*(\lambda) + C(z)(1 - A^*(\lambda))] - \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) \lambda \alpha z \delta C(z) [1 - B_i^*(g(z))] / g(z)$$

Substituting the equation (3.21) in equation (3.19), we have

$$P_i(0, z) = \lambda I_0 A^*(\lambda) \Theta_{i-1} S_{i-1}^*(g(z)) (C(z) - 1) / D(z) \quad (3.22)$$

$$I(0, z) = T(z) \lambda I_0 g(z) C(z) - \lambda I_0 g(z) z + \lambda I_0 \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) \delta \alpha z C(z) [1 - B_i^*(g(z))] / D(z) \quad (3.23)$$

- The probability generating function of the orbit size when the server is idle in the non-empty is given by

$$\begin{aligned} I(z) &= \int_0^{\infty} I(x, z) dx \\ &= \frac{I_0 [1 - A^*(\lambda)] \{T(z) g(z) C(z) - g(z) z + \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) \delta \alpha z \lambda C(z) [1 - B_i^*(g(z))]\}}{g(z) D(z)} \end{aligned} \quad (3.24)$$

- The probability generating function of the orbit size when the server is busy in  $i^{th}$  ( $i=1, 2, \dots, M$ ) stage is given by

$$\begin{aligned} P_i(z) &= \int_0^{\infty} P_i(x, z) dx \\ &= \frac{\lambda I_0 A^*(\lambda) \Theta_{i-1} S_{i-1}^*(g(z)) (C(z) - 1) [1 - B_i^*(g(z))]}{g(z) D(z)} \end{aligned} \quad (3.25)$$

Using normalizing condition

$$I_0 + \lim_{z \rightarrow 1} I(z) + \lim_{z \rightarrow 1} P_i(z) = 1$$

we obtain

$$I_0 = \frac{\lambda \delta \left\{ 1 - \frac{(1-\delta)m_1}{\delta} - T_3 + \frac{T_2 \lambda \bar{\delta} m_1}{\lambda \delta} - T_2 m_1 (1 - A^*(\lambda)) - \alpha \sum_{i=1}^M \Theta_{i-1} (1 + m_1) S_i^*(\lambda \delta) (1 - B_i^*(\lambda \delta)) \right\} - \alpha \sum_{i=1}^M \Theta_{i-1} (M_i^{(1)} (1 - B_i^*(\lambda \delta)) - S_i^*(\lambda \delta) f_i^{(1)})}{A^*(\lambda) \{ \lambda \delta T_3 - T_2 \lambda \bar{\delta} m_1 + \lambda \bar{\delta} m_1 - \lambda \delta + \lambda \alpha \delta [(1 + m_1) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda \delta) (1 - B_i^*(\lambda \delta)) + \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(1)} (1 - B_i^*(\lambda \delta)) - S_{i-1}^*(\lambda \delta) f_i^{(1)})] + \lambda m_1 \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda \delta) (1 - B_i^*(\lambda \delta)) \}}$$

(3.26)

where

$$M_i^{(1)} = \lim_{z \rightarrow 1} S_i^*(g(z)) = \sum_{j=1}^i [B_j^*(\lambda \delta) \prod_{\substack{k=1 \\ k \neq j}}^i B_k^*(\lambda \delta)]$$

The probability generating function of the orbit size is given by

$$P_q(z) = I_0 + I(z) + \sum_{i=1}^M P_i(z)$$

$$= \frac{I_0 A^*(\lambda) \left\{ z - \sum_{i=1}^M \Theta_{i-1} q_i S_i^*(g(z)) + (\lambda(C(z) - 1) - \lambda \bar{\alpha} C(z)) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z)) (1 - B_1^*(g(z))) / g(z) \right\}}{D(z)}$$

(3.27)

The probability generating function of the system size is given by

$$\begin{aligned}
P_s(z) &= I_0 + I(z) + z \sum_{i=1}^M P_i(z) \\
&= \frac{I_0 A^*(\lambda) \{z - \sum_{i=1}^M \Theta_{i-1} q_i S_i^*(g(z)) + (z\lambda(C(z)-1) - \lambda \bar{\alpha} C(z)) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(g(z))(1 - B_1^*(g(z))/g(z))\}}{D(z)}
\end{aligned} \tag{3.28}$$

### 3.3 Stability condition

The necessary and sufficient condition for the system to be stable is

$$\begin{aligned}
&\lambda \delta \left\{ 1 - \frac{(1-\delta)m_1}{\delta} - T_3 + \frac{T_2 \lambda \bar{\delta} m_1}{\lambda \delta} - T_2 m_1 (1 - A^*(\lambda)) - \alpha \sum_{i=1}^M \Theta_{i-1} (1 + m_1) S_i^*(\lambda \delta) (1 - B_i^*(\lambda \delta)) \right. \\
&\quad \left. - \alpha \sum_{i=1}^M \Theta_{i-1} (M_i^{(1)} (1 - B_i^*(\lambda \delta)) - S_i^*(\lambda \delta) f_i^{(1)}) \right\} < 1
\end{aligned} \tag{3.29}$$

### 3.4 Performance Measure

- The probability that the server is idle in the non-empty system is

$$\begin{aligned}
I &= \lim_{z \rightarrow 1} I(z) \\
&= \frac{I_0 (1 - A^*(\lambda)) (\lambda \delta T_3 - T_2 \lambda \bar{\delta} m_1 + T_2 \lambda \delta m_1 + \lambda \bar{\delta} m_1 - \lambda \delta + \lambda \alpha \delta [(1 + m_1) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda \delta) (1 - B_i^*(\lambda \delta))] + \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(1)} (1 - B_i^*(\lambda \delta)) - S_{i-1}^*(\lambda \delta) f_i^*))}{D'}
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
D' &= \lambda \delta \left\{ 1 - \frac{(1-\delta)m_1}{\delta} - T_3 + \frac{T_2 \lambda \bar{\delta} m_1}{\lambda \delta} - T_2 m_1 (1 - A^*(\lambda)) - \alpha \sum_{i=1}^M \Theta_{i-1} (1 + m_1) S_i^*(\lambda \delta) (1 - B_i^*(\lambda \delta)) \right. \\
&\quad \left. - \alpha \sum_{i=1}^M \Theta_{i-1} (M_i^{(1)} (1 - B_i^*(\lambda \delta)) - S_i^*(\lambda \delta) f_i^{(1)}) \right\}
\end{aligned}$$

$$T_2 = \sum_{i=1}^M \Theta_{i-1} q_i S_{i-1}^*(\lambda\delta) + \lambda\delta\bar{\alpha} \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) \frac{[1 - B_i^*(\lambda\delta)]}{\lambda\delta}$$

$$T_3 = \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(1)} + \bar{\alpha} \left[ \left( 1 - m_1 + \frac{\bar{\delta}m_1}{\delta} \right) \sum_{i=1}^M \Theta_{i-1} q_i S_{i-1}^*(\lambda\delta) [1 - B_i^*(\lambda\delta)] \right. \\ \left. + \sum_{i=1}^M \Theta_{i-1} (M_i^{(1)} [1 - B_i^*(\lambda\delta)] - S_{i-1}^*(\lambda\delta) f_i^{(1)}) \right]$$

- The probability that the server is busy in service is

$$P_i = \lim_{z \rightarrow 1} P_i(z) \\ = \frac{\lambda I_0 m_1 A^*(\lambda) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) [1 - B_i^*(\lambda\delta)]}{D'} \quad (3.31)$$

Expected orbit size is given by

$$L_q = \lim_{z \rightarrow 1} P_q(z) \\ = \frac{D' N_q'' - D'' N_q'}{2(D')^2} \quad (3.32)$$

where

$$N_q' = I_0 A^*(\lambda) [\lambda\delta - \lambda\bar{\delta}m_1 - \sum_{i=1}^M \Theta_{i-1} q_i (M_i^{(1)} \lambda\delta - \lambda\bar{\delta}m_1 S_i^*(\lambda\delta))] + (\lambda m_1 - \lambda\delta - \lambda\bar{\delta}m_1)$$

$$\sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) (1 - B_i^*(\lambda\delta) - \lambda\delta \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(1)} (1 - B_i^*(\lambda\delta) - S_{i-1}^*(\lambda\delta) f_i^{(1)}))]$$

$$N_q'' = I_0 A^*(\lambda) \{-2m_1 \lambda\bar{\delta} - m_2 \lambda\bar{\delta} - \sum_{i=1}^M \Theta_{i-1} q_i (M_i^{(2)} - 2m_1 \lambda\bar{\delta} M_i^{(1)} - \lambda m_2 \bar{\delta} S_i^*(\lambda\delta))\}$$

$$+ (\lambda m_2 - 2m_1 \lambda\delta - \lambda\bar{\delta}m_2) \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) (1 - B_i^*(\lambda\delta)) + 2(\lambda m_1 - \lambda\delta - \lambda\bar{\delta}m_1)$$

$$\begin{aligned} & \left( \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(1)} (1 - B_i^*(\lambda\delta)) - S_{i-1}^*(\lambda\delta) f_i^{(1)}) \right) \\ & - \lambda\delta \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(2)} (1 - B_i^*(\lambda\delta)) - 2M_{i-1}^{(1)} f_i^{(1)} - S_{i-1}^*(\lambda\delta) f_i^{(2)}) \end{aligned}$$

$$D'' = \lambda\delta[2T_3 + m_2(1 - A^*(\lambda))T_2 - T_4] + \lambda\bar{\delta}[2m_1T_3 + m_2T_2 - 2m_1^2(1 - A^*(\lambda))T_2 - 2m_1 - m_2]$$

$$\sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) (1 - B_i^*(\lambda\delta)) - \lambda\delta \sum_{i=1}^M \Theta_{i-1} (M_{i-1}^{(1)} (1 - B_i^*(\lambda\delta)) - S_{i-1}^*(\lambda\delta) f_i^{(1)})]$$

$$T_4 = \sum_{i=1}^M \Theta_{i-1} q_i M_i^{(2)} + \lambda\delta\bar{\alpha}T_5$$

$$\begin{aligned} T_5 &= \frac{1}{\lambda\delta} \sum_{i=1}^M \Theta_{i-1} S_{i-1}^*(\lambda\delta) [2m_1(1 - B_i^*(\lambda\delta)) - 2f_i^{(1)} + m_2(1 - B_i^*(\lambda\delta)) - f_i^{(2)} + \frac{f_i^{(1)} m_1 \bar{\delta}}{\delta} \\ &+ f_i^{(1)} + \frac{m_2 \bar{\delta}}{\delta} (1 - B_i^*(\lambda\delta)) + \frac{2(m_1 \bar{\delta})^2}{\delta^2} (1 - B_i^*(\lambda\delta))] + \frac{1}{\lambda\delta} \sum_{i=1}^M \Theta_{i-1} M_{i-1}^{(1)} [2(1 - B_i^*(\lambda\delta)) \\ &+ 2m_1(1 - B_i^*(\lambda\delta)) - 2m_1 f_i^{(1)} + 2m_1^2 \bar{\delta} \frac{(1 - B_i^*(\lambda\delta))}{\delta} + 2f_i^{(1)} \\ &+ 2m_1 \bar{\delta} \frac{(1 - B_i^*(\lambda\delta))}{\delta^2}] + \sum_{i=1}^M \Theta_{i-1} M_{i-1}^{(2)} \frac{(1 - B_i^*(\lambda\delta))}{\lambda\delta} \end{aligned}$$

$$M_i^{(2)} = \sum_{j=1}^i [B_j^{*''}(\lambda\delta) \prod_{\substack{k=1 \\ k \neq j}}^i B_k^*(\lambda\delta)] + 2 \sum_{j=1}^{i-1} [B_j^{*'}(\lambda\delta) B_{j+1}^{*'}(\lambda\delta) \prod_{\substack{k=1 \\ k \neq j+1}}^i B_k^*(\lambda\delta)]$$

$$f_i^{(1)} = B_i^{*'}(\lambda\delta) = \lambda\bar{\delta} m_1 \int_0^{\infty} x e^{-\lambda\delta x} b_i(x) dx$$

Expected system size is given by

$$\begin{aligned} L_s &= \lim_{z \rightarrow 1} P_s(z) \\ &= L_q + \sum_{i=1}^M P_i \end{aligned} \tag{3.33}$$

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## **SUMMARY AND CONCLUSION**

## **Summary and conclusion**

Retrial Queueing models with multi stages of services are considered in this dissertation. Chapter-2 is devoted to the study of batch arrival retrial queueing system with multi stages of services, Bernoulli vacation, server breakdown and repair. The average system and orbit size are calculated. Reliability indices are also obtained. Batch arrival multi stage retrial queueing system with priority and collision is analysed in chapter 3. Performances measures are derived.

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