

## CHAPTER – II

### THE (m, N) POLICY FOR A REPAIRABLE BULK ARRIVAL QUEUEING MODEL WITH SECOND MULTI-OPTIONAL SERVICE FACILITY UNDER J-VACATION POLICY

#### INTRODUCTION

The optimal control of queueing system is to determine when and how to change arrival or service rates to optimize some objective function. Most of the time, this corresponds to determine queue levels, at which service should start or stop. The optimal control of queues has been the subject of numerous research papers and it has received considerable attention of many investigators. The researchers working in this area have looked at the problem from different angles and this has given rise to vast and rich literature. Among many control policies, the N-policy introduced by Yadin and Naor (1963) is designed to minimize server switchovers and to avoid excessive frequent use of setups. Tadj and Choudhury (2005) have tried to compile an exhaustive list of all the research on this topic.

The N-policy  $M^X/G/1$  queueing systems with or without server vacations were studied by Lee and Srinivasan (1989) and Lee et al. (1994a, 1994b, 1995). The server set-up (also known as start-up or warm-up) corresponds to the preparatory work of the server before starting a busy period. Combination of N-policy and set-up in queueing system was first considered by Baker (1973). Lee and Park (1997) and Lee et al. (1998) considered  $M/G/1$  and  $M^X/G/1$  queueing systems with early set-up under (m, N)-policy in which the thresholds m and N are used respectively to control the starting condition of a setup time and service time. Lee and Park developed a procedure to find the optimal values of m and N that minimize a linear average cost. They have shown that the double-threshold policy is more beneficial than the single N-policy when the setup cost was excessively high compared with the holding cost.

Lee et al. (2003) have extended Lee and Park's model to batch arrival queueing system with or without server's vacations. These authors analysed the threshold policy for the queueing situations in which the server is assumed to be reliable. However in many real world situations, the server is subject to breakdowns. J.C.Ke (2004) investigated the  $(m, N)$  policy for an unreliable server. These models assume that the server provides single service to the customers and the decomposition property of vacation queues is used to derive the queue length and Probability Generating Functions. Later, Julia Rose Mary, K., (2011) and Jemila Parveen (2012) have analysed the double threshold policy for batch arrival vacation queueing models in which the service discipline involves more than one service. However these authors treated the classical single and multiple vacation policies as two different cases in their work.

The early works on vacation queues concentrated mainly on two standard vacation policies (single and multiple vacation policies) introduced by Levy and Yechalli (1975), framed under the exhaustive service discipline. Presently most of the studies have been devoted to batch arrival vacation models under different vacation policies because of their interdisciplinary characters.

Zhang and Tian (2001), Ke and Chu (2006) have considered a modified vacation policy also termed as J-vacation policy which provides additional flexibility and allows to include the cases of multiple vacation and single vacation in the same framework. This motivates us to analyze the double threshold  $(m, N)$  policy for batch arrival queueing systems under J-vacation policy along with different features in Chapters II to V.

Chapter II analyses a repairable  $M^X/G/1$  queueing system in which all the customers demand the first essential service, whereas only some of them demand second multi-optional services. Chapter III considers variant behaviour of service interrupted customers during breakdown periods. In these chapters the server operates J-vacation policy during idle period.

Chapter IV deals with the batch arrival queueing system with second optional service facility in which, the server after completing each service may take  $i^{\text{th}}$  type of vacation with probability  $p_i$  ( $i = 1$  to  $M$ ) or continue to serve the next customer with complementary probability and, when the system becomes empty the server may take a variant single optional vacation.

Chapter V investigates a reliable  $M^X/G/1$  queue in which the server operates randomized J-vacation policy during idle period and takes optional single vacations during busy period at service completion epochs. In addition to this, the customers if unsatisfied with their services can demand re services.

The main objectives in Chapters II to V are :

- To obtain the steady state results in terms of the probability generating functions for the number of customers in the system corresponding to different states of the system.
- To illustrate the Stochastic decomposition property.
- To deduce some known results as special cases of the present work.
- To study the effects of various system parameters on performance characteristics through numerical illustrations.
- To develop, a cost model in order to determine the optimal threshold values of  $(m, N)$  that correspond to minimum cost.

## **2.1 MATHEMATICAL ANALYSIS OF THE SYSTEM**

### **2.1.1 Model Description**

In the present chapter, the author analyses the bi-level control  $((m, N))$  policy for an  $M^X/G/1$  queueing system with the following specifications.

#### **Arrival Pattern**

Customers arrive in batches in accordance with a time-homogeneous Poisson process with random batch size  $X$ , group arrival rate  $\lambda$  and probability distribution  $g_k = \Pr(X = k)$ ,  $k = 1, 2, 3, \dots$  (i.e.,) the probability that a batch of  $k$  units arrive in an infinitesimal interval  $(t, t+h)$  is  $\lambda g_k h + o(h)$ . The customers who arrive in batches join the system and form a single waiting line based on

the order of the batches. The customers with in a batch are pre-ordered. There is a single server who serves the customers one by one according to the order in the queue during busy period. i.e., First Come First Served (FCFS) queue discipline is followed.

### **Idle Period and Vacation Policy**

A cycle starts whenever the system empties ; the server is deactivated, and immediately leaves the system for vacation. The server operates a bi-level threshold  $(m, N)$  policy and takes atmost  $J$  consecutive vacations. According to the policy, the server is turned off and leaves the system for a vacation of random length  $VI_1$ , as soon as the system empties. After returning from the vacation, if the server finds  $m$  or more customers waiting in the queue then, the server will be immediately activated for setup operations. Otherwise if the server finds less than  $m$  customers at the end of a vacation, then the server leaves for another vacation  $VI_2$ . The server thus repeats the vacations until, either at least  $m$  customers accumulate in the system on returning from a vacation or the number of consecutive vacations taken by the server reaches  $J$  whichever occurs earlier. If the queue length is still less than  $m$  by the end of the  $J^{\text{th}}$  vacation, the server joins the system and stays idle in the system until the queue length reaches atleast  $m$  to start the setup operation. The period during which the server is idle in the system before starting the setup work is called the buildup period. The repeated vacations  $VI_i$  ( $i = 1$  to  $J$ ) are independently and identically distributed random variables with generic representation  $VI$ . The vacation time ( $VI$ ) and setup time ( $D$ ) are assumed to follow general distributions  $VI(x)$  and  $D(x)$  with finite moments.

At the end of the setup period, if the queue length is greater than or equal to  $N$ , then the server begins to serve the customers one at a time exhaustively. Otherwise, the server remains dormant in the system, waiting for the queue length to reach atleast  $N$ , to start a service. Thus the idle period of the server is the sum of vacation period, buildup period, setup period and dormant period. And a busy period starts when the queue length reaches atleast  $N$  at the termination of idle period.

### Busy Period

During busy period, the server provides single First Essential Service (FES)  $S$  to all the arriving customers in the first phase and 'multi-optional' heterogeneous services  $S_i$  ( $1 \leq i \leq C$ ) in the second phase. As soon as the FES is completed, with probability  $r_i$  ( $1 \leq i \leq C$ ), customers may either opt for a certain second service from  $C$  ( $C \geq 1$ ) kinds of services or else with probability  $(1-r)$  ( $r = \sum_{i=1}^C r_i$ ), may opt to leave the system. The single server follows the customers in service, when they move from the first phase to second. The service time random variables  $S$  and  $S_i$  ( $i=1$  to  $C$ ) are assumed to be mutually independent of each other having heterogeneous general law of distributions with distribution functions  $S(x)$  and  $S_i(x)$ , density functions  $s(x)$  and  $s_i(x)$  and Laplace-Stieltjes Transforms (LST)  $S^*(\theta)$  and  $S_i^*(\theta)$ ,  $i=1$  to  $C$  respectively.

### Breakdown Period

The server is subjected to breakdowns at any time while serving customers. It is assumed that the server's life time follows exponential distribution with parameter  $a$  in the FES. In the second multi-optional service (SOS), the service fails at an exponential rate  $a_i$  ( $1 \leq i \leq C$ ). Whenever the breakdowns occur, the server is sent for repair immediately. The customer just being served before server breaks down, waits for the server in the service facility to complete the remaining service. The repair time distributions  $R$  and  $R_i$  ( $1 \leq i \leq C$ ) of both service phases are arbitrarily distributed with probability distribution functions  $R(y)$  and  $R_i(y)$  ( $1 \leq i \leq C$ ), probability density functions  $r(y)$  and  $r_i(y)$  with finite moments  $E(R^k)$  and  $E(R_i^k)$ ,  $k = 1, 2$  respectively. Immediately after the server is fixed, he returns to the system and serves the customer who is waiting in the service facility to complete the remaining service. It is assumed that the service time for a customer is cumulative and after repair the server is as good as new.

The busy period and breakdown period constitute a completion period and the idle period and completion period will determine a cycle.

The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The system is denoted by  $M_{(m,N)}^X / G_{(1,i(1 \leq i \leq C))} / 1 / \text{Breakdown} / J\text{-Vacation}$ .

The steady-state system size equations under the steady-state condition are derived by using supplementary variable technique. The PGF of the system size is obtained in a closed form so that various performance measures can be derived from it. The following notations are used to discuss the model.

**TABLE 2.0**

$N_S(t)$	The system size at time t
$\lambda$	Group arrival rate
$X$	Group size random variable
$g_k$	$\Pr(X = k), k = 1, 2, 3, \dots$
$g_n^{(i)}$	$i$ – fold convolution of $\{g_n\}$ with itself and $g_n^{(0)} = \delta_{n,0}$ $g_n^{(i)} = \sum_{k=1}^{n-1} g_{n-k}^{(i-1)} g_k$ where $\delta_{m,n}$ is the Kroncker – Delta function
$X(z)$	Probability generating function of $X$ .

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its  $k^{\text{th}}$  moments of the  $RV_s$  are listed below :

	RV	CDF $F(x)$	PDF $f(x)$	LST $F^*(\theta)$	$k^{\text{th}}$ moments $k = 1, 2$ $E(F^k)$
Vacation time	VI	$VI(x)$	$vI(x)$	$VI^*(\theta)$	$E(VI^k)$
Setup time	D	$D(x)$	$d(x)$	$D^*(\theta)$	$E(D^k)$
FES	S	$S(x)$	$s(x)$	$S^*(\theta)$	$E(S^k)$
Second Optional Services (SOS)	$S_i$	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k), 1 \leq i \leq C$
Repair time in FES	R	$R(y)$	$r(y)$	$R^{*1}(\theta_1)$	$E(R^k)$
Repair time in SOS	$R_i$	$R_i(y)$	$r_i(y)$	$R_i^{*1}(\theta_1)$	$E(R_i^k), 1 \leq i \leq C$

where

$$F^*(\theta) = \int_0^{\infty} e^{-\theta x} f(x) dx = \int_0^{\infty} e^{-\theta x} d(F(x))$$

Let  $VI^{\circ}(t)$ ,  $D^{\circ}(t)$ ,  $S^{\circ}(t)$ ,  $S_i^{\circ}(t)$ ,  $R^{\circ}(t)$  and  $R_i^{\circ}(t)$  denote the remaining times of the random variables ; vacation time, setup time, FES time, SOS time, repair time due to FES and repair time due to SOS respectively at time  $t$  with  $1 \leq i \leq C$ .

Further the different states of the server are denoted by :

$$Y(t) = \begin{cases} 0, & \text{if the server is on vacation} \\ 1, & \text{the server is in buildup state} \\ 2, & \text{the server is in setup state} \\ 3, & \text{the server is in dormant state} \\ 4, & \text{the server is busy with FES} \\ 5, & \text{the server is busy with SOS} \\ 6, & \text{the server is in repair mode during FES} \\ 7, & \text{the server is in repair mode during SOS} \end{cases}$$

The supplementary variables are introduced in order to obtain a bivariate Markov process  $\{N_S(t), \delta(t)\}$ , where  $N_S(t)$  denotes the system size random variable and  $\delta(t) = (VI^{\circ}(t), 0, D^{\circ}(t), 0, S^{\circ}(t), S_i^{\circ}(t), R^{\circ}(t), R_i^{\circ}(t) \ 1 \leq i \leq C)$  according as  $Y(t) = 0, 1, 2, 3, 4, 5, 6, 7$  respectively. Let  $Z(t) = j$ , ( $j = 1, 2, \dots, J$ ) denote that the server is on  $j^{\text{th}}$  vacation at time  $t$  counting from the idle period initiation point. Let,

$$QI_{n,j}(x, t) dt = \Pr \{N_S(t) = n, x < VI^{\circ}(t) \leq x + dt, Y(t) = 0, Z(t) = j\}, \ 1 \leq j \leq J, \\ n \geq 0$$

$$PI_n(t) = \Pr \{N_S(t) = n, Y(t) = 1\}, \ 0 \leq n \leq m-1$$

$$SE_n(x, t) dt = \Pr \{N_S(t) = n, x < D^{\circ}(t) \leq x + dt, Y(t) = 2\}, \ n \geq m$$

$$U_n(t) = \Pr \{N_S(t) = n, Y(t) = 3\}, \ m \leq n \leq N-1$$

$$P_{1,n}(x, t) dt = \Pr \{N_S(t) = n, x < S^{\circ}(t) \leq x + dt, Y(t) = 4\}, \ n \geq 1$$

$$P_{2,n}^i(x, t) dt = \Pr \{N_S(t) = n, x < S_i^{\circ}(t) \leq x + dt, Y(t) = 5\}, \ n \geq 1, \ 1 \leq i \leq C$$

$$BR_{1,n}(x, y, t) dt = \Pr \{N_S(t) = n, S^{\circ}(t) = x, y < R^{\circ}(t) \leq y + dt, Y(t) = 6\}, \ n \geq 1$$

$$BR_{2,n}^i(x, y, t) dt = \Pr \{N_S(t) = n, S_i^{\circ}(t) = x, y < R_i^{\circ}(t) \leq y + dt, Y(t) = 7\}, \\ n \geq 1, \ 1 \leq i \leq C$$

Then the interpretation of the probabilities at time  $t$  are given by :

	The joint probability that at time $t$ there are $n$ customers in the system and	
$QI_{n,j}(x, t) dt$	the server is in $j^{\text{th}}$ vacation with the remaining vacation time of the server is between $x$ and $x + dt$	$n \geq 0,$ $1 \leq j \leq J$
$SE_n(x, t) dt$	The server is in setup state and the remaining setup time of the server is between $x$ and $x + dt$	$n \geq m$
$P_{1,n}(x, t) dt$	the server is busy with FES and the remaining service time lies between $x$ and $x + dt$	$n \geq 1$
$P_{2,n}^i(x, t) dt$	the server is busy and a customer being served is in the $i^{\text{th}}$ (second optional service) channel with remaining service time between $x$ and $x + dt$	$n \geq 1,$ $1 \leq i \leq C$
$BR_{1,n}(x, y, t) dt$	the remaining service time for the customer under FES is equal to $x$ , and the server is being repaired with the remaining repair time between $y$ and $y + dt$	$x \neq 0, n \geq 1$
$BR_{2,n}^i(x, y, t) dt$	The remaining service time for the customer under $i^{\text{th}}$ (SOS) channel is equal to $x$ , and the server is being repaired with the remaining repair time between $y$ and $y + dt$	$x \neq 0, n \geq 1$

$PI_n(t)$  ( $0 \leq n \leq m-1$ ) denotes the probability that the server is in buildup state and  $U_n(t)$  ( $m \leq n \leq N-1$ ) denotes the probability that the server is in dormant state.

Further  $QI_{n,j}(0, t)$ ,  $SE_n(0, t)$ ,  $P_{1,n}(0, t)$ ,  $P_{2,n}^i(0, t)$ ,  $BR_{1,n}(x, 0, t)$  and  $BR_{2,n}^i(x, 0, t)$ ,  $i = 1$  to  $C$  denote the probability that there are  $n$  customers in the system at the termination of vacation period, setup period and at the termination of the corresponding service time and repair time respectively. Assuming that at steady-state, probabilities are independent of time  $t$ , we have,

$\lim_{t \rightarrow \infty} \frac{\partial}{\partial x} QI_{n,j}(x, t) = \frac{d}{dx} QI_{n,j}(x)$	$\lim_{t \rightarrow \infty} \frac{\partial}{\partial x} SE_n(x, t) = \frac{d}{dx} SE_n(x)$
$\lim_{t \rightarrow \infty} \frac{\partial}{\partial x} P_{1,n}(x, t) = \frac{d}{dx} P_{1,n}(x)$	$\lim_{t \rightarrow \infty} \frac{\partial}{\partial x} P_{2,n}^i(x, t) = \frac{d}{dx} P_{2,n}^i(x), i = 1 \text{ to } C$
$\lim_{t \rightarrow \infty} \frac{\partial}{\partial y} BR_{1,n}(x, y, t) = \frac{\partial}{\partial y} BR_{1,n}(x, y)$	$\lim_{t \rightarrow \infty} \frac{\partial}{\partial y} BR_{2,n}^i(x, y, t) = \frac{\partial}{\partial y} BR_{2,n}^i(x, y), i = 1 \text{ to } C$
$\lim_{t \rightarrow \infty} \left( \frac{\partial}{\partial t} P_{1,n}(x, t) = \frac{\partial}{\partial t} P_{2,n}^i(x, t) = \frac{\partial}{\partial t} SE_n(x, t) = 0 \right)$	$\lim_{t \rightarrow \infty} \left( \frac{\partial}{\partial t} QI_n(x, t) = \frac{\partial}{\partial t} BR_{1,n}(x, y, t) = \frac{\partial}{\partial t} BR_{2,n}^i(x, y, t) = 0 \right)$
$\lim_{t \rightarrow \infty} PI_n(t) = PI_n$	$\lim_{t \rightarrow \infty} U_n(t) = U_n$
$\lim_{t \rightarrow \infty} SE_n(x, t) = SE_n(x)$	$\lim_{t \rightarrow \infty} QI_{n,j}(x, t) = QI_{n,j}(x)$
$\lim_{t \rightarrow \infty} P_{1,n}(x, t) = P_{1,n}(x)$	$\lim_{t \rightarrow \infty} P_{2,n}^i(x, t) = P_{2,n}^i(x)$
$\lim_{t \rightarrow \infty} BR_{1,n}(x, y, t) = BR_{1,n}(x, y)$	$\lim_{t \rightarrow \infty} BR_{2,n}^i(x, y, t) = BR_{2,n}^i(x, y), x \geq 0, y \geq 0$

The steady-state equations are obtained for the queueing system, using the supplementary variable technique and following the arguments of Alfa and Srinivasa Rao (2000) and Lee et al. (1994a).

## 2.1.2 The Steady State System Size Equations

### Vacation States

$$-\frac{d}{dx} Q_{I_{0,1}}(x) = -\lambda Q_{I_{0,1}}(x) + \bar{P}(0) vI(x)$$

$$\text{where } \bar{P}(0) = (1-r) P_{1,1}(0) + \sum_{i=1}^C P_{2,1}^i(0), \quad r = \sum_{i=1}^C r_i$$

$$-\frac{d}{dx} Q_{I_{0,j}}(x) = -\lambda Q_{I_{0,j}}(x) + Q_{I_{0,j-1}}(0) vI(x), \quad 2 \leq j \leq J$$

$$-\frac{d}{dx} Q_{I_{n,j}}(x) = -\lambda Q_{I_{n,j}}(x) + \lambda \sum_{k=1}^n Q_{I_{n-k,j}}(x) g_k + (1 - \delta_{1,j}) Q_{I_{n,j-1}}(0) vI(x),$$

$$1 \leq n \leq m-1, \quad 1 \leq j \leq J$$

where  $\delta_{m,n}$  is the Kronecker delta defined by  $\delta_{m,n} = 0$ , if  $m \neq n$ ;  $\delta_{m,n} = 1$ , if  $m = n$

$$-\frac{d}{dx} Q_{I_{n,j}}(x) = -\lambda Q_{I_{n,j}}(x) + \lambda \sum_{k=1}^n Q_{I_{n-k,j}}(x) g_k, \quad n \geq m, \quad 2 \leq j \leq J$$

### Buildup states

$$\lambda P_{I_n} = Q_{I_{n,J}}(0) + (1 - \delta_{0,n}) \lambda \sum_{k=1}^n P_{I_{n-k}} g_k, \quad 0 \leq n \leq m-1$$

### Setup states

$$-\frac{d}{dx} SE_n(x) = -\lambda SE_n(x) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} SE_{n-k}(x) g_k + \sum_{j=1}^J Q_{I_{n,j}}(0) dx$$

$$+ \lambda \sum_{k=n-m+1}^n P_{I_{n-k}} g_k d(x), \quad n \geq m$$

### Dormant states

$$\lambda U_n = SE_n(0) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} U_{n-k} g_k, \quad m \leq n \leq N-1$$

## Busy States

### Busy with FES

$$-\frac{d}{dx} P_{1,n}(x) = -(\lambda + a) P_{1,n}(x) + (1 - r) P_{1,n+1}(0) s(x) + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{1,n-k}(x) g_k + \sum_{i=1}^C P_{2,n+1}^i(0) s(x) + BR_{1,n}(x, 0),$$

$$1 \leq n \leq N-1$$

$$-\frac{d}{dx} P_{1,n}(x) = -(\lambda + a) P_{1,n}(x) + (1 - r) P_{1,n+1}(0) s(x) + \lambda \sum_{k=1}^{n-1} P_{1,n-k}(x) g_k + \sum_{i=1}^C P_{2,n+1}^i(0) s(x) + BR_{1,n}(x, 0) + SE_n(0) s(x) + \lambda \sum_{k=n-N+1}^{n-m} U_{n-k} g_k s(x),$$

$$n \geq N$$

### Busy with $i^{\text{th}}$ type service in SOS

$$-\frac{d}{dx} P_{2,n}^i(x) = -(\lambda + a_i) P_{2,n}^i(x) + r_i P_{1,n}(0) s_i(x) + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{2,n-k}^i(x) g_k + BR_{2,n}^i(x, 0),$$

$$n \geq 1, 1 \leq i \leq C$$

## Breakdown states :

### Breakdown during FES

$$-\frac{\partial}{\partial y} BR_{1,n}(x, y) = -\lambda BR_{1,n}(x, y) + a P_{1,n}(x) r(y) + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{1,n-k}(x, y) g_k,$$

$$n \geq 1$$

### Breakdown during $i^{\text{th}}$ type of SOS ( $1 \leq i \leq C$ ), ( $n \geq 1$ )

$$-\frac{\partial}{\partial y} BR_{2,n}^i(x, y) = -\lambda BR_{2,n}^i(x, y) + a_i P_{2,n}^i(x) r_i(y) + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{2,n-k}^i(x, y) g_k$$

The LST of the above equations are obtained by using the definition of Laplace-Stieltjes transformation and its properties regarding derivatives. The LST of the density functions are defined in previous table. The notations of the LST of probability distribution functions are listed in the following Table.

Probability distribution	LST
$QI_n(x)$	$QI_n^*(\theta) = \int_0^{\infty} e^{-\theta x} QI_n(x) dx$
$SE_n(x)$	$SE_n^*(\theta)$
$P_{1,n}(x)$	$P_{1,n}^*(\theta)$
$P_{2,n}^i(x)$	$P_{2,n}^{i*}(\theta)$
$BR_{1,n}(x, y), BR_{2,n}^i(x, y)$	$BR_{1,n}^*(\theta, y), BR_{2,n}^{i*}(\theta, y)$
$BR_{1,n}^*(\theta, y), BR_{2,n}^{i*}(\theta, y)$	$BR_{1,n}^{**1}(\theta, \theta_1) = \int_0^{\infty} e^{-\theta_1 y} BR_{1,n}^*(\theta, y) dy$ $BR_{2,n}^{i**1}(\theta, \theta_1) = \int_0^{\infty} e^{-\theta_1 y} BR_{2,n}^{i*}(\theta, y) dy$

Thus the LST of the equations w.r. to x are respectively given by,

$$\theta QI_{0,1}^*(\theta) - QI_{0,1}(0) = \lambda QI_{0,1}^*(\theta) - \bar{P}(0) VI^*(\theta) \quad (2.1)$$

$$\theta QI_{0,j}^*(\theta) - QI_{0,j}(0) = \lambda QI_{0,j}^*(\theta) - QI_{0,j-1}(0) VI^*(\theta), \quad 2 \leq j \leq J \quad (2.2)$$

$$\theta QI_{n,j}^*(\theta) - QI_{n,j}(0) = \lambda QI_{n,j}^*(\theta) - \lambda \sum_{k=1}^n QI_{n-k,j}^*(\theta) g_k - (1 - \delta_{1,j}) QI_{n,j-1}(0) VI^*(\theta),$$

$$1 \leq n \leq m-1, 1 \leq j \leq J \quad (2.3)$$

$$\theta QI_{n,j}^*(\theta) - QI_{n,j}(0) = \lambda QI_{n,j}^*(\theta) - \lambda \sum_{k=1}^n QI_{n-k,j}^*(\theta) g_k, \quad n \geq m, 2 \leq j \leq J \quad (2.4)$$

$$\lambda PI_n = QI_{n,J}(0) + (1 - \delta_{0,n}) \lambda \sum_{k=1}^n PI_{n-k} g_k, \quad 0 \leq n \leq m-1 \quad (2.5)$$

$$\theta SE_n^*(\theta) - SE_n(0) = \lambda SE_n^*(\theta) - (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} SE_{n-k}^*(\theta) g_k$$

$$- \sum_{j=1}^J QI_{n,j}(0) D^*(\theta) - \lambda \sum_{k=n-m+1}^n PI_{n-k} g_k D^*(\theta),$$

$$n \geq m \quad (2.6)$$

$$\lambda U_n = SE_n(0) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} U_{n-k} g_k, \quad m \leq n \leq N-1 \quad (2.7)$$

$$\theta P_{1,n}^*(\theta) - P_{1,n}(0) = (\lambda + a) P_{1,n}^*(\theta) - (1-r) P_{1,n+1}(0) S^*(\theta)$$

$$- (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta) g_k - \sum_{i=1}^C P_{2,n+1}^i(0) S^*(\theta)$$

$$- BR_{1,n}^*(\theta, 0), \quad 1 \leq n \leq N-1 \quad (2.8)$$

$$\begin{aligned}
\theta P_{1,n}^*(\theta) - P_{1,n}(0) &= (\lambda + a) P_{1,n}^*(\theta) - (1 - r) P_{1,n+1}(0) S^*(\theta) \\
&\quad - \lambda \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta) g_k - \sum_{i=1}^C P_{2,n+1}^i(0) S^*(\theta) - BR_{1,n}^*(\theta, 0) \\
&\quad - SE_n(0) S^*(\theta) - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k} g_k S^*(\theta), \quad n \geq N \quad (2.9)
\end{aligned}$$

$$\begin{aligned}
\theta P_{2,n}^{i*}(\theta) - P_{2,n}^i(0) &= (\lambda + a_i) P_{2,n}^{i*}(\theta) - r_i P_{1,n}(0) S_i^*(\theta) \\
&\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{2,n-k}^{i*}(\theta) g_k - BR_{2,n}^{i*}(\theta, 0), \\
&\hspace{20em} n \geq 1, \quad 1 \leq i \leq C \quad (2.10)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial}{\partial y} BR_{1,n}^*(\theta, y) &= -\lambda BR_{1,n}^*(\theta, y) + a P_{1,n}^*(\theta) r(y) \\
&\quad + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{1,n-k}^*(\theta, y) g_k, \quad n \geq 1 \quad (2.11)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial}{\partial y} BR_{2,n}^{i*}(\theta, y) &= -\lambda BR_{2,n}^{i*}(\theta, y) + a_i P_{2,n}^{i*}(\theta) r_i(y) \\
&\quad + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{i*}(\theta, y) g_k, \quad n \geq 1 \quad (2.12)
\end{aligned}$$

Taking the LST w.r.to  $y$ , the equations (2.11) and (2.12) lead to

$$\begin{aligned}
\theta_1 BR_{1,n}^{**1}(\theta, \theta_1) - BR_{1,n}^*(\theta, 0) &= \lambda BR_{1,n}^{**1}(\theta, \theta_1) - a P_{1,n}^*(\theta) R^*(\theta_1) \\
&\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{1,n-k}^{**1}(\theta, \theta_1) g_k, \quad n \geq 1 \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
\theta_1 BR_{2,n}^{i**1}(\theta, \theta_1) - BR_{2,n}^{i*}(\theta, 0) &= \lambda BR_{2,n}^{i**1}(\theta, \theta_1) - a_i P_{2,n}^{i*}(\theta) R_i^*(\theta_1) \\
&\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{i**1}(\theta, \theta_1) g_k, \quad n \geq 1 \quad (2.14)
\end{aligned}$$

### 2.1.3 Probability Generating Functions

Now to obtain the PGFs of the number of customers in the system when the server is in different states, the following partial PGFs are defined.

$PI(z) = \sum_{n=0}^{m-1} PI_n z^n$	$U(z) = \sum_{n=m}^{N-1} U_n z^n$
$QI_j^*(z, \theta) = \sum_{n=0}^{\infty} QI_{n,j}^*(\theta) z^n$	$QI_j(z, 0) = \sum_{n=0}^{\infty} QI_{n,j}(0) z^n, 1 \leq j \leq J$
$SE^*(z, \theta) = \sum_{n=m}^{\infty} SE_n^*(\theta) z^n$	$SE(z, 0) = \sum_{n=m}^{\infty} SE_n(0) z^n$
$P_1^*(z, \theta) = \sum_{n=1}^{\infty} P_{1,n}^*(\theta) z^n$	$P_1(z, 0) = \sum_{n=1}^{\infty} P_{1,n}(0) z^n$
$BR_1^{**1}(z, \theta, \theta_1) = \sum_{n=1}^{\infty} B_{1,n}^{**1}(\theta, \theta_1) z^n$	$BR_1^*(z, \theta, 0) = \sum_{n=1}^{\infty} BR_{1,n}^*(\theta, 0) z^n$
$P_2^{i*}(z, \theta) = \sum_{n=1}^{\infty} P_{2,n}^{i*}(\theta) z^n$	$P_2^i(z, 0) = \sum_{n=1}^{\infty} P_{2,n}^i(0) z^n$
$BR_2^{i**1}(z, \theta, \theta_1) = \sum_{n=1}^{\infty} BR_{2,n}^{i**1}(\theta, \theta_1) z^n$	$BR_2^{i*}(z, \theta, 0) = \sum_{n=1}^{\infty} BR_{2,n}^{i*}(\theta, 0) z^n, 1 \leq i \leq C$

Multiplying the transformed equations by suitable powers of  $z$  and adding over  $n$  and using the identities listed in Appendix I, the partial PGFs are obtained through some algebraic manipulations.

Equations (2.1) and (2.3) at  $j = 1$  lead to,

$$(\theta - w_X(z)) QI_1^*(z, \theta) = QI_1(z, 0) - \bar{P}(0) VI^*(\theta) \quad (2.15)$$

$$\text{where } w_X(z) = \lambda (1 - X(z)) \quad (2.15.1)$$

By evaluating  $QI_1(z, 0)$  at  $\theta = w_X(z)$ , the generating functions of the system size at vacation termination epoch and at arbitrary epoch, when the server is on first vacation are given by

$$QI_1(z, 0) = \bar{P}(0) VI^*(w_X(z)) \quad (2.16)$$

$$\text{and } QI_1^*(z, \theta) = \bar{P}(0) \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} \quad (2.17)$$

similarly equations (2.2), (2.3) and (2.4) lead to

$$QI_j(z, 0) = VI^*(w_X(z)) \sum_{n=0}^{m-1} QI_{n,j-1}(0) z^n, \quad 2 \leq j \leq J \quad (2.18)$$

$$\text{and } QI_j^*(z, \theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} \sum_{n=0}^{m-1} QI_{n,j-1}(0) z^n, \quad 2 \leq j \leq J \quad (2.19)$$

Equations (2.18) and (2.19) respectively give the generating functions of the system size at vacation termination epoch and at arbitrary epoch, when the server is on  $j^{\text{th}}$  vacation. The sum of the equations (2.16) and (2.18) over  $j = 1$  to  $J$  yields

$$QI(z, 0) = \sum_{j=1}^J QI_j(z, 0) = VI^*(w_X(z))y_v(z) \quad (2.20)$$

$$\text{where } y_v(z) = \left( \sum_{j=1}^{J-1} \sum_{n=0}^{m-1} QI_{n,j}(0) z^n + \bar{P}(0) \right) \quad (2.21)$$

Similarly the sum of the equations (2.17) and (2.19) gives,

$$QI^*(z, \theta) = \sum_{j=1}^J QI_j^*(z, \theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} y_v(z) \quad (2.22)$$

Multiplying equation (2.6) by  $z^n$  and summing over  $m$  to  $\infty$ , it is found that

$$\begin{aligned} (\theta - w_X(z)) SE^*(z, \theta) &= SE(z, 0) - \sum_{n=m}^{\infty} z^n \sum_{j=1}^J QI_{n,j}(0) D^*(\theta) \\ &\quad - \lambda \sum_{n=m}^{\infty} z^n \sum_{k=n-m+1}^n PI_{n-k} g_k D^*(\theta) \end{aligned} \quad (2.23)$$

Adding equation (2.5) over  $n = 0$  to  $m - 1$ , we have,

$$\lambda PI(z) = \sum_{n=0}^{m-1} QI_{n,J}(0) z^n + \lambda \sum_{n=1}^{m-1} z^n \sum_{k=1}^n PI_{n-k} g_k \quad (2.24)$$

Now, adding the equation (2.24) multiplied by  $(-D^*(\theta))$ , with the equation (2.23), the PGF of the system size at setup termination epoch and arbitrary epoch are obtained as,

$$SE(z, 0) = D^*(w_X(z)) [(VI^*(w_X(z)) - 1) y_v(z) + \bar{P}(0) - PI(z) w_X(z)] \text{ and } (2.25)$$

$$SE^*(z, \theta) = \frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))} [(VI^*(w_X(z)) - 1) y_v(z) + \bar{P}(0) - PI(z) w_X(z)] \quad (2.26)$$

If  $\alpha I_n$  denotes the probability that  $n$  customers arrive during a vacation time (VI) then

$$\begin{aligned} \alpha I_n &= \int_0^{\infty} \sum_{k=0}^n e^{-\lambda t} \frac{(\lambda t)^k}{k!} g_n^{(k)} d(VI(t)) \text{ and} \\ VI^*(w_X(z)) &= \sum_{n=0}^{\infty} \alpha I_n z^n \text{ (Gross and Harris, 1985)} \end{aligned} \quad (2.26.0)$$

Substituting for  $VI^*(w_X(z))$ , equations (2.16) and (2.18) respectively imply,

$$\sum_{n=0}^{\infty} QI_{n,1}(0)z^n = \bar{P}(0) \sum_{n=0}^{\infty} \alpha I_n z^n \quad (2.26.1)$$

$$\text{and } \sum_{n=0}^{\infty} QI_{n,j}(0)z^n = \left( \sum_{i=0}^{m-1} QI_{i,j-1}(0)z^i \right) \left( \sum_{k=0}^{\infty} \alpha I_k z^k \right), \quad 2 \leq j \leq J \quad (2.26.2)$$

The coefficients of  $z^n$  on either sides of the equations (2.26.1) and (2.26.2) give

$$QI_{n,1}(0) = \alpha I_n \bar{P}(0), \quad n \geq 0 \quad (2.27)$$

$$QI_{n,j}(0) = \sum_{k=0}^n QI_{n-k,j-1}(0) \alpha I_k, \quad 2 \leq j \leq J, 0 \leq n \leq m-1 \quad (2.28)$$

$$\text{At } n = 0, QI_{0,j}(0) = \alpha I_0 QI_{0,j-1}(0), \quad 2 \leq j \leq J$$

$$\text{By recursion, it follows that } QI_{0,j}(0) = \bar{P}(0)(\alpha I_0)^j, \quad 1 \leq j \leq J \quad (2.28.1)$$

Equation (2.28) recursively implies

$$QI_{n,j}(0) = \alpha I_n^{(j)} \bar{P}(0), \quad 2 \leq j \leq J, 0 \leq n \leq m-1 \quad (2.28.2)$$

where the convolution product  $\alpha I_n^{(i)} = \sum_{k=0}^n \alpha I_{n-k}^{(i-1)} \alpha I_k$ ,  $1 \leq i \leq J, 0 \leq n \leq m-1$

In particular,  $\alpha I_0^{(i)} = (\alpha I_0)^i$

Now adding equations (2.27) and (2.28) over  $j = 1$  to  $J$ , for  $0 \leq n \leq m-1$ ,

$$\sum_{j=1}^J QI_{n,j}(0) = \sum_{j=1}^{J-1} \sum_{k=0}^n QI_{n-k,j}(0) \alpha I_k + \bar{P}(0) \alpha I_n \quad (2.29)$$

By letting  $\delta_0 = \bar{P}(0) + \sum_{j=1}^{J-1} QI_{0,j}(0)$  and  $\delta_n = \sum_{j=1}^{J-1} QI_{n,j}(0)$ ,  $1 \leq n \leq m-1$

the equation (2.29) leads to

$$\delta_0 = \frac{\bar{P}(0)(1 - \alpha I_0^J)}{1 - \alpha I_0}, \quad \text{at } n = 0 \quad (2.30)$$

and for  $1 \leq n \leq m-1$ ,  $\delta_n + QI_{n,J}(0) = \left( \sum_{k=0}^{n-1} \sum_{j=1}^{J-1} QI_{n-k,j}(0) \alpha I_k + \alpha I_n \delta_0 \right)$

$$\text{i.e., } \delta_n = \alpha I_n \delta_0 + \sum_{k=0}^{n-1} \delta_{n-k} \alpha I_k - QI_{n,J}(0) = \sum_{k=0}^n \alpha I_k \delta_{n-k} - QI_{n,J}(0) \quad (2.30.1)$$

For further simplification the Theorem 2.1 is used.

**Theorem : 2.1**

Let  $\beta_0 = 1 - \alpha I_0^{(j)}$  and  $\beta_n = \sum_{i=1}^n \frac{\alpha I_i \beta_{n-i}}{1 - \alpha I_0} - \alpha I_n^{(j)}$ , for  $1 \leq n \leq m-1$ .

$$\text{Then } \delta_n = \frac{\bar{P}(0) \beta_n}{1 - \alpha I_0}, \quad 0 \leq n \leq m-1 \quad (2.31)$$

**Proof**

Equation (2.30) shows that the equation (2.31) is true for  $n = 0$ . Using equation (2.28.2) in equation (2.30.1),

$$(1 - \alpha I_0) \delta_n = \sum_{k=1}^n \alpha I_k \delta_{n-k} - \alpha I_n^{(j)} \bar{P}(0)$$

Thus by induction, substituting for  $\delta_{n-k}$ ,

$$(1 - \alpha I_0) \delta_n = \bar{P}(0) \left( \sum_{k=1}^n \frac{\beta_{n-k} \alpha I_k}{1 - \alpha I_0} - \alpha I_n^{(j)} \right) = \bar{P}(0) \beta_n$$

$$\text{i.e., } \delta_n = \frac{\bar{P}(0) \beta_n}{1 - \alpha I_0}, \quad 1 \leq n \leq m-1$$

Thus equation (2.21) leads to

$$y_v(z) = \sum_{n=0}^{m-1} \delta_n z^n = \bar{P}(0) \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} \quad (2.31.1)$$

To find the PGF of the number of customers in the system when the server is in buildup period, the following result is used.

**Theorem : 2.2**

Let  $\pi_0 = 1$ ,  $\pi_n = \sum_{k=1}^n \pi_{n-k} g_k$ ,  $\psi_0 = \alpha I_0^{(j)}$  and  $\psi_n = \sum_{k=0}^n \alpha I_k^{(j)} \pi_{n-k}$ ,  $0 \leq n \leq m-1$ .

$$\text{Then } PI(z) = \bar{P}(0) \psi(z), \text{ where } \psi(z) = \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} \psi_n z^n \quad (2.31.2)$$

**Proof**

Equation (2.5) recursively implies,

$$\lambda PI_n = \sum_{k=0}^n QI_{k,j}(0) \pi_{n-k}, \quad 0 \leq n \leq m-1$$

Substituting for  $QI_{k,J}(0)$ , from (2.28.2),

$$\lambda PI_n = \bar{P}(0) \sum_{k=0}^n \pi_{n-k} \alpha I_k^{(J)} = \bar{P}(0) \psi_n, 0 \leq n \leq m-1$$

$$\text{Hence } PI(z) = \bar{P}(0) \psi(z) \quad (2.32)$$

**Corollary : For  $(0 \leq n \leq m-1)$ ,**

$$(i) \quad \sum_{i=0}^{n-1} \psi_i g_{n-i} = \sum_{i=0}^{n-1} \alpha I_i^{(J)} \pi_{n-i} \quad (ii) \quad \alpha I_n^{(J)} + \sum_{i=0}^{n-1} \psi_i g_{n-i} = \psi_n$$

Substituting for  $\psi_{n-k}$  and using the definition of  $\pi_n$ ,

$$\sum_{i=0}^{n-1} \psi_i g_{n-i} = \sum_{k=1}^n g_k \psi_{n-k} = \sum_{k=1}^n g_k \left( \sum_{i=0}^{n-k} \alpha I_i^{(J)} \pi_{n-k-i} \right) = \sum_{k=0}^{n-1} \alpha I_k^{(J)} \sum_{i=1}^{n-k} g_i \pi_{n-k-i} = \sum_{k=0}^{n-1} \alpha I_k^{(J)} \pi_{n-k}$$

$$\text{Hence } \alpha I_n^{(J)} + \sum_{i=0}^{n-1} \psi_i g_{n-i} = \sum_{k=0}^n \alpha I_k^{(J)} \pi_{n-k} = \psi_n$$

Using equations (2.31.1) and (2.32), the equations (2.20), (2.22), (2.25) and (2.26), respectively imply

$$QI(z, 0) = VI^*(w_X(z)) \beta(z) \bar{P}(0) \quad (2.33)$$

$$QI^*(z, \theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{\theta - w_X(z)} \beta(z) \bar{P}(0) \quad (2.33.1)$$

$$SE(z, 0) = D^*(w_X(z)) [(VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)] \bar{P}(0) \quad (2.33.2)$$

$$SE^*(z, \theta) = \frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))} [(VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)] \bar{P}(0) \quad (2.33.3)$$

$$\text{where } \beta(z) = \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} \quad (2.34)$$

The equations (2.13), (2.14) and (2.8) to (2.10), are used to calculate the generating functions of the system size when the server is in breakdown states and busy states, at termination epoch and at arbitrary epoch.

The equation (2.13) implies

$$(\theta_1 - w_X(z)), BR_1^{**1}(z, \theta, 0) = BR_1^*(z, \theta, 0) - a P_i^*(z, \theta) R^{*1}(\theta_1) \quad (2.35)$$

$$\text{At } (\theta_1 = w_X(z)), BR_1^*(z, \theta, 0) = a P_i^*(z, \theta) R^{*1}(w_X(z)) \quad (2.36)$$

Substituting the value of  $BR_1^*(z, \theta, 0)$  in equation (2.35),

$$BR_1^{*1}(z, \theta, \theta_1) = \frac{a P_1^*(z, \theta)(R^{*1}(w_X(z)) - R^{*1}(\theta_1))}{(\theta_1 - w_X(z))} \quad (2.37)$$

By similar argument, the following equations are obtained from (2.14),

$$BR_2^{i*}(z, \theta, 0) = a_i P_2^{i*}(z, \theta) R_i^{*1}(w_X(z)) \quad (2.38)$$

$$BR_2^{i*1}(z, \theta, \theta_1) = \frac{a_i P_2^{i*}(z, \theta)(R_i^{*1}(w_X(z)) - R_i^{*1}(\theta_1))}{(\theta_1 - w_X(z))}, \quad 1 \leq i \leq C \quad (2.39)$$

Equation (2.10) gives the generating functions of the system size when the server is busy with  $i^{\text{th}}$  type of second optional services (SOS), at the service completion epoch and at arbitrary epoch,

Multiplying the equation (2.10) by suitable powers of  $z^n$  and adding  
 $(\theta - (a_i + w_X(z))) P_2^{i*}(z, 0) = P_2^i(z, 0) - r_i S_i^*(\theta) P_1(z, 0) - BR_2^{i*}(z, \theta, 0)$

Using equation (2.38),

$$(\theta - h_{a_i}(w_X(z))) P_2^{i*}(z, \theta) = P_2^i(z, 0) - r_i S_i^*(\theta) P_1(z, 0)$$

where  $h_{a_i}(w_X(z)) = w_X(z) + a_i (1 - R_i^{*1}(w_X(z)))$

At  $\theta = h_{a_i}(w_X(z))$

$$P_2^i(z, 0) = r_i S_i^*(h_{a_i}(w_X(z))) P_1(z, 0) \quad (2.40)$$

$$P_2^{i*}(z, \theta) = \frac{r_i P_1(z, 0)(S_i^*(h_{a_i}(w_X(z))) - S_i^*(\theta))}{(\theta - h_{a_i}(w_X(z)))}, \quad 1 \leq i \leq C \quad (2.41)$$

The generating function of the system size at service completion epoch and at arbitrary epoch, when the server is busy with FES are derived from equation (2.9) :

$$\begin{aligned} (\theta - h_a(w_X(z))) P_1^*(z, \theta) &= \frac{P_1(z, 0)}{z} [z - (1 - r) S^*(\theta)] - \sum_{i=1}^C P_2^i(z, 0) \frac{S^*(\theta)}{z} \\ &\quad + S^*(\theta) P(0) - S^*(\theta) \sum_{n=N}^{\infty} SE_n(0) z^n \\ &\quad - \lambda S^*(\theta) \sum_{n=N}^{\infty} z^n \left( \sum_{k=n-N+1}^{n-m} U_{n-k} g_k \right) \end{aligned} \quad (2.42)$$

where  $h_a(w_X(z)) = w_X(z) + a (1 - R^{*1}(w_X(z)))$

With the definition of  $U(z)$ , equation (2.7) implies,

$$\lambda U(z) = \sum_{n=m}^{N-1} SE_n(0) z^n + \lambda \sum_{n=m+1}^{N-1} z^n \left( \sum_{k=1}^{n-m} U_{n-k} g_k \right) \quad (2.43)$$

Adding equations (2.42) with equation (2.43) multiplied by  $(-S^*(\theta))$  and using equation (2.40), it is found that

$$(\theta - h_a(w_X(z))) P_1^*(z, \theta) = \frac{P_1(z, 0)}{z} [z - S^*(\theta)(1 - r + \sum_{i=1}^C r_i S_i^*(h_{a_i}(w_X(z))))] - S^*(\theta)(SE(z, 0) - \bar{P}(0) - U(z) w_X(z)) \quad (2.44)$$

when  $\theta = h_a(w_X(z))$ ,

$$P_1(z, 0) = \frac{z S^*(h_a(w_X(z))) [SE(z, 0) - \bar{P}(0) - U(z) w_X(z)]}{z - H^*(w_X(z))} \quad (2.45)$$

where  $H^*(w_X(z)) = S^*(h_a(w_X(z))) ((1 - r) + \sum_{i=1}^C r_i S_i^*(h_{a_i}(w_X(z))))$

(2.45.1)

Substituting for  $P_1(z, 0)$  in equation (2.44),

$$P_1^*(z, \theta) = \frac{z (S^*(h_a(w_X(z))) - S^*(\theta)) (SE(z, 0) - \bar{P}(0) - U(z) w_X(z))}{(\theta - h_a(w_X(z))) (z - H^*(w_X(z)))} \quad (2.46)$$

Finally, the probability generating function of the system size, when the server is in dormant period is derived.

Equation (2.7) recursively gives,

$$\lambda U_n = \sum_{k=m}^n SE_k(0) \pi_{n-k}, \quad m \leq n \leq N-1 \quad (2.47)$$

For further simplification, the following result is used.

### Theorem : 2.3

(i)  $((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) = \sum_{n=m}^{\infty} \xi_n z^n$ ,

where  $\xi_n = \sum_{k=0}^{m-1} \left( \frac{\alpha I_{n-k} \beta_k}{1 - \alpha I_0} + \psi_k g_{n-k} \right)$  and  $\beta(z) = \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0}$  (2.48)

(ii)  $D^*(w_X(z)) ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) = \sum_{n=m}^{\infty} z^n \sum_{i=m}^n \xi_i h_{n-i}$  (2.49)

where  $h_k$  denotes the probability that  $k$  customers arrive during the setup period (D).

**Proof**

Substituting for  $VI^*(w_X(z))$ ,  $\beta(z)$ ,  $\psi(z)$ ,  $w_X(z)$  from equations (2.26.0), (2.48), (2.31.2) and (2.15.1),

$$\begin{aligned}
& (VI^*(w_X(z)) - 1)\beta(z) + 1 - \psi(z)w_X(z) \\
&= 1 - \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} + \left( \sum_{n=0}^{\infty} \alpha I_n z^n \right) \left( \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} \right) - \left( \sum_{n=0}^{m-1} \psi_n z^n \right) \left( 1 - \sum_{k=1}^{\infty} g_k z^k \right) \\
&= 1 - \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} + \alpha I_0 \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0} + \sum_{n=1}^{m-1} z^n \sum_{i=1}^n \frac{\alpha I_i \beta_{n-i}}{1 - \alpha I_0} \\
&\quad + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \frac{\beta_i \alpha I_{n-i}}{1 - \alpha I_0} - \sum_{n=0}^{m-1} \psi_n z^n + \sum_{n=1}^{m-1} z^n \sum_{i=0}^{n-1} \psi_i g_{n-i} + \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} \psi_k g_{n-k} \\
&= (1 - \psi_0) - \sum_{n=0}^{m-1} \beta_n z^n + \sum_{n=1}^{m-1} z^n \left( \sum_{i=1}^n \frac{\alpha I_i \beta_{n-i}}{1 - \alpha I_0} - \alpha I_n^{(J)} \right) + \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} \left( \frac{\beta_k \alpha I_{n-k}}{1 - \alpha I_0} + \psi_k g_{n-k} \right)
\end{aligned}$$

(Using corollary (ii) of Theorem 2.2 and the definition of  $\beta_n$ )

$$= \sum_{n=m}^{\infty} \xi_n z^n \text{ where } \xi_n = \sum_{k=0}^{m-1} \left( \frac{\alpha I_{n-k} \beta_k}{1 - \alpha I_0} + \psi_k g_{n-k} \right)$$

$$\text{Since } D^*(w_X(z)) = \sum_{k=0}^{\infty} h_k z^k,$$

$$\begin{aligned}
D^*(w_X(z)) \left( \beta(z) + 1 - \psi(z)w_X(z) \right) &= \left( \sum_{k=0}^{\infty} h_k z^k \right) \left( \sum_{i=m}^{\infty} \xi_i z^i \right) \\
&= \sum_{n=m}^{\infty} z^n \sum_{i=m}^n \xi_i h_{n-i}
\end{aligned}$$

This proves Theorem 2.3.

Thus for  $k \geq m$ ,  $SE_k(0)$  = the coefficient of  $z^k$  in  $SE(z, 0)$  implies,

$$\begin{aligned}
SE_k(0) &= \bar{P}(0) \text{ coefficient of } z^k \text{ in } \left\{ \beta(z) + 1 - \psi(z)w_X(z) \right\} \\
&\quad \text{(from equation (2.33.2)).}
\end{aligned}$$

$$\text{(i.e.,) } SE_k(0) = \bar{P}(0) \sum_{i=m}^k \xi_i h_{k-i} \text{ follows from Theorem (2.3) – (ii)}$$

Substituting for  $SE_k(0)$  in equation (2.47) and rearranging the summation,

$$\lambda U_n = \bar{P}(0) \sum_{r=m}^n \xi_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k}$$

$$\text{Thus } U(z) = \bar{P}(0) \sum_{n=m}^{N-1} \frac{\phi_n z^n}{\lambda} = \bar{P}(0) \phi(z) \quad (2.50)$$

$$\text{where, } \phi_n = \sum_{r=m}^n \xi_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k} \quad (2.50.1)$$

$$\text{and } \phi(z) = \left( \frac{1}{\lambda} \right) \sum_{n=m}^{N-1} \phi_n z^n$$

Substituting for SE(z, 0) and U(z) from equations (2.33.2) and (2.50),

$$SE(z, 0) - U(z) w_X(z) = -P(0) w_X(z) I_{(m, N)}(z) \quad (2.51)$$

where  $I_{m, N}(z)$

$$= \frac{(1 - VI^*(w_X(z)))}{w_X(z)} \beta(z) D^*(w_X(z)) + \frac{(1 - D^*(w_X(z)))}{w_X(z)} + \sum_{n=m}^{N-1} \frac{\phi_n z^n}{\lambda} + \psi(z) D^*(w_X(z)) \quad (2.52)$$

Thus the equations (2.46) and (2.41) at  $\theta = 0$  imply

$$P_1^*(z, 0) = \frac{z(S^*(h_a(w_X(z))) - 1)}{h_a(w_X(z))} \frac{\bar{P}(0) w_X(z) I_{(m, N)}(z)}{(z - H^*(w_X(z)))} \quad (2.53)$$

$$P_2^{i*}(z, 0) = \frac{z r_i S^*(h_a(w_X(z))) (1 - S_i^*(h_{a_i}(w_X(z))))}{h_a(w_X(z))} \frac{(-\bar{P}(0) w_X(z) I_{(m, N)}(z))}{(z - H^*(w_X(z)))}, \quad 1 \leq i \leq C \quad (2.54)$$

where  $H^*(w_X(z))$  is given by equation (2.45.1)

Hence the partial PGFs of the system size at arbitrary epoch for the model are given by

$$QI^*(z, 0) = \frac{\bar{P}(0)(1 - VI^*(w_X(z)))}{w_X(z)} \beta(z) \quad (2.55.1)$$

$$SE^*(z, 0) = \frac{\bar{P}(0)(1 - D^*(w_X(z)))}{w_X(z)} ((\beta(z) + 1 - \psi(z) w_X(z))) \quad (2.55.2)$$

$$U(z) = \frac{\bar{P}(0)}{\lambda} \sum_{n=m}^{N-1} \phi_n z^n = \bar{P}(0) \phi(z) \quad (2.55.3)$$

$$PI(z) = \bar{P}(0) \psi(z) \quad (2.55.4)$$

$$P_1^*(z, 0) = \frac{z(S^*(h_a(w_X(z))) - 1)}{h_a(w_X(z))} \frac{\bar{P}(0) w_X(z) I_{(m,N)}(z)}{(z - H^*(w_X(z)))} \quad (2.55.5)$$

$$P_2^{i*}(z, 0) = \frac{z r_i S^*(h_a(w_X(z)))(1 - S_i^*(h_{a_i}(w_X(z))))}{h_{a_i}(w_X(z))} \quad 1 \leq i \leq C \quad (2.55.6)$$

$$BR_1^{**1}(z, 0, 0) = \frac{a P_1^*(z, 0)(1 - R^*(w_X(z)))}{w_X(z)} \quad (2.55.7)$$

$$BR_2^{**1}(z, 0, 0) = \frac{a_i P_2^{i*}(z, 0)(1 - R_i^*(w_X(z)))}{w_X(z)}, \quad 1 \leq i \leq C \quad (2.55.8)$$

To derive the total PGF of the system size distribution, the following generating functions are considered.

$P_{Idle}(z)$  = Probability generating function of the system size when the server is in idle (vacation + buildup + setup + dormant) state.

$$P_{Idle}(z) = QI^*(z, 0) + PI(z) + SE^*(z, 0) + U(z) = \bar{P}(0) I_{(m,N)}(z) \quad (2.56)$$

$P_{Comp}(z)$  = The PGF of the system size when server is busy or in breakdown state

$$\begin{aligned} &= P_1^*(z, 0) + BR_1^{**1}(z, 0, 0) + \sum_{i=1}^C (P_2^{i*}(z, 0) + BR_2^{**1}(z, 0, 0)) \\ &= \frac{z \bar{P}(0) I_{(m,N)}(z) (H^*(w_X(z)) - 1)}{(z - H^*(w_X(z)))} \end{aligned} \quad (2.57)$$

Thus the total PGF of the system size distribution is given by

$$P_{(m,N)}^J(z) = P_{Idle}(z) + P_{Comp}(z) = \frac{\bar{P}(0)(z-1)H^*(w_X(z))I_{(m,N)}(z)}{(z - H^*(w_X(z)))} \quad (2.58)$$

$\bar{P}(0)$  can be calculated by using the normalizing condition

$$P_{(m,N)}^J(1) = 1, \text{ and found to be } = \frac{1 - \rho_H}{I_{(m,N)}(1)} \quad (2.59)$$

where  $\rho_H = \lambda E(X) E(H)$ , (2.59.1)

$$E(H) = [E(S)(1 + aE(R)) + \sum_{i=1}^C r_i E(S_i)(1 + a_i E(R_i))] \quad (2.60)$$

$$\text{and } I_{(m,N)}(1) = E(D) + E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} + \left( \frac{1}{\lambda} \right) \sum_{n=m}^{N-1} \phi_n + \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} \psi_n \quad (2.61)$$

$$\text{Hence } P_{(m,N)}^J(z) = P_{M^X G(1, i(1 \leq i \leq C))/1/Breakdown}(z) \frac{I_{(m,N)}(z)}{I_{(m,N)}(1)} \quad (2.62)$$

$$\text{where } P_{M^X G(1, i(1 \leq i \leq C))/1/Breakdown}(z) = \frac{(1 - \rho_H)(z - 1)H^*(w_X(z))}{(z - H^*(w_X(z)))},$$

and  $H^*(w_X(z))$ ,  $I_{(m,N)}(z)$  are given by equations (2.45.1) and (2.52) respectively. Equation (2.62) proves the decomposition property.

### 2.1.4 Decomposition Property

#### Theorem : 2.4

Under the steady-state condition  $\rho_H < 1$ , the PGF of the stationary system size of the  $(m, N)$  – policy queueing model under consideration is the product of the PGF of the system size of  $M^X / G(1, i(1 \leq i \leq c))/1/Breakdown$  queue (without threshold policy and without vacation) and the PGF of the conditional system size distribution during the server idle period.

### 2.1.5 Performance Measures

In this section, the steady-state system size probabilities and the mean system size, when the system is in different states are calculated.

#### The Server in Idle State

Let  $P_{VI}$ ,  $P_{set}$ ,  $P_{build}$  and  $P_{dor}$  denote the steady-state system size probabilities and  $L_{VI}$ ,  $L_{set}$ ,  $L_{build}$  and  $L_{dor}$  denote the mean number of customers waiting in the system, when the system is on vacation, in setup, buildup and dormant states respectively. Then these measures can be calculated using equations (2.55.1) to (2.55.4) and the results stated in Appendix 1-B.

$$P_{VI} = \lim_{z \rightarrow 1} QI^*(z, 0) = \bar{P}(0) E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} \quad (2.63)$$

$$L_{VI} = \left[ \frac{d}{dz} QI^*(z, 0) \right]_{z=1} = \bar{P}(0) \left[ \frac{\lambda E(X) E(VI^2)}{2} \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} + E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} \right] \quad (2.63.1)$$

$$P_{\text{set}} = \lim_{z \rightarrow 1} SE^*(z, 0) = \bar{P}(0)E(D) \quad (2.64)$$

$$\begin{aligned} L_{\text{set}} &= \left[ \frac{d}{dz} SE^*(z, 0) \right]_{z=1} \\ &= \bar{P}(0)\lambda E(X) \left[ \frac{E(D^2)}{2} + E(D) \left( E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} \right) \right] \end{aligned} \quad (2.64.1)$$

$$P_{\text{buildup}} = PI(z) = \bar{P}(0) \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} \psi_n \quad (2.65)$$

$$L_{\text{buildup}} = \left[ \frac{d}{dz} PI(z) \right]_{z=1} = \bar{P}(0) \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} n \psi_n \quad (2.65.1)$$

$$P_{\text{dor}} = U(z) = \bar{P}(0) \left( \frac{1}{\lambda} \right) \sum_{n=m}^{N-1} \phi_n \quad (2.66)$$

$$L_{\text{dor}} = \left[ \frac{d}{dz} U(z) \right]_{z=1} = \bar{P}(0) \left( \frac{1}{\lambda} \right) \sum_{n=m}^{N-1} n \phi_n \quad (2.66.1)$$

Thus, the probability that the server is in idle state is given by

$$\begin{aligned} P_I &= P_{VI} + P_{\text{set}} + P_{\text{build}} + P_{\text{dor}} \\ &= \bar{P}(0) I_{(m,N)}(1) = (1 - \rho_H) \end{aligned} \quad (2.67)$$

The mean number of customers accumulated in the system when the server is idle is

$$L_I = L_{VI} + L_{\text{set}} + L_{\text{build}} + L_{\text{dor}} = \frac{I'_{(m,N)}(1)}{I_{(m,N)}(1)} (1 - \rho_H) \quad (2.67.1)$$

where

$$\begin{aligned} I'_{(m,N)}(1) &= \left[ \frac{d}{dz} I_{(m,N)}(z) \right]_{z=1} \\ &= \lambda E(X) \left[ \left( \frac{E(VI^2)}{2} + E(VI)E(D) \right) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} \right. \\ &\quad \left. + E(VI) \sum_{n=0}^{m-1} \frac{n \beta_n}{1 - \alpha I_0} + \left( \frac{1}{\lambda} \right) \sum_{n=m}^{N-1} n \phi_n + \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} n \psi_n \right] \text{ and} \end{aligned} \quad (2.67.2)$$

$$I_{(m,N)}(1) \text{ is given by equation (2.61)} \quad (2.67.3)$$

### The Server in Busy State

The probability that the server is busy with FES ( $P_{FES}$ ) and SOS ( $P_{SOS}$ ) and the expected number of customers in the system ( $L_{FES}$  and  $L_{SOS}$ ) in the corresponding states are calculated using the equations (2.55.5) and (2.55.6).

One can note that,

$$\lim_{z \rightarrow 1} \left( \frac{-w_X(z)}{z - H^*(w_X(z))} \right) = \frac{\lambda E(X)}{1 - \rho_H}$$

$$\frac{d}{dz} \left( \frac{-w_X(z)}{z - H^*(w_X(z))} \right)_{z=1} = \frac{\lambda E(X(X-1)) + (\lambda E(X))^3 E(H^2)}{2(1 - \rho_H)^2}$$

where

$$E(H^2) = E(S^2) (1 + a E(R))^2 + a E(S) E(R^2)$$

$$+ 2E(S) (1 + a E(R)) \sum_{i=1}^C r_i E(S_i) (1 + a_i E(R_i))$$

$$+ \sum_{i=1}^C r_i E(S_i^2) (1 + a_i E(R_i))^2 + \sum_{i=1}^C r_i E(S_i) a_i E(R_i^2) \text{ and}$$

$\rho_H$  is given by the equation (2.59.1).

Thus,

$P_{FES}$  = Probability that the server is busy in FES

$$= \lim_{z \rightarrow 1} P_1^*(z, 0) = \lambda E(X) E(S)$$

$$L_{FES} = \left[ \frac{d}{dz} P_1^*(z, 0) \right]_{z=1}$$

$$= P_{FES} + \frac{\lambda E(X(X-1)) + (\lambda E(X))^3 E(H^2)}{2(1 - \rho_H)} E(S) + \frac{\lambda E(X) E(S) I'_{(m,N)}(1)}{I_{(m,N)}(1)}$$

$$+ \frac{(\lambda E(X))^2 E(S^2) (1 + a E(R))}{2}$$

$P_{SOS}$  = Probability that the server is busy in second multi-optional services

$$= \lim_{z \rightarrow 1} \sum_{i=1}^C P_2^{i*}(z, 0) = \sum_{i=1}^C P_2^i, \text{ where } P_2^i = \lambda E(X) r_i E(S_i) \quad (2.68)$$

$$L_{SOS} = \left[ \frac{d}{dz} \sum_{i=1}^C P_2^{i*} (z, 0) \right]_{z=1} = \sum_{i=1}^C L_2^i, \text{ where}$$

$$L_2^i = P_2^i + \frac{\lambda E(X(X-1)) + (\lambda E(X))^3 E(H^2)}{2(1-\rho_H)} r_i E(S_i) + \frac{\lambda E(X) r_i E(S_i) I'_{(m,N)}(1)}{I_{(m,N)}(1)} \\ + (\lambda E(X))^2 r_i ((1 + a E(R)) E(S) E(S_i) + \frac{E(S_i^2)}{2} (1 + a_i E(R_i))) \quad (2.68.1)$$

$$\text{Hence } P_{\text{busy}} = P_{FES} + P_{SOS} = \lambda E(X) E_{SOS} = \rho_{SOS} \quad (2.68.2)$$

$$\text{where } E_{SOS} = E(S) + \sum_{i=1}^C r_i E(S_i)$$

$$L_{\text{busy}} = L_{FES} + L_{SOS}$$

### The Server in Breakdown State

The probability that the server is in breakdown states ( $P_{BR_1}$  and  $P_{BR_2}$ ) and the expected number of customers in the system ( $L_{BR_1}$  and  $L_{BR_2}$ ) are calculated by using the equations (2.55.7) and (2.55.8).

(i.e.)

(i.e.)  $P_{BR_1}$  = Probability that the server is in breakdown state due to FES

$$= \lim_{z \rightarrow 1} BR_1^{**1} (z, 0, 0) = \lambda E(X) a E(S) E(R)$$

$$L_{BR_1} = \left[ \frac{d}{dz} BR_1^{**1} (z, 0, 0) \right]_{z=1} = a (L_{FES} E(R) + P_{FES} \frac{\lambda E(X) E(R^2)}{2})$$

$P_{BR_2}$  = Probability that the server is in breakdown state due to SOS

$$= \lim_{z \rightarrow 1} \sum_{i=1}^C BR_2^{i**1} (z, 0, 0) = \sum_{i=1}^C r_i a_i \lambda E(X) E(S_i) E(R_i)$$

$$L_{BR_2} = \left[ \frac{d}{dz} \left( \sum_{i=1}^C BR_2^{i**1} (z, 0, 0) \right) \right]_{z=1} \\ = \sum_{i=1}^C a_i \left( \frac{\lambda E(X)}{2} E(R_i^2) P_2^i + E(R_i) L_2^i \right)$$

where  $P_2^i$  and  $L_2^i$  are given by equations (2.68) and (2.68.1). Hence,

$$P_{br} = \text{Probability that the server is in breakdown state}$$

$$\begin{aligned}
&= P_{BR_1} + P_{BR_2} \\
&= \lambda E(X) (a E(S) E(R) + \sum_{i=1}^C a_i r_i E(S_i) E(R_i)) = \rho_{BR} \quad (2.69)
\end{aligned}$$

$$L_{br} = L_{BR_1} + L_{BR_2}$$

### Mean System Size

The expected system size  $L_{(m,N)}^J$  of the model is given by

$$L_{(m,N)}^J = L_I + L_{busy} + L_{br} = \frac{I'_{(m,N)}(1)}{I_{(m,N)}(1)} + L \quad (2.70)$$

$$\text{where } L = \rho_H + \frac{\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2)}{2(1-\rho_H)} \quad (2.70.1)$$

gives the mean system size of the second multi-optional service queueing model with breakdown without vacation and without threshold policy.

#### 2.1.6 Other System Characteristics

##### Definition : Busy Period

The busy period begins, when the system size becomes atleast N, soon after the setup work (or) at the end of the dormant period, (i.e.) when the server starts serving the customers and it ends, when the system next becomes empty and the server leaves for a vacation.

Let  $E(\text{cycle})$ ,  $E(\text{Busy})$ ,  $E(\text{Break})$ ,  $E(\text{Completion})$ ,  $E(I)$  and  $E(W_S)$  respectively denote the expected length of the cycle, busy period, breakdown period, completion period, idle period and the mean waiting time. Then the long run fraction of time when the server is in setup state

$$\text{implies } P_{\text{set}} = \frac{E(D)}{E(\text{cycle})} = \bar{P}(0) E(D).$$

Therefore by using equations (2.59) and (2.63) to (2.69), we get

$$(i) \quad E(\text{Cycle}) = \frac{1}{\bar{P}(0)} = \frac{I_{(m,N)}(1)}{1-\rho_H}$$

$$\begin{aligned}
\text{(ii)} \quad E(\text{Busy}) &= P_{\text{Busy}} E(\text{Cycle}) = \frac{\rho_{\text{SOS}} I_{(m,N)}(1)}{1 - \rho_H} \\
\text{(iii)} \quad E(\text{Break}) &= P_{\text{br}} E(\text{Cycle}) = \frac{\rho_{\text{BR}} I_{(m,N)}(1)}{1 - \rho_H} \\
\text{(iv)} \quad E(\text{Completion}) &= E(\text{Busy}) + E(\text{Break}) = \rho_H \frac{I_{(m,N)}(1)}{1 - \rho_H} \\
\text{(v)} \quad E(I) &= P_I E(\text{Cycle}) = I_{(m,N)}(1) \\
\text{(vi)} \quad E(w_S) &= \frac{L_{(m,N)}^J}{\lambda E(X)} \text{ (Little's formula)}
\end{aligned}$$

### 2.1.7 Queue Size Distribution at Departure Epoch

A departing customer will see  $n$  customers in the system just after the departure if and only if there are  $(n+1)$  customers in the system just before the departure (Lee et al., 1994a). Thus if  $\pi_n^+$  denotes the probability that there are  $n$  customers in the system at departure epoch, then

$$\pi_n^+ = D \left[ (1-r) P_{1,n+1}(0) + \sum_{i=1}^C P_{2,n+1}^i(0) \right], n \geq 0$$

where  $D$  is the normalizing constant.

Therefore the PGF  $\pi^+(z)$  of  $\{\pi_n^+ ; n \geq 0\}$  is given by

$$\begin{aligned}
\pi^+(z) &= \sum_{n=0}^{\infty} \pi_n^+ z^n = \frac{D}{z} \left[ (1-r) \sum_{n=1}^{\infty} P_{1,n}(0) z^n + \sum_{n=1}^{\infty} \sum_{i=1}^C P_{2,n}^i(0) z^n \right] \\
&= \frac{D}{z} \left[ (1-r) P_1(z, 0) + \sum_{i=1}^C P_2^i(z, 0) \right]
\end{aligned}$$

Substituting for  $P_1(z, 0)$  and  $P_2^i(z, 0)$  from equations (2.45) and (2.40) we

have  $\pi^+(z) = \frac{D P_{(m,N)}^J(z) w_X(z)}{1-z}$ , where  $P_{(m,N)}^J(z)$  is the PGF of the system

size of the present model at arbitrary point of time.

Evaluating  $D$  using the normalizing condition,  $D = \frac{1}{\lambda E(X)}$

Thus  $\pi^+(z) = \frac{(1-X(z))}{E(X)(1-z)} P_{(m,N)}^J(z)$

## 2.2 OPTIMAL MANAGEMENT POLICY

In this section, the total expected cost function per unit time ( $T_C(m, N)$ ) for the queue with bilevel threshold in which  $m$  and  $N$  are decision variables is developed. To obtain the optimal values of  $m$  and  $N$  for which the total expected cost function per unit time is minimum, the cost structure that have been widely used in literature (Yadin and Naor, 1963), Lee and Park (1997) and Ke (2003b) is employed. The notations used to calculate the total average cost per unit time for the system are :

$C_y$	Startup cost per cycle
$C_{build}$	Cost per unit time for keeping the server idle (buildup)
$C_h$	Holding cost per customer per unit time.
$C_{set}$	Server setup cost per unit time
$C_{dor}$	Server standby cost per unit time.
$C_{busy}$	Cost per unit time for keeping the server on and in operation
$C_{VI}$	Reward per unit time due to vacation
$C_{br}$	Breakdown cost per unit time for a failed server
$T_C(m, N)$	Total average cost per unit time of the system

$$\begin{aligned} \text{Then , } T_C(m, N) = & \frac{C_y}{E(\text{Cycle})} + C_h L_{(m,N)}^J + C_{build} P_{build} + C_{set} P_{set} + C_{dor} P_{dor} \\ & + C_{br} P_{br} + C_{busy} P_{busy} - C_{VI} P_{VI} \end{aligned} \quad (2.71)$$

Substituting the values of the performance measures, the equation can be rewritten as

$$T_C(m, N) = \bar{A} + \frac{1}{I_{(m,N)}(1)} [A_1 + z(m) + C_{dor} (1 - \rho_H) \left( \frac{1}{\lambda} \sum_{n=m}^{N-1} \phi_n + \frac{C_h}{\lambda} \sum_{n=m}^{N-1} n \phi_n \right)] \quad (2.72)$$

$$\text{where } \bar{A} = C_{busy} P_{busy} + C_h L + C_{br} P_{br}$$

$$A_1 = (1 - \rho_H) [C_y + C_{set} E(D)] + C_h [\lambda E(X) \frac{E(D^2)}{2}]$$

$$\begin{aligned} z(m) = & C_h \left\{ \lambda E(X) \left[ \left( \frac{E(VI^2)}{2} + E(VI) E(D) \right) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} + E(D) \left( \frac{1}{\lambda} \sum_{n=0}^{m-1} \psi_n \right) \right] \right. \\ & + E(VI) \sum_{n=0}^{m-1} \frac{n \beta_n}{1 - \alpha I_0} + \left. \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} n \psi_n \right\} \\ & + (1 - \rho_H) \left[ C_{build} \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} \psi_n - C_{VI} E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} \right] \end{aligned}$$

and  $I_{(m,N)}(1)$ ,  $L$  are given in equations (2.61) and (2.70.1) respectively.

A two-dimensional search over the non-negative integer space is necessary to find the optimal control values  $(m^*, N^*)$  of  $(m, N)$  which minimize  $T_C(m, N)$ . Due to the mathematical complexity it is difficult to prove the convexity or unimodality of the cost function  $T_C(m, N)$ . Thus following the concept of the dynamic optimization (due to Bellman (1957), J.C.Ke (2001, 2003b) and Lee and Park (1997)), we consider the procedure that makes it possible to calculate the optimal thresholds  $(m^*, N^*)$ .

$$T_C(m, k+1) - T_C(m, k) = \frac{\phi_k}{\lambda I_{(m, k+1)}(1) I_{(m, k)}(1)} H_{(m, k)}, \quad (m \leq k \leq N-1) \quad (2.73)$$

where  $\phi_k$  is given by the equation (2.50.1).

$$\text{and } H_{(m, k)} = C_h[k d_m + \sum_{n=m}^{k-1} \frac{(k-n)}{\lambda} \phi_n] + C_{dor} d_m (1 - \rho_H) - (A_1 + z(m)) \quad (2.74)$$

$$\text{with } d_m = E(D) + E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0} + \left( \frac{1}{\lambda} \right) \sum_{n=0}^{m-1} \psi_n$$

The equation (2.73) implies, for a given  $m$ , the sign of  $H(m, k)$  determines whether  $T_C(m, k)$  increases or decreases with respect to  $k$ .

Let  $n$  be the first  $k$  for which  $H(m, k) > 0$ . Then

$$H(m, n+1) = H(m, n) + C_h(d_m + \sum_{i=m}^n \frac{\phi_i}{\lambda}) > 0 \text{ implies}$$

$$T_C(m, k) > T_C(m, n) \text{ for } k > n$$

That is, for a given  $m$ , the optimal value  $N^*(m)$  of  $N$  is given by the first  $k$  for which  $H(m, k) > 0$  and that, once  $T_C(m, N(m))$  increases with respect to  $N$ , it keeps on increasing thereafter. Therefore for a given  $m$ ,  $T_C(m, N(m))$ , is conditionally unimodal and thereby  $N^*(m)$  is conditionally optimal.

Thus  $N^*(m) = \min \{k \geq 1 / H(m, k) > 0\}$ , where  $H(m, k)$  is given by equation (2.74). Therefore for each  $m$ ,  $T_C(m, N)$  has a relative minimum value at  $(m, N^*(m))$ . Thus the pair  $(m, N^*(m))$  gives the relative optimal policy for a given  $m$ . Though it is difficult to prove mathematically that  $T_C(m, N)$  is convex (or) unimodal, the computer experiments show that, the expected cost

function is convex. Thus the optimal  $(m^*, N^*)$  can be obtained by using the following algorithm.

### Algorithm

**Step 1** : Set  $m = 1$ , Determine  $N^*(m)$  from equation (2.74). Calculate  $T_C(m, N^*(m))$ . GOTO Step 2.

**Step 2** : Calculate  $N^*(m+1)$  and  $T_C(m+1, N^*(m+1))$ . GOTO Step 3

**Step 3** : If  $T_C(m+1, N^*(m+1)) > T_C(m, N^*(m))$ , STOP

The optimal threshold is  $(m^*, N^*) = (m, N^*(m))$  and  $T_C(m, N^*(m))$  is the required optimal value. Otherwise set  $m = m+1$  and GOTO Step 2.

## 2.3 PARTICULAR CASES

In this section the PGF of the system size for various queueing models are derived as special cases. In the first two cases ((i), (ii)), the PGF of the system size for the unreliable second stage multi-optional  $M^X/G/1$  queueing system for the classical single and multiple vacation policies are obtained respectively.

(i)  $J = 1$  implies,  $\beta_i = 0 \forall i \geq 1$  and  $\beta(z) = 1$ . This identification in (2.62) gives the PGF of the system size for the single vacation queueing model

$$P_{(m, N)}(z) = \frac{(1 - \rho_H)(z - 1)H^*(w_X(z))}{z - H^*(w_X(z))} \frac{I_{(m, N)}^S(z)}{I_{(m, N)}^S(1)} \quad (2.75)$$

(Jemila Parveen (2012)).

$$\text{where } I_{(m, N)}^S(z) = \frac{1 - VI^*(w_X(z)) D^*(w_X(z))}{w_X(z)} + \psi(z) D^*(w_X(z)) + \quad (2.76)$$

$$\phi_n^S = \sum_{r=m}^n \xi_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k} \text{ and } \xi_r = \sum_{k=0}^{m-1} \psi_k g_{r-k} + \alpha I_r$$

(ii) As  $J \rightarrow \infty$ ,  $\alpha I_n^{(J)} \rightarrow 0$  and  $\psi_n = 0$ ,  $\beta_n = \sum_{i=1}^n \frac{\alpha I_i \beta_{n-i}}{1 - \alpha I_0}$  for  $1 \leq n \leq m-1$  with

$\psi_0 = 0$ ,  $\beta_1 = 1$ . With this substitution, the PGF of the system size for the multiple vacation queueing model is obtained from (2.62) :

$$P_{(m,N)}(z) = \frac{(1 - \rho_H)(z-1)H^*(w_X(z))}{z - H^*(w_X(z))} \frac{I_{(m,N)}^R(z)}{I_{(m,N)}^R(1)} \quad (2.77)$$

where  $I_{(m,N)}^R(z) = \frac{1 - VI^*(w_X(z))}{w_X(z)} \beta(z)D^*(w_X(z)) + \frac{1 - D^*(w_X(z))}{w_X(z)} +$  (2.78)

$$\phi_n^R = \sum_{r=m}^n \xi_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k} \text{ and } \xi_r = \sum_{k=0}^{m-1} \frac{\alpha I_{r-k} \beta_k}{1 - \alpha I_0}$$

The  $I_{(m,N)}^S(z)$  and  $I_{(m,N)}^R(z)$  exactly coincide with the conditional PGFs of the queue length under the condition that the system is idle, for the single and multiple vacation bi level threshold models of Lee et al. (2003).

(iii) By choosing  $r = 0$  in equation (2.75), the PGF of the system size for the single service  $M_{(m,N)}^X/G/1/SV$  breakdown queueing model of J.C.Ke (2004) is deduced.

(iv) By taking  $a = a_i = 0 \forall i$  and  $r = 0$  in cases (i) and (ii), the PGFs of the system size for the single and multiple vacation models of Lee et al. (2003) are obtained.

In addition if  $\Pr(VI = 0) = 1$ , the result of non-vacation model of Lee et al. (2003) can be obtained.

(v) The PGF of the system size for the  $(N, N)$  policy model  $M_{(m,N)}^X/G/1$  breakdown/J Vacation can be obtained with  $m = N$ . In this case  $\phi_n = 0$  ( $m \leq n \leq N-1$ ) and

$$I_{(m,N)}(z) = \frac{(1 - VI^*(w_X(z)))}{w_X(z)} \beta(z)D^*(w_X(z)) + \frac{(1 - D^*(w_X(z)))}{w_X(z)} + \psi(z)D^*(w_X(z))$$

(vi) It is also verified that, the results of

- (i) N-policy, single and multiple vacation models of Lee et al. (1995 and 1994b) can be deduced by setting  $r$ ,  $a$ ,  $a_i$ ,  $E(D)$  to 0 and  $N = m$  in cases (i) and (ii) respectively.
- (ii) The result of the Lee and Park (1997) for the M/G/1 bi-level threshold model can be derived with the substitution  $r$ ,  $a_i$ ,  $a$ ,  $g_j$  to 0 (for every  $i$  and  $j \geq 2$ ) with  $\Pr(VI = 0) = 1$  and  $g_1 = 1$ .
- (iii) The results of the Markovian  $(m, N)$  policy model of Lee et al. (1998) can be deduced by taking  $S^*(\theta) = \frac{\mu}{\mu + \theta}$   $D^*(\theta) = \frac{\gamma}{\gamma + \theta}$  in case (iv) and  $\Pr(VI = 0) = 1$ .

## 2.4 NUMERICAL ANALYSIS

In this section, some numerical examples are presented to indicate the effects of various parameters namely (i) the batch arrival rate  $\lambda$ , (ii) mean setup time ( $E(D)$ ), (iii) mean vacation time ( $E(VI)$ ), (iv) the probability  $r_i$ , with which customers select one of the second optional services, on the behaviour of mean system size ( $L_{(m,N)}^J$ ) and on the system size probabilities of the model.

The optimal threshold values  $(m^*, N^*)$  of  $(m, N)$  and the minimum average cost  $T_C(m^*, N^*)$  per unit time are also calculated using the procedure given in section 2.2. The bilevel policy is compared with the corresponding single threshold N-policy through the total expected cost per unit time. The values of the parameters are chosen such that they satisfy the stability condition  $\rho_H = \lambda E(X) E(H) < 1$ . For the computation purpose, the distributions assumed for different random variables are given by :

Random variables (Y)	Distribution F(Y)	Mean E(Y)	Second order moments E(Y <sup>2</sup> )
FES (S)	Two-stage hyper-exponential	$\frac{b_1}{\mu_1} + \frac{1-b_1}{\mu_2}$	$2 \left( \frac{b_1}{\mu_1^2} + \frac{1-b_1}{\mu_2^2} \right)$

			$0 \leq b_1 \leq 1$	
SOS $1 \leq i \leq C=3$	i=1	Erlang-3 type	$1 / \mu_{21}$	$4 / 3 \mu_{21}^2$
	i=2	Deterministic	$1 / \mu_{22}$	$1 / \mu_{22}^2$
	i=3	Exponential	$1 / \mu_{23}$	$2 / \mu_{23}^2$
Setup time (D)		Erlang-3 type	$1 / \gamma$	$4 / 3 \gamma^2$
Vacation time (VI)		Gamma-3 type	$3 / \eta$	$12 / \eta^2$
Repair time in FES (R)		Erlang-2 type	$1 / rp$	$3 / 2rp^2$
Repair time in SOS ( $R_i$ )	i=1	Exponential	$1 / rp_1$	$2 / rp_1^2$
	i=2	Uniform in [0, 1]	$1 / 2$	$1 / 3$
	i=3	Deterministic	$1 / rp_3$	$1 / rp_3^2$
Batch size (X)	Geometric (Geo( $p_1$ ))		$1 / 1 - p_1$	$E(X(X - 1)) = 2p_1 / (1 - p_1)^2$
	Binomial (B, 4, $p_1$ )		$4 p_1$	$12 p_1^2$

The server's life time in FES and SOS follow exponential distribution with parameters  $a$  and  $a_i$  ( $i = 1$  to  $3$ ) respectively. In all the tables except Table 2.4, the batch size distribution is assumed to follow geometric distribution Geo ( $p_1$ ).

Table 2.1 gives the total expected cost values  $T_C(m, N)$  per unit time with respect to  $m$  and  $N$  for two cases. The table values show that the optimum values are given by

- (i)  $m^*=9, N^*=13, T_C(m^*, N^*(m)) = 493.088$ , for  $J = 17, C_h = 30$  (Figure 2.1.a)
- (ii)  $m^* = 1, N^*=7, T_C(m^*, N^*(m)) = 950.674$ , for  $J = 1, C_h = 100$  (Figure 2.1.b). Figures 2.1.a and 2.1.b give the surface representations of  $T_C(m, N)$  for different values of  $m$  and  $N$ . The graphical representation of  $T_C(m, N^*(m))$  for the different values of  $m$ , corresponding to  $J = 17, C_h = 30$  of Table 2.1 is depicted in Figures 2.1.c.

**Table 2.1 The Total Expected Cost  $T_C$  (m, N) for Different Values of m and N**

$(C_{\text{build}}, C_{\text{set}}, C_{\text{dor}}, C_{\text{busy}}, C_y, C_{V1}, C_{\text{br}})$  = (8, 1000, 60, 10, 10000, 8, 20)  
 $(\rho_1, \lambda, \gamma, \eta, a, a_1, a_2, a_3, r_p, r_{p_1}, r_{p_3}, C)$  = (0.55, 0.21, 0.2, 1, 0.5, 2, 4.6, 2.6, 0.6, 3, 7, 3)  
 $(b_1, \mu_1, \mu_2, \mu_{21}, \mu_{22}, \mu_{23})$  = (0.5, 1.5, 3.5, 5.2, 3.7, 4)

m \ N	1	4	7	10	12	13	14	16	20	23
1	760.222	655.819	564.182	518.353	506.416	504.307	504.315	509.118	532.246	557.246
	1145.252	1003.431	950.674	984.316	1034.134	1064.210	1096.868	1168.160	1326.773	1454.540
2		631.245	557.028	515.302	504.136	502.256	502.348	507.286	530.485	555.468
		998.980	952.252	984.720	1033.873	1063.716	1096.193	1167.231	1325.556	1453.188
3		605.446	549.342	512.254	501.914	500.222	500.429	505.478	528.707	553.650
		994.769	955.577	986.180	1034.293	1063.776	1095.975	1166.629	1324.541	1451.990
5			532.852	506.482	497.953	496.615	497.003	502.161	525.274	550.054
			969.189	993.565	1037.968	1066.162	1097.375	1166.702	1323.293	1450.216
7			516.000	501.549	495.086	494.055	494.530	499.574	522.239	546.708
			994.817	1009.913	1047.333	1073.003	1102.279	1169.043	1323.283	1449.398
9				497.959	493.856	493.088	493.527	498.136	519.857	543.815
				1040.098	1066.122	1087.220	1112.877	1174.792	1324.796	1449.651
10				496.865	494.033	493.391	493.765	498.004	518.996	542.600
				1061.887	1080.752	1098.547	1121.464	1179.582	1326.276	1450.229

■  $C_h = 30$  for  $J = 17$

■  $C_h = 100$  for  $J = 1$

Figure 2.1.a. The expected cost function  $T_c(m, N)$  for different values of  $m$  and  $N$  for  $C_h = 30, J = 17$

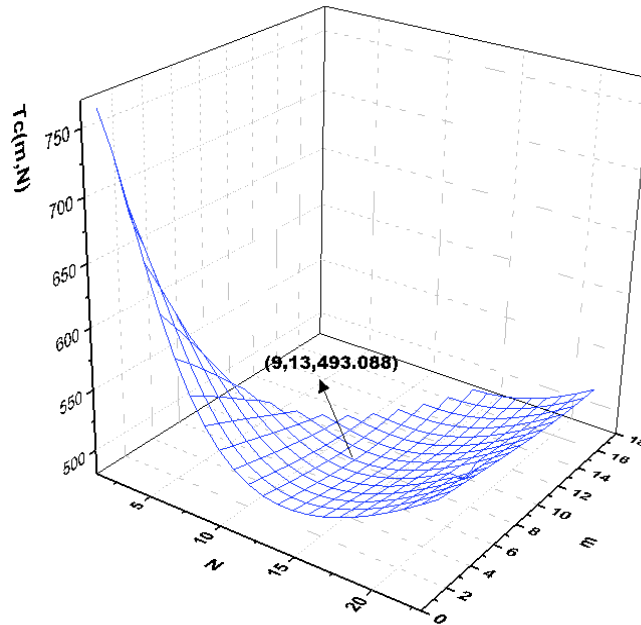


Figure 2.1.b. The expected cost function  $T_c(m, N)$  for different values of  $m$  and  $N$  for  $C_h = 100, J = 1$

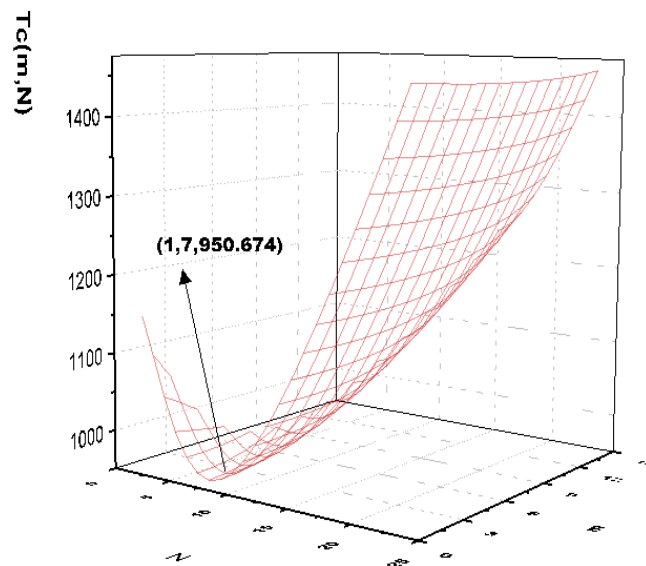


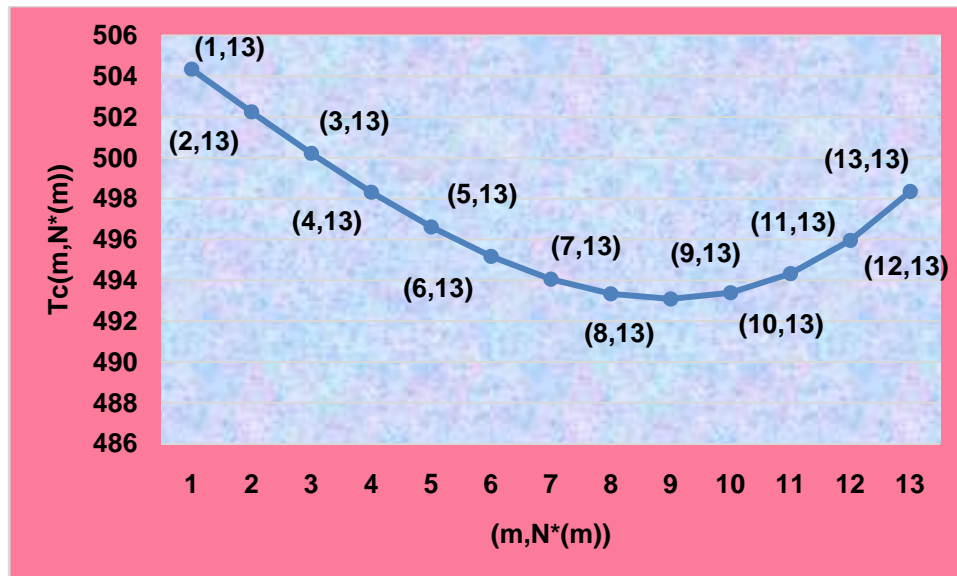
Figure 2.1.c.  $T_C(m, N^*(m))$  Vs  $(m, N^*(m))$ 

Table 2.2 gives the optimal values as the standby cost  $C_{dor}$  changes. For the queueing system with  $(N, N)$ -policy, there is no standby period. Thus the optimal values  $N^*$  and  $T_C(N^*)$  of  $(N, N)$ -policy do not vary with  $C_{dor}$ . The table values show that as  $C_{dor}$  increases, the optimal values of  $(m, N)$ -policy approach the optimal values of  $(N, N)$ -policy. It is interesting to note that in tables (2.2), (2.3), (2.3.a) and (2.10)),  $(m, N)$ -policy provides lower average cost than the single threshold  $(N, N)$ -policy. Except the parametric values mentioned in each table, the other values are referred from Table 2.1.

Table 2.2 Optimal Policy as Dormant Cost Changes ( $C_h = 30, J = 17$ )

$C_{dor}$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L^J_{(m^*, N^*)}$	E(Cycle)	$N^*$	$T_C(N^*, N^*)$	$L^J_{(N^*, N^*)}$	E(Cycle)
5	(4, 14)	487.242	8.510	66.596	11	495.710	8.923	66.999
10	(5, 14)	488.148	8.558	66.908	11	495.710	8.923	66.999
20	(6, 14)	489.681	8.624	67.329	11	495.710	8.923	66.999
100	(10, 13)	494.810	8.884	68.064	11	495.710	8.923	66.999
200	(11, 11)	495.710	8.923	66.999	11	495.710	8.923	66.999

Table 2.3 presents optimal values as the holding cost ( $C_h$ ) increases. The Table values show that as the holding cost increases, the system starts setup operation and service earlier than in the case of lower holding cost. In Table 2.3(a) it is shown that when the setup cost is high, the setup operation and service are delayed.

**Table 2.3 Optimal Policy as Holding Cost Changes****( $C_{Set} = 500, r = 0.15, J = 17, C_{dor} = 100$ )**

$C_h$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$	E(Cycle)	$N^*$	$T_C(N^*, N^*)$	$L_{(N^*, N^*)}^J$	E(Cycle)
30	(8, 12)	470.684	8.523	64.873	10	473.040	8.942	66.510
60	(2, 8)	688.995	6.257	45.281	6	705.935	7.246	49.993
100	(1, 7)	926.560	5.803	40.974	4	970.689	6.436	41.360
200	(1, 5)	1478.421	5.419	35.144	2	1556.164	5.709	32.580
300	(1, 4)	2019.438	5.332	32.708	1	2101.324	5.412	28.226
500	(1, 4)	3085.861	5.332	32.708	1	3183.750	5.412	28.226

**Table 2.3 (a). Optimal Policy as  $C_{Set}$  Changes ( $J = 17, C_h = 500$ )**

$C_{Set}$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$	E(Cycle)	$N^*$	$T_C(N^*, N^*)$	$L_{(N^*, N^*)}^J$	E(Cycle)
2000	(11, 15)	563.048	9.573	74.488	13	565.210	9.781	74.721
3000	(13, 18)	625.427	10.76	84.84	16	627.267	11.057	85.840
4000	(15, 19)	682.467	11.35	89.52	18	684.103	11.901	92.984
6000	(19, 23)	785.078	13.010	103.804	21	786.411	13.160	103.382
7000	(21, 23)	832.789	13.458	106.258	23	833.394	13.910	110.153

The effects of the probability  $r$  with which the customers opt SOS facility, on the mean system size and optimal cost values  $T_C(m^*, N^*)$  are shown in Table 2.4. It is found from the table values that both  $L_{(m^*, N^*)}^J$  and  $T_C(m^*, N^*)$  increase as  $r$  increases. The numerical values are obtained for different batch size distributions namely (Geometric)  $Geo(p_1)$  and (Binomial)  $B(4, p_1)$ . The case  $r = 0$  of the table corresponds to the single service queueing model whereas  $r_1 = 1$  corresponds to the model in which each customer undergoes both types of services one after the other.

**Table 2.4 Optimal Policy with respect to  $r$** 

$(p_1, \lambda, a, a_1, a_2, a_3, r_p, r_{p_1}, r_{p_3}, J, C_h) = (0.5, 0.4, 0.3, 0.2, 0.4, 0.6, 0.3, 0.6, 0.7, 17, 30)$

$(r_1, r_2, r_3)$	$r$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$
(0, 0, 0)	0	(3, 13)	690.892	15.678
		(3, 13)	666.892	14.878
(0.05, 0.04, 0.05)	0.14	(1, 12)	721.897	17.299
		(1, 12)	692.013	16.303
(0.5, 0.2, 0.1)	0.8	(1, 7)	1638.221	51.87
		(1, 7)	1509.716	47.586
(0.6, 0.2, 0.1)	0.9	(1, 6)	2488.459	80.974
		(1, 6)	2279.398	74.005
(1, 0, 0)	1.0	(1, 6)	2614.023	85.230
		(1, 6)	2394.023	78.018

■  $Geo(p_1)$

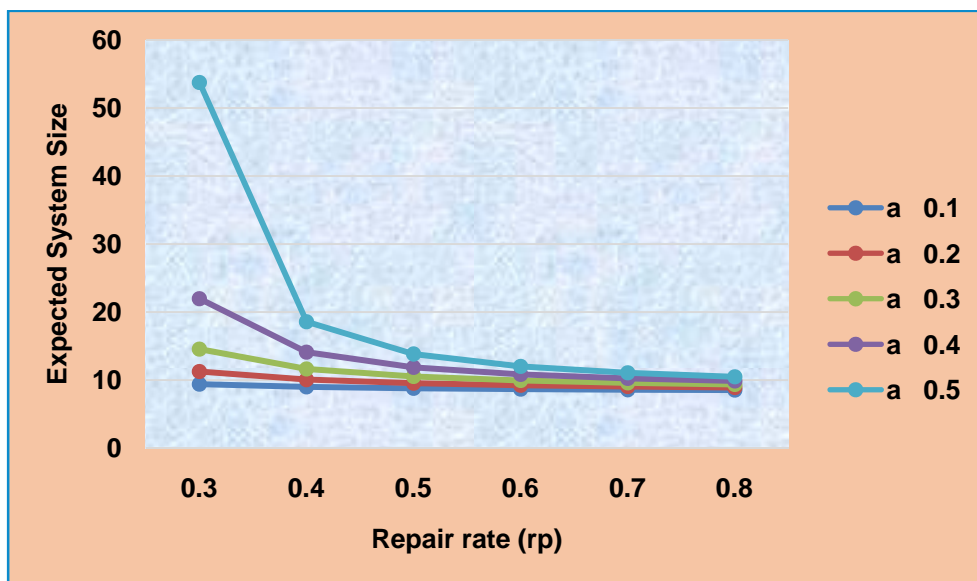
■  $B(4, p_1)$

The Tables 2.5 and 2.6 show that the mean system size increases with the breakdown rates  $a$  ( $a_i$ ) and decreases with the repair rates  $r_p$  ( $r_{p_i}$ ). Figure 2.5 illustrates the graphical representation of Table 2.5.

**Table 2.5 Expected System Size as Repair Rate ( $r_p$ ) for Different Values of Breakdown Rate ( $a$ ) Changes ( $m = 9, N = 13, \lambda = 0.28, J = 17$ )**

$a \backslash r_p$	0.3	0.4	0.5	0.6	0.7	0.8
0.1	9.406	9.002	8.798	8.677	8.596	8.540
0.2	11.262	10.078	9.540	9.237	9.045	8.912
0.3	14.553	11.639	10.518	9.936	9.583	9.348
0.4	21.957	14.106	11.865	10.831	10.242	9.865
0.5	53.767	18.573	13.837	12.016	11.065	10.487

**Figure 2.5. Expected System Size Vs. Repairable rate ( $r_p$ ) for Different Values of  $a$**



**Table 2.6 Expected System Size Vs Breakdown rate  $a_i$  for Different values of  $r_{p_i}$**

$a = 0.5, r_p = 0.3, r_{p_2} = 1/2, J = 17, m = 9, N = 13$

$(r_{p_1}, r_{p_3}) \backslash (a_1, a_2, a_3)$	(0.2, 0.4)	(0.25, 0.45)	(0.3, 0.5)	(0.35, 0.55)	(0.4, 0.6)	(0.5, 0.7)
(0.1, 0.2, 0.15)	51.479	48.719	46.959	45.731	44.821	43.556
(0.15, 0.25, 0.2)	60.364	55.026	51.835	49.700	48.166	46.099
(0.2, 0.3, 0.25)	73.222	63.370	57.948	54.501	52.109	48.993
(0.35, 0.35, 0.3)	142.509	95.713	78.071	68.796	63.068	56.353
(0.45, 0.4, 0.35)	484.365	155.273	105.801	85.835	75.035	63.618

Table 2.7 presents the influence of different parameters on the system size probabilities and mean system size.

**Table 2.7 Various Performance Measures with Respect to  $r_1, r_2, r_3, \lambda$  and Two Different Distributions of X**

$(\rho_1, \gamma, \eta, a, a_1, a_2, a_3, r_p, \Gamma_{p_1}, \Gamma_{p_3}, J) = (0.75, 0.2, 1, 0.5, 0.4, 0.2, 0.6, 0.6, 0.4, 0.7, 17)$ ;  $(m, N) = (9, 13)$

Case 1 :  $(r_1, r_2, r_3) = (0.05, 0.03, 0.04)$ ,  
 Case 2 :  $(r_1, r_2, r_3) = (0.5, 0.2, 0.1)$ ,  
 Case 3 :  $(r_1, r_2, r_3) = (0.05, 0.03, 0.04)$ ,  
 Case 4 :  $(r_1, r_2, r_3) = (0.1, 0.04, 0.1)$ ,  
 Case 5 :  $(r_1, r_2, r_3) = (0.05, 0.03, 0.04)$ ,  
 Case 6 :  $(r_1, r_2, r_3) = (0.5, 0.2, 0.1)$ ,  
 Case 7 :  $(r_1, r_2, r_3) = (0.05, 0.03, 0.04)$ ,  
 Case 8 :  $(r_1, r_2, r_3) = (0.1, 0.04, 0.1)$ ,  
 Geo (0.75) Case B (2, 0.75)

	$P_{br}$	$P_{busy}$	$P_{build\ up}$	$P_{set}$	$P_{dor}$	$P_{VI}$	$L_{(m, N)}^J$	$\lambda$
Case 1	0.249	0.302	0.007	0.074	0.049	0.318	11.803	0.15
Case 2	0.309	0.391	0.004	0.050	0.033	0.213	16.431	0.15
Case 3	0.415	0.504	0.000	0.020	0.005	0.056	60.831	0.25
Case 4	0.437	0.531	0.000	0.008	0.002	0.022	151.570	0.25
Case 5	0.093	0.113	0.012	0.132	0.087	0.563	4.740	0.15
Case 6	0.116	0.147	0.010	0.122	0.081	0.524	4.873	0.15
Case 7	0.156	0.189	0.000	0.163	0.040	0.452	5.289	0.25
Case 8	0.164	0.199	0.000	0.158	0.039	0.439	5.354	0.25

It is noted from the table (2.7) that as  $\lambda$  increases, the probability that the server is busy ( $P_{busy}$ ) and expected queue length ( $L_{(m, N)}^J$ ) also increase.

Table 2.8 reveals that as the mean setup time increases, an early setup is more beneficial than usual N-policy. This is because more customers are expected to arrive during the setup time and dormant period is minimized. It is observed from the Tables 2.8 and 2.9 that the system size decreases as the mean setup time  $E(D)$  or mean vacation time  $E(VI)$  decreases. For Tables 2.8 and 2.9,  $(r_1, r_2, r_3) = (0.05, 0.03, 0.04)$ .

**Table 2.8. Optimal Policy as Mean Setup Time  $E(D)$  Changes with  $J = 17$**

$E(D) = 1/\gamma$	$(m^*, N^*)$	$T_c(m^*, N^*)$	$L_{(m^*, N^*)}^J$	$N^*$	$T_c(N^*, N^*)$	$L_{(N^*, N^*)}^J$
18.182	(8, 20)	695.670	11.398	13	707.221	12.451
13.333	(9, 18)	635.175	10.492	13	643.306	11.225
10.000	(12, 17)	591.165	10.424	13	595.241	10.369
6.667	(12, 15)	540.536	9.387	13	542.784	9.499
5.000	(12, 14)	513.083	8.853	13	514.527	9.058

**Table 2.9 Optimal Policy as Mean Vacation Time E(VI) Changes with J = 17**

$E(VI) = 3/\eta$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$	$N^*$	$T_C(N^*, N^*)$	$L_{(N^*, N^*)}^J$
60.000	(1, 23)	760.95	21.078	3	773.645	21.403
30.000	(1, 16)	573.230	12.034	7	589.501	12.860
6.897	(9, 15)	520.934	8.714	12	524.326	9.120
6.000	(9, 15)	519.287	8.654	12	522.162	9.011

Table 2.10 exhibits the effect of starting cost per cycle ( $C_y$ ) on  $T_C(m^*, N^*)$  and  $L_{(m^*, N^*)}^J$ . The values show that as ( $C_y$ ) increases the optimum cost and expected system size also increase.

**Table 2.10 Optimal Policy as Cycle Cost Changes****( $C_{Set} = 500, r = 0.15, J = 17, C_{dor} = 100, C_h = 200$ )**

$C_y$	$(m^*, N^*)$	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$	E(Cycle)	$N^*$	$T_C(N^*, N^*)$	$L_{(N^*, N^*)}^J$	E(Cycle)
10000	(1, 5)	1478.421	5.419	35.144	2	1556.161	5.709	32.580
20000	(1, 6)	1749.087	5.579	37.924	3	1847.315	6.057	36.976
30000	(1, 8)	1993.402	6.082	44.233	4	2097.858	6.436	41.360
40000	(1, 9)	2216.193	6.406	47.650	5	2319.652	6.835	45.703
60000	(1, 11)	2614.881	7.149	54.811	7	2705.349	7.664	54.222
80000	(3, 12)	2964.700	7.707	59.549	9	3040.942	8.514	62.482
90000	(3, 13)	3123.716	8.093	63.085	10	3195.998	8.942	66.510

It is calculated that the length of the cycle for the 1-policy model ( $m = N = 1$ ) corresponding to the parametric values of Table 2.10 is 28.226 and this is less than that of ( $m, N$ )-policy as well as N-policy.