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CONTRA CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. Contra continuous functions on generalized topological spaces are introduced, and their characterizations and properties are investigated.

1. Introduction

The theory of generalized topological spaces, which was founded by Császár in recent years, is one of the most important developments of general topology (see [1–6]). In [1], Császár introduced the concept of continuity on GTS's and gave characterizations and properties of this notion. W. K. Min [7,8] introduced the concepts of weak continuity and almost continuity on GTS's, investigated properties of these notions and obtained relationships among continuity, almost continuity and weak continuity on GTS's. S. Bai and Y. Zuo [11] introduced the concepts of g-α-irresolute functions.

The purpose of this paper is to investigate contra continuity on GTS's. We introduce the concept of contra continuity on GTS's and give characterizations and properties of this notion.

2. Preliminaries

We recall some basic concepts and results.

Let X be a nonempty set and let g be a family of subsets of X. g is called a generalized topology (briefly GT) on X if $\emptyset \in g$ and $G_i \in g$ for each $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} G_i \in g$. We call the pair (X, g) a generalized topological

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space (briefly GTS). The elements of g are called g-open subset of X and the complements are called g-closed subset of X.

Let (X, g) be a GTS. For $A \subset X$, the closure of A and the interior of A in X are defined as the following:

$$\mathrm{cl}_g(A) = \bigcap\{F : F \text{ is } g\text{-closed in } X \text{ and } A \subset F\},$$

$$\mathrm{int}_g(A) = \bigcup\{V : V \text{ is } g\text{-open in } X \text{ and } V \subset A\}.$$

In this paper, spaces always mean generalized topological spaces on which no separation axiom is assumed. The family of all g-open subsets of X and the family of all g-closed subsets of X are denoted by $gO(X)$ and $gC(X)$, respectively. For $x \in X$, the family of all g-open subsets containing x and the family of all g-closed subsets containing x are denoted by $gO(X, x)$ and $gC(X, x)$, respectively. We simply use cA and iA instead of $\mathrm{cl}_g(A)$ and $\mathrm{int}_g(A)$, respectively. Sometimes, the generalized topology on X also is denoted by g_X, i.e., $g_X = gO(X)$.

DEFINITION 2.1 [1]. A function $f : (X, g_X) \to (Y, g_Y)$ is called (g_X, g_Y)-continuous, if $f^{-1}(V) \in gO(X)$ for every $V \in gO(Y)$.

THEOREM 2.2 [1]. *Let* (X, g_X) *be a GTS and* $A \subset X$. *Then*

$$cA = X - i(X - A); \tag{1}$$

$$iA = X - c(X - A). \tag{2}$$

3. Contra continuity on GTS's

DEFINITION 3.1. A function $f : (X, g_X) \to (Y, g_Y)$ is called contra (g_X, g_Y)-continuous, if $f^{-1}(V) \in gC(X)$ for each $V \in gO(Y)$.

EXAMPLE 3.2. Let $X = Y = \{a, b, c, d\}$, $g_X = g_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. We define the function $f : (X, g_X) \to (Y, g_Y)$ such that

$$f(a) = f(b) = a, \quad f(c) = c, \quad f(d) = d.$$

Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a\}) = \{a, b\}$, $f^{-1}(\{a, b\}) = \{a, b\}$ and $f^{-1}(\{a, b, c\}) = \{a, b, c\}$ are g-open subsets of X, then f is (g_X, g_Y)-continuous.

Since $\{a, b\} \in gO(Y)$ and $f^{-1}(\{a, b\}) = \{a, b\} \notin gC(X)$, then f is not contra (g_X, g_Y)-continuous.

Example 3.3. Let $X = Y = \{a, b, c, d\}$, $g_X = \{\emptyset, \{a, b, c\}, X\}$ and $g_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}\}$. We define the function $f : (X, g_X) \to (Y, g_Y)$ such that

$$f(a) = f(b) = f(c) = d, \quad f(d) = c.$$

Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a\}) = \emptyset$, $f^{-1}(\{a, b\}) = \emptyset$, $f^{-1}(\{a, b, c\}) = \{d\}$ and $f^{-1}(\{a, c, d\}) = X$ are g-closed subsets of X, then f is (g_X, g_Y)-contra (g_X, g_Y)-continuous.

Since $\{a, b, c\} \in gO(Y)$ and $f^{-1}(\{a, b, c\}) = \{d\} \notin gO(X)$, then f is not (g_X, g_Y)-continuous.

Then f is contra (g_X, g_Y)-continuous. But f is not (g_X, g_Y)-continuous.

From Examples 3.2 and 3.3, we have the following relations:

$$\begin{tikzcd} (g_X, g_Y)\text{-continuity} \arrow[r, shift left, "/" marking] & \text{contra } (g_X, g_Y)\text{-continuity.} \arrow[l, shift left, "\backslash" marking] \end{tikzcd}$$

Definition 3.4. A function $f : (X, g_X) \to (Y, g_Y)$ is called contra (g_X, g_Y)-continuous at some $x \in X$, if for $V \in gC(Y, f(x))$, there exists $U \in gO(X, x)$ such that $f(U) \subset V$.

Theorem 3.5. *Let* $f : (X, g_X) \to (Y, g_Y)$ *be a function. Then the following are equivalent*:

(1) f *is contra* (g_X, g_Y)*-continuous.*

(2) $f^{-1}(F) \in gO(X)$ *for any* $F \in gC(Y)$.

(3) *For each* $x \in X$ *and each* $V \in gO(Y)$ *with* $f(x) \notin V$, *there exists* $U \in gC(X)$ *such that* $x \notin U$ *and* $f^{-1}(V) \subset U$.

(4) f *is contra* (g_X, g_Y)*-continuous at any* $x \in X$.

(5) $f^{-1}(F) \subset if^{-1}(F)$ *for any* $F \in gC(Y)$.

(6) $cf^{-1}(V) \subset f^{-1}(V)$ *for any* $V \in gO(Y)$.

(7) $cf^{-1}(iB) \subset f^{-1}(iB)$ *for any* $B \subset Y$.

(8) $f^{-1}(cB) \subset if^{-1}(cB)$ *for any* $B \subset Y$.

Proof. (1) $\Rightarrow$ (2). Let $F \in gC(Y)$. Then $Y - F \in gO(Y)$. By (1),

$$f^{-1}(Y - F) = X - f^{-1}(F) \in gC(X).$$

Thus $f^{-1}(F) \in gO(X)$.

(1) $\Rightarrow$ (3). Let $x \in X$ and $V \in gO(Y)$ with $f(x) \notin V$. Then $x \notin f^{-1}(V)$. By (1), $f^{-1}(V) \in gC(X)$. Put $U = f^{-1}(V)$. Then $f^{-1}(V) \subset U$ and $x \notin U$.

(3) $\Rightarrow$ (1). Let $V \in gO(Y)$. For each $x \in f^{-1}(Y - V)$, $f(x) \notin V$. By (3), there exists $U_x \in gC(X)$ such that $x \notin U_x$ and $f^{-1}(V) \subset U_x$. Then $x \in X - U_x \subset X - f^{-1}(V) = f^{-1}(Y - V)$. We have

$$\bigcup_{x \in f^{-1}(Y-V)} \{x\} \subset \bigcup_{x \in f^{-1}(Y-V)} (X - U_x) \subset f^{-1}(Y - V).$$

Thus $f^{-1}(Y-V)=\bigcup_{x\in f^{-1}(Y-V)}(X-U_x)\in gO(X)$. This implies $f^{-1}(V)\in gC(X)$. Hence f is contra (g_X,g_Y)-continuous.

(2) ⇒ (4). Let $x\in X$ and $V\in gC\big(Y,f(x)\big)$. By (2), $f^{-1}(V)\in gO(X)$. Put $U=f^{-1}(V)$. We have $U\in gO(X,x)$ and $f(U)\subset V$.

(4) ⇒ (5). Let $F\in gC(Y)$. For each $x\in f^{-1}(F)$, $f(x)\in F$. By (4), there exists $U\in gO(X,x)$ such that $f(U)\subset F$. Since $x\in U\subset f^{-1}(F)$, we have $x\in if^{-1}(F)$. This implies $f^{-1}(F)\subset if^{-1}(F)$.

(5) ⇒ (6). Let $V\in gO(Y)$. Then $Y-V\in gC(Y)$. By (5) and Theorem 2.2,

$$f^{-1}(Y-V)\subset if^{-1}(Y-V)=i\big(X-f^{-1}(V)\big)=X-cf^{-1}(V).$$

Thus $cf^{-1}(V)\subset f^{-1}(V)$.

(6) ⇒ (7). Let $B\subset Y$. Since $iB\in gO(Y)$, by (6), we have

$$cf^{-1}(iB)\subset f^{-1}(iB).$$

(7) ⇒ (8). Let $B\subset Y$. By (7), $cf^{-1}\big(i(Y-B)\big)\subset f^{-1}\big(i(Y-B)\big)$. By Theorem 2.2,

$$cf^{-1}\big(i(Y-B)\big)=cf^{-1}(Y-cB)=c\big(X-f^{-1}(cB)\big)=X-if^{-1}(cB),$$

and $f^{-1}\big(i(Y-B)\big)=X-f^{-1}(cB)$. Thus $f^{-1}(cB)\subset if^{-1}(cB)$.

(8) ⇒ (1). Let $B\in gO(Y)$. Then $Y-B\in gC(Y)$. By (8),

$$X-f^{-1}(B)=f^{-1}(Y-B)=f^{-1}\big(c(Y-B)\big)\subset if^{-1}\big(c(Y-B)\big)$$

$$=if^{-1}(Y-B).$$

By Theorem 2.2,

$$if^{-1}(Y-B)=i\big(X-f^{-1}(B)\big)=X-cf^{-1}(B).$$

Then $f^{-1}(B)\supset cf^{-1}(B)$. Thus $f^{-1}(B)\in gC(X)$. This shows that f is contra (g_X,g_Y)-continuous. □

Definition 3.6. Let (X,g_X) be a GTS and $A\subset X$. Then

(1) The set $\bigcap\{U\in g_X : A\subset U\}$ is called the kernel of A in X. We denote it by $\ker(A)$.

(2) The set $cA\cap c(X-A)$ is called the frontier of A in X. We denote it by $\mathrm{Fr}(A)$.

Example 3.7. Let $X=\{a,b,c,d\}$ and $g_X=\big\{\emptyset,\{a\},\{a,c\},\{b,c\},\{a,b,c\}\big\}$. Let $A=\{a,b\}$. Since $\{U\in g_X : A\subset U\}=\big\{\{a,b,c\}\big\}$, then $\ker(A)=\{a,b,c\}$. Since $gC(X)=\big\{\{d\},\{a,d\},\{b,d\},\{b,c,d\},X\big\}$, then

$\{F \in gC(X) : A \subset F\} = \{X\}$ and $\{F \in gC(X) : X - A \subset F\} = \{\{b, c, d\}, X\}$. Then $cA = X$ and $c(X - A) = \{b, c, d\} \cap X = \{b, c, d\}$. Thus

$$\mathrm{Fr}\,(A) = cA \cap c(X - A) = \{b, c, d\}.$$

LEMMA 3.8. *Let (X, g_X) be a GTS, $A, B \subset X$ and $x \in X$. Then the following properties hold.*

(1) $A \subset \ker(A)$.

(2) *If $A \in gO(X)$, then $A = \ker(A)$.*

(3) *If $A \subset B$, then $\ker(A) \subset \ker(B)$.*

(4) *$x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in gC(X, x)$.*

PROOF. (1)–(2) Obvious.

(3) Suppose that $\ker(A) - \ker(B) \neq \emptyset$. Pick $x \in \ker(A) - \ker(B)$. $x \notin \ker(B)$ implies $B \subset U$ and $x \notin U$ for some $U \in g_X$. Since $A \subset B$, then $A \subset U$. Note that $x \in \ker(A)$, $x \in U$, a contradiction. Thus $\ker(A) - \ker(B) = \emptyset$. So $\ker(A) \subset \ker(B)$.

(4) Let $x \in \ker(A)$. Suppose that $F \in gC(X, x)$ and $A \cap F = \emptyset$, then $A \subset X - F$. Since $X - F \in gO(X)$, by (2) and (3),

$$\ker(A) \subset \ker(X - F) = X - F.$$

Then $x \in X - F$. So $x \notin F$, a contradiction. Thus $A \cap F \neq \emptyset$ for any $F \in gC(X, x)$.

On the other hand, suppose that $x \notin \ker(A)$, then there exists $U \in g_X$ such that $A \subset U$ and $x \notin U$. Since $X - U \in gC(X, x)$, then $A \cap (X - U) \neq \emptyset$. So $A \not\subset U$, a contradiction. Thus $x \in \ker(A)$. $\square$

THEOREM 3.9. *Let $f : (X, g_X) \to (Y, g_Y)$ be a function. Then the following are equivalent.*

(1) *f is contra (g_X, g_Y)-continuous;*

(2) *$f(cA) \subset \ker\big(f(A)\big)$ for any $A \subset X$;*

(3) *$cf^{-1}(B) \subset f^{-1}\big(\ker(B)\big)$ for any $B \subset Y$.*

PROOF. (1) $\Rightarrow$ (2). Let $A \subset X$. Suppose that $f(cA) - \ker\big(f(A)\big) \neq \emptyset$. Pick $y \in f(cA) - \ker\big(f(A)\big)$. By $y \notin \ker\big(f(A)\big)$, there exists $F \in gC(Y, y)$ such that $f(A) \cap F = \emptyset$. Then $A \cap f^{-1}(F) = \emptyset$ and $cA \cap f^{-1}(F) = \emptyset$. This implies that $f(cA) \cap F = \emptyset$ and $y \notin f(cA)$. Thus $f(cA) \subset \ker\big(f(A)\big)$.

(2) $\Rightarrow$ (3). Let $B \subset Y$. By (2),

$$f\big(cf^{-1}(B)\big) \subset \ker\Big(f\big(f^{-1}(B)\big)\Big) \subset \ker(B).$$

Thus $cf^{-1}(B) \subset f^{-1}\big(\ker(B)\big)$.

(3) ⇒ (1). Let $B \in gO(Y)$. By (3), $cf^{-1}(B) \subset f^{-1}\big(\ker(B)\big)$. By Lemma 3.8, $B = \ker(B)$. Thus $cf^{-1}(B) \subset f^{-1}(B)$. This implies $f^{-1}(B) \in gC(X)$. □

Theorem 3.10. *Let $f : (X, g_X) \to (Y, g_Y)$ be a function. Put*

$$P = \{\, x \in X : f \textit{ is not } (g_X, g_Y)\textit{-continuous at } x \},$$

then

$$P = \bigcup \Big\{ \operatorname{Fr}\big(f^{-1}(B)\big) : B \in gC\big(Y, f(x)\big) \textit{ and } x \in X \Big\}.$$

Proof. Let $x_0 \in P$. Since f is not contra (g_X, g_Y)-continuous at x_0, then there exists $B_0 \in gC\big(Y, f(x_0)\big)$ such that $f(U) \not\subset B_0$ for any $U \in gO(X, x_0)$. So $U \not\subset f^{-1}(B_0)$ for any $U \in gO(X, x_0)$. Thus $U \cap \big(X - f^{-1}(B_0)\big) \neq \emptyset$ for any $U \in gO(X, x_0)$. This implies $x_0 \in c\big(X - f^{-1}(B_0)\big)$. Note that $f(x_0) \in B_0$. Then $x_0 \in cf^{-1}(B_0)$. Thus $x_0 \in \operatorname{Fr}\big(f^{-1}(B_0)\big)$. Hence

$$x_0 \in \bigcup \Big\{ \operatorname{Fr}\big(f^{-1}(B)\big) : B \in gC\big(Y, f(x)\big) \text{ and } x \in X \Big\}.$$

On the other hand, let $x_0 \in \bigcup \Big\{ \operatorname{Fr}\big(f^{-1}(B)\big) : B \in gC\big(Y, f(x)\big)$ and $x \in X \Big\}$. Then $x_0 \in X$ and $x_0 \in \operatorname{Fr}\big(f^{-1}(B_0)\big)$ for some $B_0 \in gC\big(Y, f(x_0)\big)$. Since $x_0 \in c\big(X - f^{-1}(B_0)\big)$, then $U \cap \big(X - f^{-1}(B_0)\big) \neq \emptyset$ for any $U \in gO(X, x_0)$. Thus, there exists $B_0 \in gC\big(Y, f(x_0)\big)$ such that $f(U) \not\subset B_0$ for any $U \in gO(X, x_0)$. Hence f is not (g_X, g_Y)-continuous at x_0, that is $x_0 \in P$.

Therefore

$$P = \bigcup \Big\{ \operatorname{Fr}\big(f^{-1}(B)\big) : B \in gC\big(Y, f(x)\big) \text{ and } x \in X \Big\}. \quad \square$$

Corollary 3.11. *A function $f : (X, g_X) \to (Y, g_Y)$ is (g_X, g_Y)-continuous at some $x \in X$ if and only if $x \notin \operatorname{Fr}\big(f^{-1}(B)\big)$ for any $B \in gC\big(Y, f(x)\big)$.*

Corollary 3.12. *A function $f : (X, g_X) \to (Y, g_Y)$ is (g_X, g_Y)-continuous if and only if*

$$\bigcup \Big\{ \operatorname{Fr}\big(f^{-1}(B)\big) : B \in gC(Y, f(x) \textit{ and } x \in X) \Big\} = \emptyset.$$

4. Properties of contra continuity on GTS's

By Theorem 3.5, we have the following

Theorem 4.1. *Let* $f : (X, g_X) \to (Y, g_Y)$ *and* $g : (Y, g_Y) \to (Z, g_Z)$ *be two functions. Then the following properties hold*:

(1) *If* f *is contra* (g_X, g_Y)*-continuous and* g *is* (g_Y, g_Z)*-continuous, then the composition* $g \circ f$ *is contra* (g_X, g_Z)*-continuous.*

(2) *If* f *is* (g_X, g_Y)*-continuous and* g *is contra* (g_Y, g_Z)*-continuous, then the composition* $g \circ f$ *is contra* (g_X, g_Z)*-continuous.*

Definition 4.2. A function $f : (X, g_X) \to (Y, g_Y)$ is called g-open, if $f(U) \in gO(Y)$ for each $U \in gO(X)$.

Theorem 4.3. *If* $f : (X, g_X) \to (Y, g_Y)$ *is a surjective* g*-open function and* $g : (Y, g_Y) \to (Z, g_Z)$ *is a function such that the composition* $g \circ f : (X, g_X) \to (Z, g_Z)$ *is contra* (g_X, g_Z)*-continuous, then* g *is contra* (g_Y, g_Z)*-continuous.*

Proof. Let $F \in gC(Z)$. Since $g \circ f : X \to Z$ is contra (g_X, g_Z)-continuous, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)) \in gO(X)$. Since f is surjective and g-open, we have $f\big(f^{-1}(g^{-1}(F))\big) = g^{-1}(F) \in gO(Y)$. Thus g is contra (g_Y, g_Z)-continuous. □

Lemma 4.4 [4]. *Let* $\{X_\alpha : \alpha \in \Lambda\}$ *be a family of GTS's. Then the projection* $p_\alpha : X \to X_\alpha$ *is* (g_X, g_{X_α})*-continuous for any* $\alpha \in \Lambda$*, where* $X = \prod_{\alpha\in\Lambda} X_\alpha$.

By Theorem 4.1 and Lemma 4.4, we have the following

Theorem 4.5. *Let* $\{Y_\alpha : \alpha \in \Lambda\}$ *be a family of GTS's. If* $f : X \to Y$ *is contra* (g_X, g_Y)*-continuous, then* $p_\alpha \circ f : X \to Y_\alpha$ *is contra* (g_X, g_{Y_α})*-continuous for any* $\alpha \in \Lambda$*, where* $Y = \prod_{\alpha\in\Lambda} Y_\alpha$ *and* $p_\alpha : Y \to Y_\alpha$ *is the projection.*

Theorem 4.6. *Let* $f : (X, g_X) \to (Y, g_Y)$ *be a function and* $H \in gC(X)$*. If* f *is contra* (g_X, g_Y)*-continuous, then the restriction* $f|_H : (H, g_X|_H) \to (Y, g_Y)$ *is contra* $(g_X|_H, g_Y)$ *-continuous.*

Proof. Let $F \in gO(Y)$. Since f is contra (g_X, g_Y)-continuous, then $f^{-1}(F) \in gC(X)$. By $H \in gC(X)$, $(f|_H)^{-1}(F) = f^{-1}(F) \cap H \in gC(X)$. Thus $f|_H$ is contra $(g_X|_H, g_Y)$ -continuous. □

Lemma 4.7. *Let* (X, g_X) *and* (Y, g_Y) *be two GTS's. If* $A \in gC(X)$ *and* $B \in gC(Y)$*, then* $A \times B \in gC(X \times Y)$.

Proof. By hypothesis, $cA = A$, $cB = B$. By Proposition 2.3 in [4], $A \times B = cA \times cB = c(A \times B)$. Hence $A \times B \in gC(X \times Y)$. □

THEOREM 4.8. *Let* $f:(X,g_X)\to(Y,g_Y)$ *be a function and let* $g:(X,g_X)\to(X\times Y,g_{X\times Y})$ *be the graph function of* f, *defined by* $g(x)=(x,f(x))$ *for each* $x\in X$. *If* g *is contra* $(g_X,g_{X\times Y})$*-continuous, then* f *is contra* (g_X,g_Y)*-continuous.*

PROOF. Let $V\in gC(Y)$. By Lemma 4.7, $X\times V\in gC(X\times Y)$. Since g is contra $(g_X,g_{X\times Y})$-continuous, by Theorem 3.5, $g^{-1}(X\times V)\in gO(X)$. Note that $g^{-1}(X\times V)=f^{-1}(V)$. By Theorem 3.5, f is contra (g_X,g_Y)-continuous. □

DEFINITION 4.9 [9,10]. Let (X,g_X) be a GTS.

(1) X is called g-connected, if there are no nonempty disjoint g-open subsets U, V of X such that $U\cup V=X$.

(2) X is called g-hyperconnected, if every nonempty g-open subset U of X is g-dense (i.e., $cU=X$).

REMARK 4.10 [9]. (X,g_X) is g-hyperconnected ⇒ (X,g_X) is g-connected.

The following Lemma 4.11 can be easily proved.

LEMMA 4.11. *Let* (X,g_X) *be a GTS. If* U, V *are nonempty disjoint* g*-open subsets of* X *and* $U\cup V=X$, *then* $U,V\in gC(X)$.

THEOREM 4.12. *Contra continuous functions on GTS's preserve* g*-connectedness.*

PROOF. Let $f:(X,g_X)\to(Y,g_Y)$ be a contra (g_X,g_Y)-continuous surjection and let X be g-connected. Suppose that Y is not g-connected. Then there are nonempty disjoint g-open subsets V_1, V_2 of Y such that $V_1\cup V_2=Y$. By Lemma 4.11, $V_1,V_2\in gC(Y)$. Since f is contra (g_X,g_Y)-continuous, then $f^{-1}(V_1),f^{-1}(V_2)\in gO(X)$. Note that $f^{-1}(V_1)\cap f^{-1}(V_2)\neq\emptyset$ and $X=f^{-1}(V_1)\cup f^{-1}(V_2)$. Then X is not g-connected, contradiction. Thus Y is g-connected. □

COROLLARY 4.13. *Contra continuous images of* g*-hyperconnected spaces are* g*-connected.*

DEFINITION 4.14. Let $f:(X,g_X)\to(Y,g_Y)$ be a function. The graph $G(f)=\big\{(x,f(x)):x\in X\big\}$ of f is called contra-(g_X,g_Y)-graph, if for each $(x,y)\in X\times Y-G(f)$, there are $A\in gO(X,x)$, $B\in gC(Y,y)$ such that $(A\times B)\cap G(f)=\emptyset$.

THEOREM 4.15. *Let* $f:(X,g_X)\to(Y,g_Y)$ *be a function and let* $G(f)$ *be the graph of* f. *Then the following are equivalent.*

(1) $G(f)$ *is contra-*(g_X,g_Y)*-graph.*

(2) *For each* $(x,y)\in X\times Y-G(f)$, *there are* $A\in gO(X,x)$ *and* $B\in gC(Y,y)$ *such that* $f(A)\cap B=\emptyset$.

PROOF. The proof follows from the fact that $(A \times B) \cap G(f) = \emptyset$ if and only if $f(A) \cap B = \emptyset$ for any $A \subset X$ and $B \subset Y$. $\square$

DEFINITION 4.16. Let (X, g_X) be a GTS.

(1) X is called g-Urysohn, if for each pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gO(X, x_1)$, $V \in gO(X, x_2)$ such that $cU \cap cV = \emptyset$.

(2) X is called g-T_2 [6,9], if for each pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gO(X, x_1)$, $V \in gO(X, x_2)$ such that $U \cap V = \emptyset$.

(3) X is called g-T_1 [6], if for each pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gO(X, x_1)$, $V \in gO(X, x_2)$ such that $x_2 \notin U$ and $x_1 \notin V$.

THEOREM 4.17. *Let $f : (X, g_X) \to (Y, g_Y)$ be contra (g_X, g_Y)-continuous. If Y is g-Urysohn, then $G(f)$ is a contra-(g_X, g_Y)-graph and closed in $X \times Y$.*

PROOF. Let $(x, y) \in X \times Y - G(f)$. Then $f(x) \neq y$. Since Y is g-Urysohn, then there are $U \in gO\big(Y, f(x)\big)$, $V \in gO(Y, y)$ such that $cU \cap cV = \emptyset$. Note that $cU \in gC\big(Y, f(x)\big)$. Since f is contra (g_X, g_Y)-continuous, by Theorem 3.5, there are $A \in gO(X, x)$ such that $f(A) \subset cU$. Put $B = cV$. Obviously, $B \in gC(Y, y)$ and $f(A) \cap B = \emptyset$. By Theorem 4.15, $G(f)$ is a contra-(g_X, g_Y)-graph.

Note that $f(x) \in U$ and $cU \cap cV = \emptyset$. Then $f(x) \notin cV$. Thus $x \in X - f^{-1}(cV)$. So $(x, y) \in \big(X - f^{-1}(cV)\big) \times V \in gO(X) \times gO(Y)$. We claim that $(X - f^{-1}(cV) \times V) \cap G(f) = \emptyset$. Otherwise, pick $(a, b) \in (X - f^{-1}(cV) \times V) \cap G(f)$. $(a, b) \in G(f)$ implies $b = f(a)$. $(a, b) \in X - f^{-1}(cV) \times V$ implies $f(a) \notin cV$ and $b \in V$. Then $f(a) \in V \subset cV$, a contradiction. Thus

$$(x, y) \in \big(X - f^{-1}(cV)\big) \times V \subset X \times Y - G(f).$$

This implies $X \times Y - G(f) \in gO(X \times Y)$. Hence $G(f)$ is g-closed in $X \times Y$. $\square$

THEOREM 4.18. *Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (g_X, g_Y)-continuous injection. If Y is g-Urysohn, then X is g-T_2.*

PROOF. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Put $y_1 = f(x_1)$, $y_2 = f(x_2)$. Then $y_1 \neq y_2$. Since Y is g-Urysohn, then there are $V_1 \in gO(Y, y_1)$, $V_2 \in gO(Y, y_2)$ such that $cV_1 \cap cV_2 = \emptyset$.

Since f is contra (g_X, g_Y)-continuous, then $f^{-1}(cV_1), f^{-1}(cV_2) \in gO(X)$. Put $U_1 = f^{-1}(cV_1)$, $U_2 = f^{-1}(cV_2)$. Then $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Thus X is g-T_2. $\square$

Finally, we give Theorem 4.19 in order to compare with the results in Theorem 4.17 and Theorem 4.18.

THEOREM 4.19. *Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (g_X, g_Y)-continuous injection. If $G(f)$ is contra-(g_X, g_Y)-graph, then X is g-T_1.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, then $f(x_1) \neq f(x_2)$. We have $(x_1, f(x_2)) \in X \times Y - G(f)$. By Theorem 4.15, there are $U \in gO(X, x_1)$ and $B \in gC(Y, f(x_2))$ such that $f(U) \cap B = \emptyset$. Since f is contra (g_X, g_Y)-continuous, then $f^{-1}(B) \in gO(X, x_2)$.

Put $f^{-1}(B) = V$. $x_1 \in U$ and $f(U) \cap B = \emptyset$ imply $f(x_1) \notin B$. Thus $x_1 \notin V$. $f(U) \cap B = \emptyset$ implies $U \cap f^{-1}(B) = \emptyset$. By $f(x_2) \in B$, $x_2 \notin U$. Thus X is g-T_1. $\square$

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