Chapter 6

Weaker Forms of δP_S-continuity

6.1 Introduction

Continuity and multi-functions are basic topics in several branches of mathematics such as in general topology, set valued analysis. Several different forms of continuous multi-functions have been introduced and studied over the years. Many authors have researched and studied several stronger and weaker forms of continuous functions and multi-functions. The aim of this chapter is to give a new weaker form of some types of continuity including almost δP_S-continuity and weakly δP_S-continuity. In this chapter, almost δPs-continuity and weakly δP_S-continuity is introduced and studied. Also, the notion of almost δP_S-continuous functions and weakly δP_S-continuousfunctions is introduced and characterizations and some relationships are investigated and obtained.

6.2 Almost δP_S-Continuous Functions

Definition 6.2.1: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called **almost δP_S-continuous function** at a point $x \in$ X if for each open set V of Y containing $f(x)$, there exists a δP_S-open set U of X containing x such that f(U) ⊆ Int(δCl(V)). If f is almost δP_S-continuous at every point of X, then it is called almost δP_S-continuous.

Note 6.2.2. For an open set $\delta Cl(V) = Cl(V)$ [Proposition 1.3.18(b)]. Hence in the definition $f(U) \subseteq IntCl(V)$.

Theorem 6.2.3: The following results supervene from their definitions directly:

a) Every δP_S-continuous functions is almost δP_S-continuous.

b) Every almost P_S-continuous function is almost δP_S-continuous.

Proof: (a) Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be δP_S-continuous. Then for $x \in X$ and $V \in \sigma$ containing $f(x)$ there exists a δP_S-open set U in X containing x such that $f(U) \subseteq V$ ⟶ (1)

Then $V \subseteq Cl(V)$. Since V is open, $V = Int\, V \subseteq Int\ Cl(V)$ ⟶ (2)

∴ From (1) & (2) $V \subseteq Int\ Cl(V)$

Hence from Note 6.2.2, f is almost δP_S-continuous function.

(b) Every almost P_S-continuous function is almost δP_S-continuous function

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be almost P_S-continuous function. Then for $x \in X$ and $V \in \sigma$ containing $f(x)$ there exists a δP_S-open set U in X containing x such that $f(U) \subseteq IntCl(V)$. Since every P_S-open set is δP_S-open, from Note 6.2.2, f is almost δP_S-continuous.

Proposition 6.2.4: If a function $f:(X,\tau)\to(Y,\sigma)$ is δ-continuous, then f is almost δP_S-continuous.

Proof. From Proposition 1.3.9, Every δ-continuous is almost δP_S-continuous. From Theorem 6.2.3(b) every almost δP_S-continuous functions is almost δP_S-continuous. Therefore, every δ-continuous function is almost δP_S-continuous functions.

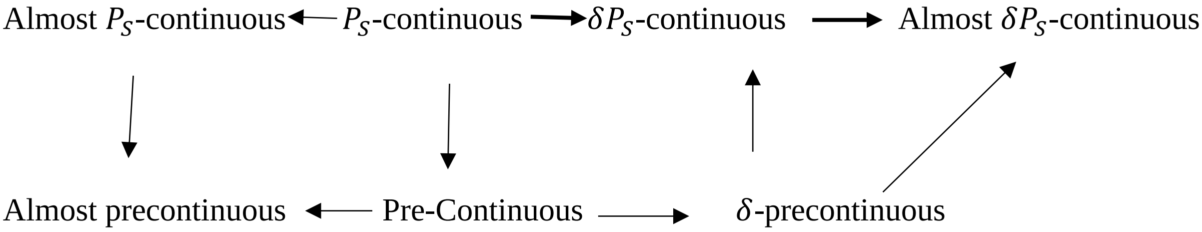

Figure 6.1

And we have

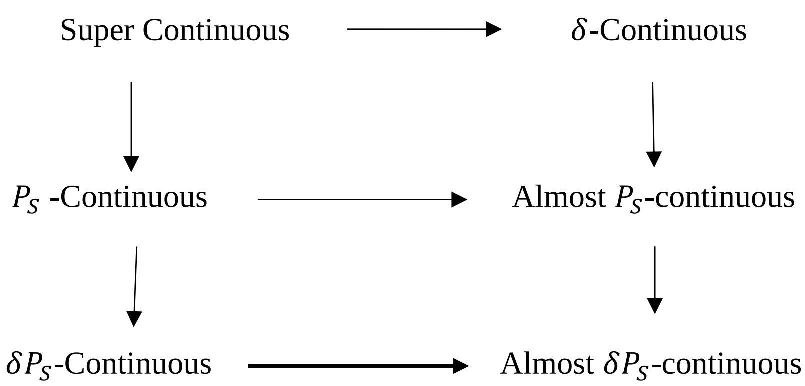

Figure 6.2

The following examples substantiate the converse of Theorem 6.2.3(a) is generally not true.

Example 6.2.5. Let $X=\{a,b,c\}$ with the two topologies $\tau=\{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}\}$ and $\sigma=\{X,\emptyset,\{a\},\{a,b\}\}$; then the $\delta P_S O(X)=\{X,\emptyset,\{b\},\{c\},\{a,c\},\{b,c\}\}$ with respect to τ. Let $f:(X,\tau)\to(X,\sigma)$ be the identity function, for $a\in V=\{a\}$ or $\{a,b\}$, then there exists $U=\{a,c\}$ such that $f(U)=\{a,c\}=Int(Cl(\{a,c\})=X$. Then f is almost δP_S-continuous, but it is not δP_S-continuous, because $f(U)=\{a,c\}\nsubseteq V=\{a\}$ or $\{a,b\}$.

The following example substantiate the Theorem 6.2.3(b) is not true in general.

Example 6.2.6. Let $X=\{a,b,c,d\}$ with the two topologies $\tau=\{X,\emptyset,\{a\},\{a,b\},\{a,b,c\},\{a,b,d\}$ and $\sigma=\{X,\emptyset,\{a\},\{c,d\},\{a,c,d\}\}$. Let $f:(X,\tau)\to(X,\sigma)$ be defined by $f(a)=f(b)=b, f(c)=c$ and $f(d)=d$. Then for a and b, $f(a)=f(b)=b\in X$ and $Int\ Cl(V)=X$. Therefore, there exists $\delta P_S O(X)=X\subseteq a$ and $\delta P_S O(X)=\{b\}\subseteq b$ such that $f(U)=X\subseteq X$ and $f(U)=\{b\}\subseteq X$, for

$f(c) \in \{c,d\}, \{a,c,d\} = V$ and but there exists $f(c) = U$ such that $f(U) = \{c\} \subseteq \{c,d\}$ and X. Hence f is almost δP_S-continuous but not almost P_S-continuous.

The following example shows that almost δP_S-continuous but not δ-continuous.

Example 6.2.7. Let $X = \{a,b,c,d\}$ with the two topologies $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$; Let $f: (X,\tau) \to (X,\sigma)$ be identity functions for $b \in X$ and $V = \{b\} \in \sigma$ there exists $U = \{a,b\}$ containing b which is δP_S-open in X such that $f(U) = \{a,b\} \subseteq Int\ \delta Cl(V) = Int(X) = X$. Here f is almost δP_S-continuous but not δ-continuous. Since $f(Int\ Cl(U)) = f(Int(Cl\{a,b\}) = f(X) = Y \nsubseteq Int\ Cl(V) =$
$Int\ Cl(\{b\}) = Int\{b,c,d\} = \emptyset$.

Remark 6.2.8: The following theorem gives the characterizations to almost δP_S-continuous functions.

Theorem 6.2.9. For a function $f: (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.
b) For each $x \in X$ and each δ-open set V of Y containing $f(x)$ there exists a δP_S-open set U in X containing x such that $f(U) \subseteq sClV$.
c) For each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a δP_S-open set U in X containing x such that $f(U) \subseteq V$.
d) For each $x \in X$ and each open set V of Y containing $f(x)$, there exists a δP_S-open set U in X containing x such that $f(U) \subseteq V$.

Proof. (a) ⇒ (b). Let $x \in X$ and let V be any open set of Y containing f (x). By (a), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq sCl(V)$.

$(b) \Rightarrow (c)$. Let $x \in X$ and let V be any regular open set of Y containing f (x). Then V is an open set of Y containing f (x). By (b), there exists a δP_S-open set U in X containing x such that $f(U) \subseteq sCl(V)$. Since V is regular open and hence is an open set. Therefore, by Lemma 1.3.10, $f(U) \subseteq Int\ Cl(V)$. Since V is regular open, $f(U) \subseteq V$.

$(c) \Rightarrow (d)$. Let $x \in X$ and let V be any δ-open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists an open set G containing $f(x)$ such that $G \subseteq Int\ Cl(G) \subseteq V$. Since $Int\ Cl(G)$ is a regular open set of Y containing f (x), by (c), there exists a δP_S-open set U in X containing x such that $f(U) \subseteq Int\ Cl(G) \subseteq V$.

$(d) \Rightarrow (a)$. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then $Int\ Cl(V)$ is open set of Y containing f (x). By (d), there exists a δP_S-open set U in X containing x such that $f(U) \subseteq Int\ Cl(V)$. Therefore, f is almost δP_S-continuous.

Theorem 6.2.10. For a function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.

b) $f^{-1}(Int\, Cl(V))$ is δP_S-open in X, for each open set V in Y.

c) $f^{-1}(Cl\, Int(F))$ is δP_S-closed set in X, for each closed set F in Y.

d) $f^{-1}(F)$ is a δP_S-closed set in X, for each regular closed set F of Y.

e) $f^{-1}(V)$ is a δP_S-open set in X, for each regular open set V of Y.

Proof. (a) ⇒ (b). Let V be any open set in Y. We have to show that $f^{-1}(Int\, Cl(V))$ is a δP_S-open set in X. Let $x \in f^{-1}(Int\, Cl(V)$ and $Int\, Cl(V)$ is a regular open set in Y. Since f is almost δP_S-continuous, by Theorem 6.2.9, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Int\, Cl(V)$, which implies that $x \in U \subseteq f^{-1}(Int\, Cl(V))$. Therefore, $f^{-1}(Int\, Cl(V)$ is a δP_S-open set in X.

$(b) \Rightarrow (c)$. Let F be any closed set of Y. Then $Y \backslash F$ is an open set of Y. By (b), $f^{-1}(Int\, Cl(Y \backslash F))$ is a δP_S-open set in X and $f^{-1}(Int\, Cl(Y \backslash F) = f^{-1}\big(Int(Y \backslash F)\big) = f^{-1}(Y \backslash Cl\, Int(F)) = X \backslash f^{-1}(Cl\, Int(F))$ is a δP_S-open set in X and hence $f^{-1}(Cl\ \ Int(F))$ is a δP_S-closed set in X .

$(c) \Rightarrow (d)$. Let F be any regular closed set of Y . Then F is a closed set of Y . By (c), $f^{-1}(Cl\, Int(F))$ is a δP_S-closed set in X. Since F is a regular closed set, then $f^{-1}(Cl\, Int(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a δP_S -closed set in X.

$(d) \Rightarrow (e)$. Let V be any regular open set of Y. Then $Y \backslash V$ is a regular closed set of Y and by (d), we have $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is a δP_S-closed set in X and hence $f^{-1}(V)$ is δP_S-open in X.

$(e) \Rightarrow (a)$. Let $x \in X$ and let V be any regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (e), we have $f^{-1}(V)$ is a δP_S -open set in X. Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Theorem 6.2.9, f is almost δP_S-continuous.

The following result can be proved easily from the above Proposition.

Proposition 6.2.11. If f is almost δP_S -continuous then $f^{-1}(F)$ is a δP_S-closed set for each clopen set F.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$ be almost δP_S-continuous and F be a clopen set. By (c) of Theorem 6.2.10, $f^{-1}(Cl\, Int(F))$ is δP_S-closed. Since every clopen set is open, $int(F) = F$ and since F is closed we get $Cl\, Int(F) = Cl(F) = F$.

$\therefore f^{-1}(F)$ is a δP_S-closed in X.

Theorem 6.2.12. For a function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.

b) $f(\delta P_S Cl(A)) \subseteq Cl_\delta f(A)$, for each $A \subseteq X$.

c) $\delta P_S Cl f^{-1}(B) \subseteq f^{-1} Cl_\delta(B)$, for each $B \subseteq Y$.

d) $f^{-1}(F)$ is δP_S-closed in X, for each δ-closed set F of Y.

e) $f^{-1}(V)$ is δP_S-open set in X, each δ-open set V of Y.

f) $f^{-1}(Int_\delta B) \subseteq \delta P_S f^{-1}(B)$, for each $B \subseteq Y$.

Proof. (a) $\Rightarrow$ (b). Let A be a subset of X. Since $Cl_\delta f(A)$ is δ-closed set in Y, so $\delta Cl\, f(A) = \cap \{F_\alpha / \alpha \in \Lambda$ and $F_\alpha \in RC(Y)\}$.

Now, $f(A) \subseteq \delta\ Cl\ f(A) = \cap \{F_\alpha\}$

$A \subseteq f^{-1}\big(\delta\ Cl\ f(A)\big) = f^{-1}(\cap \{F_\alpha\}) = \cap\, (f^{-1}\{F_\alpha\})$ ⟶ (1)

Then $A \subseteq f^{-1}\{F_\alpha\}\ \forall \alpha \in \Lambda$

$f(A) \subseteq F_\alpha, \alpha \in \Lambda$, Then by Theorem 6.2.10(d) and by (1) $f^{-1}(F_\alpha)$ is δP_S-closed in X, since F_α is regular closed and $\bigcap_{\alpha \in \Lambda}(f^{-1}(F_\alpha))$ is δP_S-closed. ⟶ (2)

Hence $f^{-1}(\delta\ Cl(f(A)))$ is δP_S-closed. ⟶ (3)

Now, $A \subseteq f^{-1}(\delta\ Cl\ f(A))$

$\delta P_S\ Cl(A) \subseteq \delta P_S\ Cl(f^{-1}\big(\delta\ Cl\ f(A)\big))$

$\delta P_S\ Cl(A) \subseteq f^{-1}(\delta\ Cl\ f(A))$ from (3)

$\therefore f(\delta P_S\ Cl\ (A) \subseteq\ \delta\ Cl\ f(A)$

(b)⇒ (c). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (b), we have $f(\delta P_S Cl f^{-1}(B)) \subseteq Cl_\delta(f\big(f^{-1}(B)\big) = Cl_\delta(B)$. Hence $\delta P_S Cl(f^{-1}(B) \subseteq f^{-1}(Cl_\delta(B))$.

(c)⇒ (d). Let F be any δ-closed set of Y. Then $F = \delta\ Cl(F)$ ⟶ (1)

Then by (c), $\delta P_S\ Cl\ f^{-1}(F) \subseteq f^{-1}(\delta\ Cl\ F) = f^{-1}(F)$ from (1) and hence $f^{-1}(F)$ is δP_S-closed in X.

(d)⇒ (e). Let V be any δ-open set of Y. Then Y \ V is δ-closed set of Y and by (d), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is δP_S-closed set in X. Hence $f^{-1}(V)$ is δP_S -open set in X.

(e)⇒ (f). For each subset B of Y. We have $\delta\ Int(B) \subseteq B$. Then

$f^{-1}(\delta\ Int(B)) \subseteq f^{-1}(B)$ ⟶ (1)

By (e), since $\delta\ Int\ (B)$ is δ-open, we get $f^{-1}(\delta\ Int\ B)$ is δP_S-open.

Then (1) $\Rightarrow \delta P_S\ Int(f^{-1}(\delta\ Int\ (B)) \subseteq \delta P_S\ Int\big(f^{-1}(B)\big)$

$$\Rightarrow f^{-1}\big(\delta\ Int(B)\big) \subseteq \delta P_S\ Int(f^{-1}(B))$$

(f)⇒ (a). Let $x \in X$ and V be any regular open set of Y containing x. Hence V is δ-open.

$\therefore \delta Int(V) = V \longrightarrow$ (1)

Moreover, by (f), $f^{-1}(\delta\ Int(V)) \subseteq \delta P_S Int(f^{-1}(V)) \longrightarrow$ (2)

From (1) & (2), $f^{-1}(V) \subseteq \delta P_S\ Int f^{-1}(V)$

$\therefore f^{-1}(V)$ is a δP_S-open set in X which contains x and we know $f(f^{-1}(V)) \subseteq V$. Hence by Theorem 6.2.9 (c) we get f is almost δP_S-continuous.

Theorem 6.2.13. For a function $f: (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.
b) $\delta P_S Cl f^{-1}(V) \subseteq f^{-1}(Cl\,V)$, for each β-open set V of Y.
c) $f^{-1}(Int\ F) \subseteq \delta P_S Int\ f^{-1}(F)$, for each β-closed set F of Y.
d) $f^{-1}(Int\ F) \subseteq \delta P_S Int\ f^{-1}(F)$, for each semi-closed set F of Y.
e) $\delta P_S Cl f^{-1}(V) \subseteq f^{-1}(Cl\,V)$, for each semi-open set V of Y.

Proof. (a) ⇒ (b). Let V be any β-open set of Y. It follows from Lemma 1.3.11 that $Cl(V)$ is regular closed set in Y. Since f is almost δP_S-continuous, by Theorem 6.2.10(d), $f^{-1}(Cl(V))$ is δP_S-closed set in X. Therefore, we obtain

$f^{-1}(Cl(V) = \delta P_S f^{-1} Cl(V) \longrightarrow$ (1)

Now $V \subseteq Cl(V) \Rightarrow f^{-1}(V) \subseteq f^{-1}\big(Cl(V)\big) \Rightarrow \delta P_S Cl\big(f^{-1}(V)\big) \subseteq \delta P_S Cl\big(f^{-1}Cl(V)\big) =$
$f^{-1}Cl(V)$ [From (1)]

Hence $\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$.

(b)⇒ (c). Let F be any β-closed set of Y. Then $Y\backslash F$ is β-open set of Y and by (b), we have $\delta P_S Cl\, f^{-1}(Y\backslash F) \subseteq f^{-1}(Cl(Y\backslash F))$ and $\delta P_S Cl(X\backslash f^{-1}(F)) \subseteq f^{-1}(Y\backslash Int\ F)$ and hence, $X\backslash \delta P_S Int\ f^{-1}(F) \subseteq X\backslash f^{-1}(Int\ F)$. Therefore, $f^{-1}(Int\ F) \subseteq \delta P_S Int f^{-1}(F)$.

(c)⇒ (d). Obvious since every semi-closed set is β-closed.

(d)⇒ (e). Let V be any semi-open set of Y. Then Y \ V is semi-closed set in Y and by (d), we have $f^{-1}(Int(Y\backslash V)) \subseteq \delta P_S Int\ f^{-1}(Y\backslash V)$ and $f^{-1}(Y\backslash Cl\ V) \subseteq \delta P_S Int(X\backslash f^{-1}(V))$ and hence, $X\backslash f^{-1}(Cl\ V) \subseteq X\backslash \delta P_S Cl\ f^{-1}(V)$. Therefore, $\delta P_S Cl\ f^{-1}(V) \subseteq f^{-1}(Cl\ V)$.

(e)⇒ (a). Let F be any regular closed set of Y. Then F is a semi-open set of Y. By (e), we have $\delta P_S Cl\ f^{-1}(F) \subseteq f^{-1}(\ Cl\ F) = f^{-1}(F)$ [Since every regular closed set is closed]. This shows that $f^{-1}(F)$ is a δP_S -closed set in X. Therefore, by Theorem 6.2.10(d), f is almost δP_S-continuous.

Corollary 6.2.14. For a function $f: (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.
b) $\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(\alpha Cl(V))$, for each β-open set V of Y.

c) $\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(Cl_\delta(V))$, for each β-open set V of Y.

d) $\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(\delta P_S Cl(V))$, for each semi-open set V of Y.

e) $\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(pCl(V))$, for each semi-open set V of Y.

Proof. (a) ⇒ (b). Follows from Proposition 6.2.13 and Theorem 1.1.19(c)

(b)⇒ (c). Follows from the fact that $\alpha Cl(V) \subseteq Cl_\delta V$.

(c)⇒ (d) and (d) ⇒ (e). Follows from Proposition 6.2.13 and Lemma 2.4.17.

(e) ⇒ (f). Follows from Proposition 6.2.13 and Lemma 2.4.17.

The following result also can be concluded directly.

Corollary 6.2.15. For a function $f:(X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.

b) $f^{-1}(\alpha \text{Int } F) \subseteq \delta P_S \text{Int } f^{-1}(F)$, for each β-closed set F of Y.

c) $f^{-1}(\text{Int}_\delta F) \subseteq \delta P_S \text{Int } f^{-1}(F)$, for each β-closed set F of Y.

d) $f^{-1}(\delta P_S \text{Int } F) \subseteq \delta P_S \text{Int } f^{-1}(F)$, for each semi-closed set F of Y.

e) $f^{-1}(p \text{ Int } F) \subseteq \delta P_S \text{Int } f^{-1}(F)$, for each semi-closed set F of Y.

Proposition 6.2.16. A function $f:(X,\tau) \to (Y,\sigma)$ is almost δP_S-continuous if and only if $f^{-1}(V) \subseteq \delta P_S \text{Int } f^{-1}(\text{IntCl } V)$ for each preopen set V of Y.

Proof. Necessity. Let V be any preopen set of Y. Then V ⊆ IntCl V and IntCl V is a regular open set in Y. Since f is almost δP_S-continuous, by Theorem 6.2.10(e), $f^{-1}(\text{IntCl } V)$ is δP_S-open set in X and hence we obtain that $f^{-1}(V) \subseteq f^{-1}(\text{IntCl } V) = \delta P_S \text{Int} f^{-1}(\text{IntCl } V)$.

Sufficiency. Let V be any regular open set of Y. Then V is a preopen set of Y. By hypothesis, we have $f^{-1}(V) \subseteq \delta P_S Int f^{-1}\big(Int\ Cl(V)\big) = \delta P_S Int\ f^{-1}(V)$. Therefore, $f^{-1}(V)$ is δP_S-open set in X and hence by Theorem 6.2.10(e), f is almost δP_S-continuous.

We obtain the following corollary.

Corollary 6.2.17. The following statements are equivalent for a function $f:(X,\tau) \to (Y,\sigma)$:

a) f is almost δP_S-continuous.

b) $f^{-1}(V) \subseteq \delta P_S \text{Int } f^{-1}(\text{sCl } V)$ for each preopen set V of Y. [Theorem 1.3.17]

c) $\delta P_S \text{Cl } f^{-1}(\text{Cl Int } F) \subseteq f^{-1}(F)$ for each preclosed set F of Y.

d) $\delta P_S \text{Cl } f^{-1}(\text{s Int } F) \subseteq f^{-1}(F)$ for each preclosed set F of Y .

Corollary 6.2.18. For a function $f:(X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

a) f is almost δP_S-continuous.

b) For each neighborhood V of f(x), $x \in \delta P_S \text{ Int } f^{-1}(\text{sCl } V)$.

c) For each neighborhood V of f (x), $x \in \delta P_S \text{Int } f^{-1}(\text{IntCl } V)$.

Proof. Follows from Proposition 6.2.16 and Corollary 6.2.17.

Proposition 6.2.19. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be an almost δP_S-continuous function and let V be any open subset of Y. If x ∈ δP_SCl $f^{-1}(V) \setminus f^{-1}(V)$, then f (x) ∈ δP_SCl V.

Proof. Let x ∈ X be such that $x \in \delta P_S$Cl $f^{-1}(V) \setminus f^{-1}(V)$ and suppose $f(x) \notin \delta P_S Cl(V)$. Then there exists a δP_S-open set H containing f (x) such that H ∩ V = ∅. Then Cl H ∩ V =∅ implies IntCl H ∩V = ∅ and IntCl H is a regular open set. Since f is almost δP_S-continuous, by Theorem 6.2.9(c), there exists a δP_S-open set U in X containing x such that f (U) ⊆IntCl H. Therefore, f (U) ∩ V = ∅. However, since $x \in \delta P_S Cl(f^{-1}(V)), U \cap f^{-1}(V) \neq \emptyset$ for every δP_S-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. It follows that $f(x) \in \delta P_S Cl(V)$.

Theorem 6.2.20. If a function $f:(X,\tau)\rightarrow(Y,\sigma)$ is almost precontinuous. Then the following statements are equivalent:

a) f is almost δP_S-continuous.
b) For each x ∈ X and each open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that f (F) ⊆ Int Cl V.
c) For each x ∈ X and each open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that f (F) ⊆ sCl V.
d) For each x ∈ X and each regular open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that f (F) ⊆ V.
e) For each x ∈ X and each δ-open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that f (F) ⊆ V.

Proof. (a) ⇒ (b). Let x ∈ X and let V be any open set of Y containing f (x). By (a), there exists a δP_S-open set U of X containing x such that f (U) ⊆ Int Cl V. Since U is δP_S-open set, so for each x ∈ U there exists a semi-closed set F in X such that x ∈ F ⊆ U. Therefore, we have f (F) ⊆ Int δCl V.

(b)⇒ (c). Obvious as $IntCl(V) \subseteq sCl(V)$.

(c)⇒ (d). Let x ∈ X and let V be any regular open set of Y containing f (x). Then V is an open set of Y containing f (x). By (c), there exists a semi-closed set F in X containing x such that f (F) ⊆ sCl V. Since V is regular open and hence is preopen. Therefore, by Lemma 1.3.12,

f (F) ⊆ IntCl V. Since V is regular open, then f (F) ⊆ V.

(d)⇒ (e). Let x ∈ X and let V be any δ-open set of Y containing f (x). Then for each f (x) ∈ V, there exists an open set G containing f (x) such that G ⊆ IntClG ⊆ V. Since IntClG is a regular open set of Y containing f (x), by (d), there exists a semi-closed set F in X containing x such that f (F) ⊆ IntClG ⊆V. This completes the proof.

(e)⇒ (a). Let V be any δ-open set of Y. We have to show that $f^{-1}(V)$ is a δP_S-open set in X. Since f is almost precontinuous, by Lemma 1.3.13(c), $f^{-1}(V)$ is a preopen set in X. Since every preopen set is δ-preopen we get $f^{-1}(V)$ is δ-preopen. Let x ∈ $f^{-1}(V)$, then f (x) ∈ V. By (e), there exists a semi-closed set F of X containing x such that f (F) ⊆ V. Which implies that x ∈ F ⊆ $f^{-1}(V)$. Therefore, $f^{-1}(V)$ is δP_S-open set in X. Hence by Theorem 6.2.12(e), f is almost δP_S-continuous.

Proposition 6.2.21. A function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous if and only if $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Proof. Necessity. Let H ∈ σ_S, then H is a regular open set in (Y, σ). Since $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous, by Theorem 6.2.10(e), $f^{-1}(H)$ is δP_S-open set in X. Therefore, $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Sufficiency. Let G be any regular open set in (Y, σ). Then $G \in \sigma_S$. Since $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous, by Definition 5.2.1, $f^{-1}(G)$ is δP_S-open set in X. Therefore, by Theorem 6.2.10(e), $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous.

Proposition 6.2.22. Let X be a locally indiscrete space. Then the function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S -continuous if and only if $f: (X, \tau) \to (Y, \sigma_S)$ is continuous.

Proof. Necessity. Let $H \in \sigma_S$, then H is a regular open set in (Y, σ). Since $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous, by Theorem 6.2.10(e), $f^{-1}(H)$ is δP_S -open set in X. Since X is locally indiscrete space, by Proposition 2.2.31, $f^{-1}(H)$ is open set in X. Therefore, $f: (X, \tau) \to (Y, \sigma_S)$ is continuous.

Sufficiency. Let G be any regular open set in (Y, σ). Then G ∈ σ_S. Since $f: (X, \tau) \to (Y, \sigma_S)$ is continuous, so $f^{-1}(G)$ is open set in X. Since X is locally indiscrete space, by Proposition 2.2.31, $f^{-1}(G)$ is δP_S-open set in X. Therefore, by Theorem 6.2.10(e), $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous.

Corollary 6.2.23. If $f: (X, \tau) \to (Y, \sigma)$is almost δP_S-continuous function if and only if f is almost continuous where X is locally indiscrete space.

6.2.1 Properties and Comparisons

In this section, we give some properties of almost δP_S-continuous functions and compared it with other types of continuous functions.

Proposition 6.2.1.1. Let $f: (X, \tau) \to (Y, \sigma)$ be an almost δP_S-continuous function. If A is either open or regular semi-open subset of X, then f |A : A → Y is almost δP_S-continuous in the subspace A.

Proof. Let V be any regular open set of Y. Since f is almost δP_S-continuous, by Theorem 6.2.10(e), f $^{-1}$(V) is δP_S-open set in X. Since A is either open or regular semi-open subset of X. By Lemma 2.3.5, (f |A^{-1}(V) = f $^{-1}$(V)∩ A is a δP_S-open subset of A. This shows that f |A:A → Y is almost δP_S-continuous.

Corollary 6.2.1.2. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be almost δP_S-continuous function. If A is regular open subset of X, then f |A: A → Y is almost δP_S-continuous in the subspace A.

Proof. Follows from Proposition 6.2.1.1, as every regular open set is regular semi-open

Proposition 6.2.1.3. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost δP_S-continuous, if for each x ∈ X, there exists a regular open set A of X containing x such that f |A: A → Y is almost δP_S-continuous.

Proof. Let x ∈ X, then by hypothesis, there exists a regular open set A containing x such that f |A: A → Y is almost δP_S-continuous. Let V be any open set of Y containing f (x), there exists a δP_S-open set U in A containing x such that (f |A) (U) ⊆ $\mathrm{Int}\ \delta\mathrm{Cl}$ V. Since A is regular open, by Lemma 1.3.14, U is δP_S-open set in X and hence f (U) ⊆ $\mathrm{Int}\ \delta\mathrm{Cl}$ V. This shows that f is almost δP_S-continuous.

Corollary 6.2.1.4. Let $\{U_\gamma : \gamma \in \Delta\}$ be a regular open cover of a topological space X. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is almost δP_S-continuous if and only if f $|U_\gamma : U_\gamma \rightarrow$ Y is almost δP_S-continuous for each $\gamma \in \Delta$.

Proof. This is an immediate consequence of Corollary 6.2.1.2 and Proposition 6.2.1.3.

Proposition 6.2.1.5. If X = R ∪ S, where R and S are regular open sets and $f: (X, \tau) \rightarrow (Y, \sigma)$ is a function such that both f |R and f |S are almost δP_S-continuous, then f is almost δP_S-continuous.

Proof. Let V be any regular open set of Y. Then f $^{-1}$(V) = (f |R)$^{-1}$(V) ∪ (f |S)$^{-1}$(V). Since f |R and f |S are almost δP_S-continuous, by Theorem 6.2.10(e), (f |R)$^{-1}$(V) and (f |S)$^{-1}$(V) are δP_S-open sets in R and S, respectively. Since R and S are regular open sets in X, then by Proposition 2.3.2, (f |R)$^{-1}$(V) and (f |S)$^{-1}$(V) are δP_S-open sets in X . Since a union of two δP_S-open sets is δP_S-open, hence f $^{-1}$(V) is a δP_S-open set in X. Therefore, by Theorem 6.2.10(e), f is almost δP_S-continuous.

In general, if X = $\cup\{K_\gamma : \gamma \in \Delta\}$, where each K_γ is a regular open set and $f: (X, \tau) \rightarrow (Y, \sigma)$ is a function such that f $|K_\gamma$ is almost δP_S-continuous for each γ, then f is almost δP_S-continuous.

Proposition 6.2.1.6. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be almost δP_S-continuous and let A be either open or regular semi- open subset of X such that f (A) is dense in Y. Then f |A: A → f (A) is almost δP_S-continuous.

Proof. Let O be a regular open subset of f(A). Then by Lemma 1.3.14. O = f (A) ∩ IntCl O. Thus, $(f|A)^{-1}(O)$ = $(f|A)^{-1}$(f (A)∩IntCl O) = $(f|A)^{-1}$(f (A))∩$(f|A)^{-1}$(IntCl O) = A∩$(f|A)^{-1}$(IntCl O)= A∩ f^{-1}(IntCl O) = A∩ f^{-1}(O). Since f is almost δP_S-continuous, by Theorem 6.2.10(b), $f^{-1}(O)= f^{-1}$(IntCl O) is δP_S-open in X. Since A is either an open or a regular semi-open subset of X. Then by Corollary 2.3.7, $(f|A)^{-1}(O)$ is δP_S-open set in the subspace A. Therefore, by Theorem 6.2.10(e), f |A: A →f (A) is almost δP_S-continuous.

The following is the pasting lemma for almost $\boldsymbol{\delta P_S}$-continuity

Lemma 6.2.1.7. Let X = $R_1 \cup R_2$, where R_1 and R_2 are regular open sets in X. Let f : R_1 → Y and g : R_2 → Y be almost δP_S-continuous. If f (x) = g (x) for each x ∈ $R_1 \cap R_2$. Then h : $R_1 \cup R_2$ → Y such that h(x) = f (x) for x ∈ R_1 and h(x) = g (x) for x ∈ R_2 is almost δP_S-continuous.
Proof. Let O be a regular open set of Y. Now $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$. Since f and g are almost δP_S-continuous, by Theorem 6.2.10(e), $f^{-1}(O)$ and $g^{-1}(O)$ are δP_S-open set in R_1 and R_2 respectively. But R_1 and R_2 are both regular open sets in X. Then by Proposition 2.3.2, $f^{-1}(O)$ and $g^{-1}(O)$ are δP_S-open sets in X. Since union of two δP_S-open sets is δP_S-open, so $h^{-1}(O)$ is a δP_S-open set in X. Hence by Theorem 6.2.10(e), h is almost δP_S-continuous.
Proposition 6.2.1.8. Let $f:(X,\tau) \to (Y,\sigma)$ be almost δP_S-continuous surjection and A be either δ-open or regular semi-open subset of X. If f is an open function, then the function g : A → f (A), defined by g (x) = f (x) for each x ∈ A, is almost δP_S-continuous.

Proof. Suppose that H = f (A). Let x ∈ A and V be any open set in H containing $g(x)$. Since H is open in Y and V is open in H, so V is open in Y. Since f is almost δP_S-continuous, hence there exists a δP_S-open set U in X containing x such that $f(U) \subseteq Int(\delta Cl(V))$. Taking $W = U \cap A$, since A is either open or regular semi-open subset of X, by Proposition 2.3.5, W is a δP_S-open set in A containing x and $g(W) \subseteq Int_Y Cl_Y(V) \cap H = Int_H Cl_H(V)$. Then $g(W) \subseteq Int_H Cl_H(V)$. This shows that g is almost δP_S-continuous.
Proposition 6.2.1.9. Let $f:(X,\tau) \to (Y,\sigma)$ Y be almost δP_S-continuous. If Y is a preopen subset of Z, then $f:(X,\tau) \to (Z,\eta)$ is almost δP_S-continuous.
Proof. Let V be any regular open set of Z. Since Y is preopen, by Lemma 1.3.15, V ∩ Y is a regular open set in Y. Since f: X → Y is almost δP_S-continuous, by Theorem 6.2.10(e), $f^{-1}(V \cap Y)$ is a δP_S-open set in X. But f (x) ∈ Y for each x ∈ X. Thus $f^{-1}(V) = f^{-1}(V \cap Y)$ is a δP_S-open set of X. Therefore, by Theorem 6.2.10(e), $f:(X,\tau) \to (Z,\eta)$ is almost δP_S-continuous.

Proposition 6.2.1.10. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ and $g:(Y,\sigma)\rightarrow(Z,\eta)$ be functions. Then the composition function $g\circ f:(X,\tau)\rightarrow(Z,\eta)$ is almost δP_S -continuous if f and g satisfy one of the following conditions:

a) f is δP_S -continuous and g is almost continuous.

b) f is almost δP_S -continuous and g is δ-continuous.

c) f is continuous and open and g is almost δP_S-continuous.

d) f is almost δP_S-continuous and g is RC-continuous.

Proof. (a). Let W be any regular open subset of Z. Since g is almost continuous, $(f\circ g)^{-1}$(W) is open subset of Y. Since f is δP_S-continuous, by Definition 5.2.1, $(g\circ f)^{-1}$(W) = $f^{-1}(g^{-1}(W))$ is δP_S-open subset in X. Therefore, by Theorem 6.2.10(e), $g\circ f$ is almost δP_S-continuous.

(b). Let W be any δ-open subset of Z. Since g is δ-continuous, $(f\circ g)^{-1}$(W) is δ-open subset of Y. Since f is almost δP_S-continuous, by Theorem 6.2.12(e), $(g\circ f)^{-1}$(W) = $f^{-1}(g^{-1}(W))$ is δP_S-open subset in X . Therefore, by Theorem 6.2.12(e), $g\circ f$ is almost δP_S-continuous.

(c). Let W be any regular open subset of Z. Since g is almost δP_S-continuous, by Theorem 6.2.10(e), g^{-1}(W) is δP_S-open subset of Y. Since f is continuous and open, by Proposition 5.2.9, $f^{-1}(g^{-1}(W))$ = $(g\circ f)^{-1}$(W) is a δP_S-open set in X. Hence by Theorem 6.2.10(e), $g\circ f$ is almost δP_S- continuous.

d) Let W be any open subset of Z. Since g is RC-continuous, $g^{-1}(W)$ is regular closed subset of Y. Since f is almost δP_S-continuous, then $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))$ is δP_S-closed subset in X. Therefore, $g\circ f$ is contra δP_S-continuous.

Proposition 6.2.1.11. If $f:(X,\tau)\rightarrow(Y,\sigma)$ is almost δP_S-continuous and $g:(Y,\sigma)\rightarrow(Z,\eta)$ is super continuous functions. Then the composition function $g\circ f:(X,\tau)\rightarrow(Z,\eta)$ is δP_S-continuous.

Proof. Let W be any open subset of Z. Since g is super continuous, then g^{-1}(W) is δ-open subset of Y. Since f is almost δP_S-continuous, by Theorem 6.2.12(e), $(gof)^{-1}$(W) = $f^{-1}(g^{-1}(W))$ is δP_S-open subset in X. Therefore, by Definition 5.2.1, $g\circ f$ is δP_S-continuous.

Proposition 6.2.1.12. If $f:(X,\tau)\rightarrow(Y,\sigma)$ is an almost δP_S-continuous function and Y is semi-regular. Then f is δP_S-continuous.

Proof. Let x ∈ X and let V be any open set of Y containing f (x). By the semi-regularity of Y, there exists a regular open set G of Y such that f (x) ∈ G ⊆ V. Since f is almost δP_S-continuous, by Theorem 6.2.9, there exists a δP_S-open set U of X containing x such that f (U) ⊆ G ⊆ V. Therefore, f is δP_S-continuous.

Corollary 6.2.1.13. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and X is locally indiscrete space. Then f is almost δP_S-continuous if and only if f is almost continuous.

Proof. Follows from the definition and Proposition 2.2.31. By Theorem 6.2.9(c), for all $x \in X$ and regular open V of Y such that $f(x) \in V$, there exists a δP_S-open set U continue x such that $f(U) \subseteq V$. But Since, $\delta P_S O(\tau) = \tau$ in a locally indiscrete space, the set U becomes open. Hence by Theorem 1.3.16, f becomes almost continuous.

Corollary 6.2.1.14. If X is a locally indiscrete space and Y is semi-regular space, then the following statements are equivalent for a function $f:(X,\tau) \rightarrow (Y,\sigma)$:

a) f is δP_S-continuous.
b) f is almost δP_S-continuous.
c) f is almost continuous.
d) f is continuous.

Proof. $(a) \Rightarrow (b)$ True for any space by Theorem 6.2.3(a).

$(b) \Rightarrow (c)$ By Corollary 6.2.1.13.

$(c) \Rightarrow (d)$ By Lemma 1.3.13(b)

$(d) \Rightarrow (a)$ In a locally indiscrete space, $\delta P_S O(\tau) = \tau$, by Proposition 2.2.31.

Corollary 6.2.1.15. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and X is s-regular space. If f is almost continuous, then f is almost δP_S-continuous.

Proof. By a result, for all $x \in X$ and regular open V of Y such that $f(x) \in V$ there exists an open set U containing x such that $f(U) \subseteq V$. But by Proposition 2.2.35, $\tau \subseteq \delta P_S O(\tau)$.

$\therefore$ U is δP_S-open in (X,τ). Hence f is almost δP_S-continuous

Corollary 6.2.1.16. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and X is semi-T_1 space. Then f is almost δP_S-continuous if and only if f is almost δ-precontinuous.

Proof. In a semi- T_1 space, $\delta P_S O(\tau) = \delta PO(\tau)$, by Proposition 2.2.23. Hence from definitions of almost δ-pre-continuous and almost δP_S-pre continuous the proof follows.

Corollary 6.2.1.17. If X is a semi-T_1 space and Y is semi-regular space, then the following statements are equivalent for a function $f:(X,\tau) \rightarrow (Y,\sigma)$:

a) f is δP_S-continuous.
b) f is almost δP_S-continuous.
c) f is almost δ-precontinuous.
d) f is precontinuous.

Proof. $(a) \Rightarrow (b)$ True for any space by 6.2.2(a)

$(b) \Rightarrow (c)$ By Corollary 6.2.1.13.

$(c) \Rightarrow (d)$ By Lemma 1.3.13(c).

$(d) \Rightarrow (a)$ In a locally indiscrete space, $\delta P_S O(\tau) = \tau$, by Proposition 2.2.31.

Proposition 6.2.1.18. If Y is a hyperconnected space, then every function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous.

Proof. Let x ∈ X and V be any open set of Y containing f (x). Since Y is a hyperconnected space, then Cl V = Y and hence IntCl V = Y. Therefore, we have f (U) ⊆ IntCl V ⊆ $Int\delta Cl(V)$, for any δP_S-open set U in X. This shows that f is almost δP_S-continuous.

Proposition 6.2.1.19. If a function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous and semi-open, then f (δP_SCl V) ⊆ δP_SCl f (V) for each open set V of X.

Proof. Let V be any open set of X. Since f is semi-open, then f (V) is semi-open set in Y. Since f is almost δP_S-continuous, then by Corollary 6.2.14(d), we obtain that

$\delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(\delta P_S Cl(f(V))$ which implies that $f(\delta P_S Cl(V)) \subseteq \delta P_S Cl(f(V))$

Corollary 6.2.1.20. If a function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous and semi-open, then δP_SInt f (F)⊆ f (δP_SInt F) for each closed set F of X.

Proposition 6.2.1.21. If a function $f: (X, \tau) \to (Y, \sigma)$ is irresolute and preopen. Then f is almost δP_S-continuous if and only if δP_SCl f^{-1}(V) = f^{-1}(δP_SCl V) for each semi-open set V of Y.

Proof. Necessity. Let V be any semi-open set of Y. Since f is almost δP_S-continuous, by Corollary 6.2.14(d), δP_SCl f^{-1}(V) ⊆ f^{-1}(δP_SCl V). Since V is semi-open set of Y, by Proposition 2.4.18, δP_SCl V = Cl V which implies that f^{-1}(δP_SCl V) ⊆ f^{-1}(Cl V). Since V is semi-open set of Y and f is preopen, by Theorem 1.3.17, we have f^{-1}(δP_SCl V) ⊆ f^{-1}(Cl V) ⊆ Cl f^{-1}(V) and hence f^{-1}(δP_SCl V) ⊆ Cl f^{-1}(V). Since V is semi-open set of Y and f is irresolute, so f^{-1}(V) is semi- open set in X. Then by Proposition 2.4.18, we obtain that

f^{-1}(δP_SCl V) ⊆ $\delta P_S Cl(f^{-1}(V)$. Therefore, we have $\delta P_S Clf^{-1}(V) = f^{-1}(\delta P_S Cl(V))$.

Sufficiency. Follows from Proposition 6.2.14.

From the above Proposition and Proposition 2.4.18, we obtain the following results:

Corollary 6.2.1.22. If a function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous, irresolute and preopen, then $\delta P_S Intf^{-1}(F) = f^{-1}(\delta P_S IntF)$ for each semi-closed set F of Y.

Corollary 6.2.1.23. If a function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous, irresolute and preopen, then $Clf^{-1}(V) = f^{-1}(Cl(V))$ for each semi-open set V of Y.

Definition 6.2.1.24: A function $f: (X, \tau) \to (Y, \sigma)$ is said to be **$\boldsymbol{\delta P_S}$-preopen** if for every δP_S-open subset U of X, $f(U)$ is preopen.

Proposition 6.2.1.25: If $f: (X, \tau) \to (Y, \sigma)$ is δP_S-preopen and contra δP_S-continuous, then f is almost δP_S-continuous.

Proof: Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then by Theorem 5.5.9(c) there exists a δP_S-open subset U of X for which $x \in U$ and $f(U) \subseteq Cl(V)$. Since f is δP_S-preopen, $f(U)$ is preopen, which proves f is almost δP_S-continuous.

Corollary 6.2.1.26: If $f: (X, \tau) \to (Y, \sigma)$ has the property that images of semi-closed sets are preopen and f is contra δP_S-continuous, then f is almost δP_S-continuous.

Proof: The proof follows from Proposition 6.2.1.25 and the fact that every δP_S-open set is union of semi-closed sets.

6.3 WEAKLY δP_S- CONTINUOUS FUNCTIONS

In this section, we introduce the concept of weakly δP_S-continuous functions by using δP_S- open sets. We give some characterizations of weakly δP_S-continuous functions with several relations between this function and other types of continuous functions and spaces.

Definition 6.3.1. A function $f: (X, \tau) \to (Y, \sigma)$ is called **weakly δP_S-continuous** if for each $x \in X$ and each open set V of Y containing f(x), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq \delta Cl(V)$.[By Lemma 1.1.42]. Hence, in the above definition we can have $f(U) \subseteq Cl(V)$.

Lemma 6.3.2. The following results supervene from their definitions directly:

a) Every almost δP_S-continuous functions is weakly δP_S-continuous.

b) Every weakly δP_S-continuous function is weakly δ-pre-continuous.

c) Every weakly P_S-continuous function is weakly δP_S-continuous.

Proof: a) Let $f: (X, \tau) \to (Y, \sigma)$ be almost δP_S-continuous. Let $x \in X$ and each open set V of Y containing $f(x)$. Since f is almost δP_S-continuous, there exists a δP_S-open set U of X contained in x such that $f(U) \subseteq Int\ Cl(V)$

We know that $Int\ Cl(V) \subseteq Cl(V)$

Hence $f(U) \subseteq Cl(V)$

b) Let $f: (X, \tau) \to (Y, \sigma)$ be weakly δP_S-continuous. Let $x \in X$ and V be an open set in Y containing x. Since f is weakly δP_S-continuous, there exists a δP_S-open set V contained in $f(x)$ such that $f(U) \subseteq V$. Since every δP_S-open set is δP-open set f is weakly δ-precontinuous.

c) Follows from the fact that every P_S-open set is δP_S-open set.

Therefore, from the above Proposition we have:

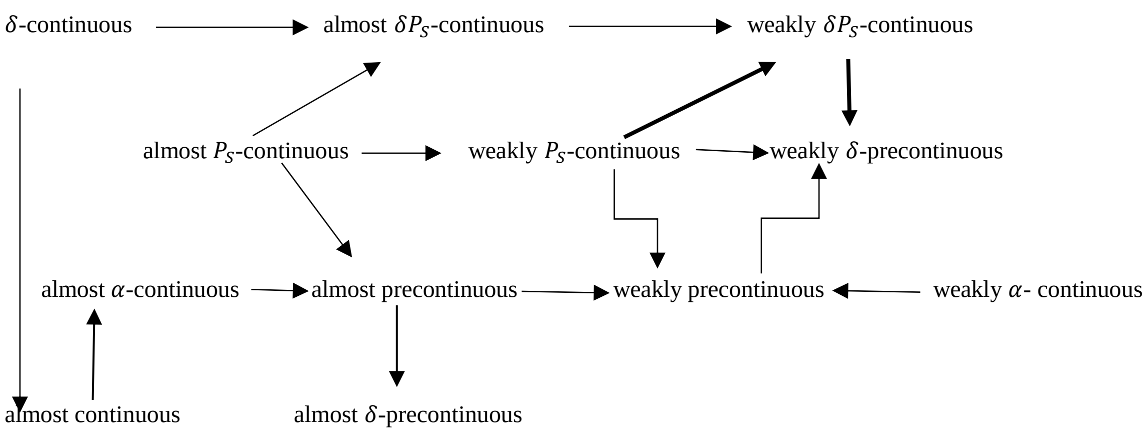

Figure 6.3

In the sequel, we shall show that none of the implications that concerning weakly δP_S- continuity in Diagram 6.3.1 is reversible.

Example 6.3.3. Let $X = \{a, b, c, d\}$ with two topologies $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, b\}\}$.Let $f(a) = d, f(b) = b = f(d)$ and $f(c) = c$. Let $x = a$ and $V = \{a\} \in \sigma$ then there exists $U = \{a\}, \delta P_S$-open in X such that $f(U) = \{d\} \subseteq Cl(V) = \{a, d\}$ and for $a \in \{a, b, c\} \in \sigma, f(U) \subseteq Cl(V) = X$, for $b \in \{b, c\}, \{a, b, c\}; f(U) \subseteq \{b, c, d\} = Cl(V)$.Therefore, f is weakly δP_S-continuous, but $f(U) = \{d\} \nsubseteq Int\, Cl(V) = Int\{(\{a, d\}) = \{a\}$.$\therefore f$ is not almost δP_S-continuous Functions.

Example 6.3.4: Let $X = \{a, b, c, d\}$ with the two topologies

$\tau = \{\mathrm{X}, \emptyset, \{a, \mathrm{b}\}, \{a, \mathrm{b, c}\}, \{a, \mathrm{b, d}\}$ and $\sigma = \{\mathrm{X}, \emptyset, \{a\}, \{\mathrm{c}\}, \{a, \mathrm{b}\}, \{a, \mathrm{c}\}, \{a, \mathrm{b, c}\}, \{a, \mathrm{c, d}\}\}$. Let $f: (X, \tau) \to (X, \sigma)$ be defined by $f(a) = f(b) = f(c) = c$ and $f(d) = d$, there exists $\delta P_S O(X, \tau) = \{c\} \subseteq x$ such that $f(U) \subseteq IntCl(V)$. but there exists no $P_S O(X, \tau)$ in τ, such that $f(U) \subseteq IntCl(V)$. Hence f is weakly δP_S-continuous but not weakly P_S-continuous.

Example 6.3.5. Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $f: (X, \tau) \to (X, \sigma)$ be the identity function. Then f is not weakly δP_S-continuous. Since for $a \in X$ such that $V = \{a\} \subseteq f(a)$ in Y but there exists no δP_S-open set U in τ such that $f(U) \subseteq Cl(V) = \{a, d\}$. But f is weakly δ-precontinuous.

Remark 6.3.6. We notice that a function $f: (X, \tau) \to (Y, \sigma)$ is weakly δP_S-continuous if either X is discrete or Y is indiscrete.

Proof: Case-(i) X is discrete

Proof: When X is discrete, (ie.,) $\tau = \mathcal{P}(X)$. Hence for every $x \in X, \{x\}$ is a δP_S-open in X. Therefore, the δP_S-open set $U = \{x\}$ containing x such that $f(U) = f(\{x\}) = f(x) \in V \subseteq Cl(V)$. Thus f is weakly δP_S-continuous

Case –(ii) Y is indiscrete

Proof: When Y is indiscrete, $\sigma = \{Y, \sigma\}$ then $\delta P_S O(\sigma) = \{Y, \sigma\}$. Any open set V in σ is Y. and $Cl(V) = Y$. Hence for any $U, f(U) \subseteq Y = Cl(V)$

$\therefore f$ is weakly δP_S-continuous.

Proposition 6.3.7. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then for each x $\in$ X and each θ-openset V of Y containing f(x), there exists a δP_S- U in X containing x such that $f(U) \subseteq V$.

Proof. Let $x \in X$ and let V be any θ-open set of Y containing f(x). Then for each f(x) $\in$ V, there exists an open set G containing f(x) such that $G \subseteq Cl(G) \subseteq V$. Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Cl(G) \subseteq V$. This completes the proof.

Corollary 6.3.8. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then for each x $\in$ X and each θ- open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that $f(F) \subseteq V$.

Proof. Let $x \in X$ and let V be any θ-open set of Y containing f(x). Since f is weakly δP_S-continuous, then by Proposition 6.3.7, there exists a δP_S-open set U in X containing x such that

$f(U) \subseteq V$. Since U is a δP_S-open set in X, then for each x $\in$ U, there exists a semi-closed set F of X such that $x \in F \subseteq U$. Therefore, we obtain $f(F) \subseteq f(U) \subseteq V$. Hence $f(F) \subseteq V$.

Proposition 6.3.9. Let $f:(X,\tau) \to (Y,\sigma)$ be a function. If for each $x \in X$ and each regular closed set R of Y containing $f(x)$, there exists a δP_S-open set U in X containing x such that $f(U) \subseteq R$, then f is weakly δP_S-continuous.

Proof. Let $x \in X$ and let V be any open set of Y containing f(x). Then put $R = Cl(V)$ which is a regular closed set of Y containing f(x). By hypothesis, there exists a δP_S-open set U in X containing x such that $f(U) \subseteq R = Cl(V)$. Hence f is weakly δP_S-continuous.

Proposition 6.3.10. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then the inverse image of each θ- open set of Y is a δP_S-open set in X.

Proof. Let V be any θ-open set in Y. We have to show that $f^{-1}(V)$ is a δP_S-open set in X. Let $x \in f^{-1}(V)$ Then $f(x) \in V$. Since f is weakly δP_S- continuous, then by Proposition 6.3.7, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq V$, which implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is a δP_S-open set in X.

Corollary 6.3.11. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then the inverse image of each θ-closed set of Y is a δP_S-closed set in X.

Proposition 6.3.12. Let $f:(X,\tau) \to (Y,\sigma)$ be a function. If $f^{-1}(Cl(V))$ is a δP_S-open set in X for each open set V in Y, then f is weakly δP_S-continuous.

Proof. Let $x \in X$ and let V be any open set of Y containing f (x). Then $x \in f^{-1}(V) \subseteq f^{-1}(Cl(V))$. By hypothesis, we have $f^{-1}(Cl(V))$ is a δP_S-open set in X containing x. Therefore, we obtain $f(f^{-1}(Cl(V))) \subseteq Cl(V)$.

Hence f is weakly δP_S-continuous.

Corollary 6 . 3 . 13. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. If $f^{-1}(Int(F))$ is a δP_S-closed set in X for each closed set F in Y, then f is weakly δP_S-continuous.

Proposition 6.3.14. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. If the inverse image of each regular closed set of Y is a δP_S-open set in X, then f is weakly δP_S-continuous.

Proof. Let V be any open set of Y. Then $Cl(V)$ is a regular closed set in Y. By hypothesis, we have $f^{-1}(Cl(V))$ is a δP_S-open set in X. Therefore, by Proposition 6.3.12, f is weakly δP_S-continuous.

Corollary 6.3.15. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. If the inverse image of each regular open set of Y is a δP_S-closed set in X, then f is weakly δP_S- continuous.

Proposition 6.3.16. If a function $f:(X,\tau) \rightarrow (Y,\sigma)$ is weakly δP_S-continuous, then for each $x \in X$ and each open set V of Y containing f (x), there exists a semi-closed set F in X containing x such that $f(F) \subseteq Cl(V)$.

Proof. Let $x \in X$ and let V be any open set of Y containing f (x). Since f is weakly δP_S-continuous, then there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Since U is δP_S-open set, then for each $x \in U$, there exists a semi-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq Cl(V)$.

The following result is a characterization of weakly δP_S-continuous functions:

Theorem 6.3.17. For a function $f:(X,\tau) \rightarrow (Y,\sigma)$, thefollowing statements are equivalent:

a) f is weakly δP_S-continuous.

b) $\delta P_S Cl f^{-1}(IntCl(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$

c) $f^{-1}(Int(B)) \subseteq \delta P_S Int f^{-1}(Cl(Int(B)))$ for each $B \subseteq Y$

d) $f^{-1}(Int(Cl\,V)) \subseteq \delta P_S Int f^{-1}(ClV)$ for each open setV of Y

e) $f^{-1}(V) \subseteq \delta P_S Int(f^{-1}(Cl(V)))$ for each regular open set V of Y.

f) $\delta P_S(Cl(f^{-1}(Int(F)))) \subseteq f^{-1}(F)$, for each regular closed set F of Y.

g) $\delta P_S(Cl(f^{-1}(Int(F)))) \subseteq f^{-1}(Cl(Int(F)))$, for each closed set F of Y.

h) $\delta P_S(Cl(f^{-1}(V))) \subseteq f^{-1}(Cl(V))$, for each open set V of Y.

i) $f^{-1}(Int(F)) \subseteq \delta P_S(Int(f^{-1}(F)))$, for each closed set F of Y.

Proof. $(a) \Rightarrow (b)$ Let B be any subset of Y. Assume that $x \notin f^{-1}(Cl(B))$. Then $f(x) \notin Cl(B)$ and there exists an open set V containing $f(x)$such that $V \cap B = \emptyset$, hence $Cl(V) \cap Int(Cl(B)) = \emptyset$. By(a), there exists a δP_S-open set U of X containingx such that $f(U) \subseteq cl(V)$. Therefore, we have $f(U) \cap Int(Cl(B)) = \emptyset$ which implies that $U \cap f^{-1}(Int(Cl(B))) = \emptyset$ and hence $x \notin \delta P_S(Cl(f^{-1}(Int(Cl(B)))))$. Therefore, we obtain $\delta P_S(Cl(f^{-1}(Int(Cl(B))) \subseteq f^{-1}(Cl(B))$.

$(b) \Rightarrow (c)$**.** Let B be any subset of Y. Then apply (b) to Y\B we obtain $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{IntCl}(Y \setminus B)) \subseteq f^{-1}(\mathrm{Cl}(Y \setminus B)) \Rightarrow \delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}(Y \setminus \mathrm{Int}B)) \subseteq f^{-1}(Y \setminus \mathrm{Int}B) \Rightarrow \delta P_S \mathrm{Cl} f^{-1}(Y \setminus \mathrm{ClInt}B) \subseteq f^{-1}(Y \setminus \mathrm{Int}B) \Rightarrow \delta P_S Clf^{-1}(Y \setminus ClInt(B)) \subseteq X \setminus f^{-1}(\mathrm{Int}B) \Rightarrow$ $X \setminus \delta P_S \mathrm{Int}(f^{-1}(\mathrm{ClInt}B)) \subseteq X \setminus f^{-1}(\mathrm{Int}B) \Rightarrow f^{-1}(\mathrm{Int}B) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{ClInt}B)$. Therefore, we obtain $f^{-1}(\mathrm{Int}B) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{ClInt}B)$.

$(c) \Rightarrow (d)$. Let V be any open set of Y. Then apply (c) to $cl(V)$ we obtain $f^{-1}(\mathrm{IntCl}V) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{ClIntCl}V) = \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V)$. Therefore, we obtain $f^{-1}(\mathrm{IntCl}V) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V)$.

$(d) \Rightarrow (e)$ Let V be any regular open set of Y. Then V is an open set of Y. By (d), we have $f^{-1}(V) = f^{-1}(\mathrm{IntCl}V) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V))$. Therefore, we obtain $f^{-1}(V) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V)$.

$(e) \Rightarrow (f)$**.** Let F be any regular closed set of Y. Then Y\F is a regular open set of Y. By (e), we have $f^{-1}(Y \setminus F) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}(Y \setminus F)) \Rightarrow X \setminus f^{-1}(F) \subseteq \delta P_S \mathrm{Int} f^{-1}(Y \setminus \mathrm{Int}F) \Rightarrow X \setminus f^{-1}(F) \subseteq \delta P_S \mathrm{Int}(X \setminus f^{-1}(\mathrm{Int}F)) \Rightarrow X \setminus f^{-1}(F) \subseteq X \setminus \delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}F) \Rightarrow \delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}F) \subseteq f^{-1}(F)$. Hence $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}F) \subseteq f^{-1}(F)$.

$(f) \Rightarrow (g)$.Let F be any closed set of Y. Then ClIntF is a regular closed set of Y. By (f) we have $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{IntClInt}F) = \delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}F) \subseteq f^{-1}(\mathrm{ClInt}F)$. Therefore, we obtain $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{Int}F) \subseteq f^{-1}(\mathrm{ClInt}F)$.

$(g) \Rightarrow (h)$. Let V be any open set of Y. Then by (g) we have $\delta P_S \mathrm{Cl} f^{-1}(V) \subseteq \delta P_S \mathrm{Cl} f^{-1}(\mathrm{IntCl}V) \subseteq f^{-1}(\mathrm{ClIntCl}V) = f^{-1}(\mathrm{Cl}V)$. Therefore, $\delta P_S \mathrm{Cl} f^{-1}(V) \subseteq f^{-1}(\mathrm{Cl}V)$.

$(h) \Rightarrow (i)$. Let F be any closed set of Y. Then Y\F is an open set of Y. By (h), we have $\delta P_S \mathrm{Cl} f^{-1}(Y \setminus F) \subseteq f^{-1}(\mathrm{Cl}(Y \setminus F)) \Rightarrow \delta P_S \mathrm{Cl}(X \setminus f^{-1}(F)) \subseteq f^{-1}(Y \setminus \mathrm{Int}F) \Rightarrow X \setminus \delta P_S \mathrm{Int} f^{-1}(F) \subseteq X \setminus f^{-1}(\mathrm{Int}F) \Rightarrow f^{-1}(\mathrm{Int}F) \subseteq \delta P_S \mathrm{Int} f^{-1}(F)$. Therefore, $f^{-1}(\mathrm{Int}F) \subseteq \delta P_S \mathrm{Int} f^{-1}(F)$.

$(i) \Rightarrow (a)$**.** Let x be any point of X and let V be any open set in Y containing f(x). Then $x \in f^{-1}(V)$ and ClV is a closed set in Y. By (i), we have $x \in f^{-1}(V) \subseteq f^{-1}(\mathrm{IntCl}V) \subseteq \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V)$. Put $U = \delta P_S \mathrm{Int} f^{-1}(\mathrm{Cl}V)$. Then we obtain $x \in U \in \delta P_S O(X)$ and $f(U) \subseteq \mathrm{Cl}V$. Therefore, f is weakly δP_S-continuous.

Proposition 6.3.18. If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is continuous, then f is weakly δP_S-continuous.

Proof. Let V be any open set of Y. Since f is continuous, then f^{-1}(V) is an open set and hence it is a semi-open set. By Proposition 2.4.18, we have δP_SClf^{-1}(V) = Clf^{-1}(V). Also, since f is continuous, then Clf^{-1}(V) ⊆ f^{-1}(ClV). Therefore, we obtain that δP_SClf^{-1}(V) ⊆ f^{-1}(ClV) and hence by Theorem 6.3.17(h), f is weakly δP_S-continuous.

Another characterization theorem of weakly δP_S-continuous functions is the following:

Theorem 6.3.19. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$, thefollowing statements are equivalent:

a) f is weakly δP_S-continuous.

b) $f(\delta P_S Cl(A)) \subseteq Clf(A)$, for each subset A of X.

c) Int$_\theta f$ (A) ⊆ f (δP_SIntA), for each subset A of X.

d) f^{-1}(Int$_\theta$B) ⊆ δP_SIntf^{-1}(B), for each subset B ofY.

e) δP_SClf^{-1}(B) ⊆ f^{-1}(Cl$_\theta$B), for each subset B of Y.

Proof. (a) ⇒ (b). Let A be a subset of X. Suppose that f(δP_SClA) ⊈ Cl$_\theta$f(A). Then there exists y ∈ f(δP_SClA) such that y ∉ Cl$_\theta$f(A), then there exists an open set G in Y containing y such that ClG ∩ f(A) = ∅. If $f^{-1}(y) = \emptyset$, then there is nothing to prove. Suppose that x be any arbitrary point of f^{-1}(y), so f(x) ∈ G. Since G is an open set in Y, by (a), there exists a δP_S-open set H in X containing x such that f(H) ⊆ ClG. Therefore, wehave f(H) ∩ f(A) = ∅. Then $y \notin \delta P_S Cl(f(A)) \Rightarrow x \notin \delta P_S Cl(A)$. Hence $y \notin \delta P_S Cl(A)$ which is a contradiction. Therefore, we have f(δP_SClA) ⊆ Cl$_\theta f$(A).

(b) ⇒ (c). Let A be any subset of X. Then apply (b) to X\A we obtain f(δP_SCl(X\A)) ⊆ Cl$_\theta$f(X\A) ⇒ f(X\ δP_SIntA) ⊆ Cl$_\theta$(Y\ f(A)) ⇒ Y\ f(δP_SIntA) ⊆ Y\Int$_\theta$f(A) ⇒ Int$_\theta$f(A) ⊆ f(δP_SIntA). Therefore, we obtain that Int$_\theta$f(A) ⊆f(δP_SIntA).

(c) ⇒ (d). Let B be a subset of Y. Then f $^{-1}$(B) is a subset of X. By (c), we have Int$_\theta$f(f $^{-1}$(B)) ⊆ f(δP_SIntf $^{-1}$(B)). Then IntθB ⊆ f(δP_SIntf $^{-1}$(B)) and hence f $^{-1}$(Int$_\theta$B) ⊆ δP_SIntf $^{-1}$(B).

(d) ⇒ (e). Let B be any subset of Y. Then apply (d) to Y\B we obtain f $^{-1}$(Int$_\theta$(Y\B)) ⊆ δP_SInt f $^{-1}$(Y\B) ⇒ f $^{-1}$(Y\Cl$_\theta$B) ⊆ δP_SInt(X\f $^{-1}$(B)) ⇒ X\f $^{-1}$(Cl$_\theta$B) ⊆ X\ δP_SClf $^{-1}$(B) ⇒ δP_SClf $^{-1}$(B) ⊆ f $^{-1}$(Cl$_\theta$B). Therefore, we obtain δP_SClf $^{-1}$(B) ⊆ f $^{-1}$(Cl$_\theta$B).

(e) ⇒ (a). Let V be any open set of Y. By (e), we have δP_SClf $^{-1}$(V) ⊆ f $^{-1}$(Cl$_\theta$V). By Proposition 2.4.18, we have $\delta P_S Clf^{-1}(V) \subseteq f^{-1}(Cl(V))$.

Therefore, by Theorem 6.3.17(h), f is weakly δP_S-continuous.

Proposition 6.3.20. A function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous if and only if $\delta P_S Clf^{-1}(IntCl_\theta(B))\subseteq f^{-1}(Cl_\theta(B))$ for each subset B of Y.

Proof. Necessity. Let B be any subset of Y. Assume that x $\notin f^{-1}(Cl_\theta B)$. Then f (x) $\notin Cl_\theta B$ and hence there exists an open set H containing f (x) such that B ∩ ClH = ∅. This implies that $Cl_\theta B$ ∩ H = ∅ and so H ⊆ $Y \setminus Cl_\theta B$ and hence ClH ⊆ $Cl(Y\setminus Cl_\theta B)$. Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that f (U) ⊆ ClH ⊆ $Cl(Y\setminus Cl_\theta B) = Y\setminus IntCl_\theta B$. This implies that f (U) ∩ $IntCl_\theta B$ = ∅ and hence U ∩ $f^{-1}(IntCl_\theta B)$ = ∅. Then x $\notin \delta P_S Clf^{-1}(IntCl_\theta B)$. Therefore, $\delta P_S Clf^{-1}(IntCl_\theta B)\subseteq f^{-1}(Cl_\theta B)$.

Sufficiency. Let V be any open set of Y. Then by hypothesis and Proposition 1.3.18, we have $\delta P_S Clf^{-1}(IntClV)=\delta P_S Clf^{-1}(IntCl_\theta V)\subseteq f^{-1}(Cl_\theta V) = f^{-1}(ClV)$. Therefore, $\delta P_S Cl(f^{-1}(IntCl(V)\subseteq f^{-1}(Cl(V))$. Hence by Theorem 6.3.17(b), f is weakly δP_S-continuous.

From Proposition 6.3.20, we obtain that:

Corollary 6.3.21. A function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous if and only if $f^{-1}(Int_\theta(B))\subseteq \delta P_S(Int(f^{-1}(Cl(Int_\theta(B)))$ for each subset B of Y.

Proposition 6.3.22. A function $f:(X,\tau)\to(Y,\sigma)$is weakly δP_S-continuous if and only if $f^{-1}(V)\subseteq\delta P_S(Int(f^{-1}(Cl(V))))$ for each open set V of Y.

Proof. Necessity. Let f be weakly δP_S-continuous and let V be any open set of Y. Then $V\subseteq Int(Cl(V))$. Therefore, by Theorem 6.3.17(b), $f^{-1}(V)\subseteq f^{-1}(Int(Cl(V))\subseteq \delta P_S Int(f^{-1}(Cl(V)))$. Hence $f^{-1}(V)\subseteq\delta P_S(Int(f^{-1}(Cl(V))$.

Sufficiency. Let V be any regular open set of Y. Then V is an open set of Y. By hypothesis, we have $f^{-1}(V)\subseteq\delta P_S f^{-1}(Cl(V))$. Therefore, by Theorem 6.3.17(c), f is weakly δP_S-continuous.

From Proposition 6.3.22, we obtain that:

Corollary 6.3.23. A function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous if and only if $\delta P_S Cl(f^{-1}(Int(F)))\subseteq f^{-1}(F)$ for each closed set F of Y.

Proposition 6.3.24. A function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous if and only if $\delta P_S Cl(f^{-1}(V))\subseteq f^{-1}(\delta P_S(Cl(V)))$ for each open set V of Y.

Proof. Necessity. Let V be any open set of Y. Since f is weakly δP_S-continuous, then by Theorem 6.3.17(h), we have $\delta P_S Cl(f^{-1}(V))\subseteq f^{-1}(\delta P_S(Cl(V)))$. Since V is an open set and hence V is a semi-open set. Therefore, by Proposition 2.4.18, we obtain $\delta P_S(Cl(f^{-1}(V)))\subseteq f^{-1}(\delta P_S(Cl(V)))$.

Sufficiency. Let F be any closed set of Y. Then $Int(F)$ is an open set in Y. By hypothesis, we have $\delta P_S(Cl\left(f^{-1}(Int(F))\right)) \subseteq f^{-1}(\delta P_S(Cl(Int(F)))$. Since $Int(F)$ is a semi-open set, then by Proposition 2.4.18, $\delta P_S(Cl(f^{-1}(Int(F)))) \subseteq f^{-1}(Cl(Int(F)))$. Therefore, by Theorem 6.3.17(g), f is weakly δP_S-continuous.

From Proposition 6.3.24, we obtain that:

Corollary 6.3.25. A function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous if and only if $f^{-1}(\delta P_S \mathrm{Int} F) \subseteq \delta P_S \mathrm{Int} f^{-1}(F)$ for each closed set F of Y.

Proposition 6.3.26. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then $f:(X,\tau) \to (Y,\sigma_\theta)$ is δP_S-continuous.

Proof. Let H $\in \sigma_\theta$, then H is θ-open set in (Y,σ). Since $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, then by Proposition 6.3.10, $f^{-1}(H)$ is a δP_S-open set in X. Therefore, $f:(X,\tau) \to (Y,\sigma_\theta)$ is δP_S-continuous.

Proposition 6.3.27. Let X be a locally indiscrete space. Then the function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous if and only if $f:(X,\tau) \to (Y,\sigma_\theta)$ is continuous.

Proof. Let H $\in \sigma_\theta$, then H is θ-open set in (Y,σ). Since $f:(X,\tau) \to (Y,\sigma)$is weakly δP_S-continuous, then by Proposition 6.3.10, $f^{-1}(H)$ is a δP_S-open set in X. Since X is a locally indiscrete space, then by Proposition 2.2.31, $f^{-1}(H)$ is open set in X. Therefore, $f:(X,\tau) \to (Y,\sigma_\theta)$ is continuous.

Proposition 6.3.28. Let $f:(X,\tau) \to (Y,\sigma)$ be a function. Let $\mathcal{B}$ be any basis for τ_θ in Y. If f is weakly δP_S-continuous, then for each $B \in \mathcal{B}$, $f^{-1}(B)$ is a δP_S-open set of X.

Proof. Suppose that f is weakly δP_S-continuous. Since each $B \in \mathcal{B}$ is a θ-open subset of Y, therefore, by Proposition 6.3.10, $f^{-1}(B)$ is a δP_S-open subset of X.

Proposition 6.3.29: If a function $f:(X,\tau) \to (Y,\sigma)$ is contra δP_S-continuous, then f is weakly δP_S-continuous.

Proof: Let V be any regular open set of Y. Then V is open. Since f is contra δP_S-continuous. Then $f^{-1}(V)$ is δP_S-closed set of X. Therefore, by Proposition 6.3.14, f is weakly δP_S-continuous.

Example 6.3.30: Let $X = \{a,b,c\}$ with the two topologies $\tau = \{X,\emptyset,\{a\},\{c\},\{a,c\}\}$ and $\sigma = \{X,\emptyset,\{a\},\{a,b\}\}$ then $\delta P_S O(X,\tau) = \{X,\emptyset,\{a\},\{c\},\{a,c\}\}$. Let $f:(X,\tau) \to (X,\sigma)$ be the identity function. Then f is weakly δP_S-continuous, but it is not contra δP_S-continuous, since $\{b,c\}$ is a closed set of (X,σ) but $\{b,c\}$ is not δP_S-open in (X,τ).

6.3.1 PROPERTIES AND COMPARISONS

In this section, we give some properties of weakly δP_S-continuous functions and we compare them with other types of continuous functions.

Proposition 6.3.1.1. Let $f:(X,\tau) \to (Y,\sigma)$ be weakly δP_S-continuous function. If A is a regular semi-open subset of X, then the restriction $f|A: A \to Y$ is weakly δP_S-continuous in the subspace A.

Proof. Let x ∈ A and V be an open set of Y containing f(x). Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that f(U) ⊆ ClV. Since A is a regular semi-open subset of X, by Proposition 2.3.5, A ∩ U is a δP_S-open subset of A containing x and (f|A)(A ∩ U) = f(A ∩ U) ⊆ f(U) ⊆ ClV. This show that f|A is weakly δP_S-continuous.

Corollary 6.3.1.2. Let $f:(X,\tau) \to (Y,\sigma)$ be a weakly δP_S-continuous function. If A is a regular open subset of X, then the restriction f|A:A→Y is weakly δP_S-continuous in the subspace A.

Proof. Since every regular open set is regular semi-open, this is an immediate consequence of Proposition 6.3.1.1.

Proposition 6.3.1.3. Let $f:(X,\tau) \to (Y,\sigma)$ be a function. If for each x ∈ X, there exists a regular open set A of X containing x such that the restriction f|A:A→Y is weakly δP_S-continuous, then f is weakly δP_S-continuous.

Proof. Let x ∈ X, then by hypothesis, there exists a regular open set A containing x such that f|A:A→Y is weakly δP_S-continuous. Let V be any open set of Y containing f(x), there exists a δP_S-open set U in A containing x such that (f|A)(U) ⊆ ClV. Since A is regular open set, by Proposition 2.3.2, U is δP_S-open set in X and hence f(U) ⊆ ClV. This shows that f is weakly δP_Scontinuous.

As an immediate consequence of Corollary 6.3.1.2 and Proposition 6.3.1.3, we obtain that:

Corollary 6.3.1.4. Let {Uα : α ∈ Δ} be a regular open cover of a topological space X. A function $f:(X,\tau) \to (Y,\sigma)$is weakly δP_S-continuous if and only if the restriction f|U_α :U_α →Y is weakly δP_Scontinuous for each α ∈ Δ.

Remark 6.3.1.5. If $f:(X,\tau) \to (Y,\sigma)$ is a weakly δP_Scontinuous function and A, B are any subsets of X. Then the restriction f|A:A→f(A) need not be weakly δP_S-continuous in general.

Moreover, f|(A ∪ B): A∪B→f(A ∪ B) is not always weakly δP_S-continuous even if f|A:A→f(A), f|B:B→f(B) and f are all weakly δP_S-continuous.

Proposition 6.3.1.6. If X = R ∪ S, where R and S are regular open sets and f:X→Y is a function such that both f|R and f|S are weakly δP_S-continuous, then f is weakly δP_S-continuous.

Proof. Let x ∈ X and V be an open set of Y containing f (x). Since f|R and f|S are weakly δP_S-continuous, there exist δP_S-open sets U of R and W of S with x ∈ U and x ∈ W, such that (f|R)(U) ⊆ ClV and (f|S)(W) ⊆ ClV. Then f(U ∪ W) = (f|R)(U) ∪ (f|S)(W) ⊆ ClV. Since R and S are regular open sets in X, then by Proposition 2.3.2, U and W are δP_S-open sets in X. Since union of two δP_S-open sets is δP_S-open, then U ∪ W is a δP_S-open set of X containing x. Therefore, f is weakly δP_S-continuous.

In general, if X = ∪{K_α : α ∈ Δ}, where each K_α is a regular open set and $f:(X,\tau) \to (Y,\sigma)$ is a function such that the restriction f|K_α is weakly δP_S-continuous for each α, then f is weakly δP_S-continuous.

Proposition 6.3.1.7. Let X = R_1 ∪ R_2, where R_1 and R_2 are regular open sets in X. Let f:R_1→Y and g:R_2→Y be weakly δP_S-continuous. If f(x) = g(x) for each x ∈ $R_1 \cap R_2$, then h:$R_1 \cup R_2$→Y such that

$$h(x) = \begin{cases} f(x) & if\ x \in R_1\ and\ x \notin R_2 \\ g(x) & if\ x \in R_1 and\ x \notin R_2 \\ f(x) = g(x) & if\ x \in R_1 \cap R_2 \end{cases}$$

is weakly δP_S-continuous.

Proof. Let x ∈ X and V be an open set of Y containing h (x). Then x ∈ $R_1 \cup R_2$ and V is an open set of Y containing f (x) and g (x). Since $f$is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that f (U) ⊆ ClV. Then f^{-1}(ClV) is a δP_S-open set of R_1 containing x. But R_1 is a regular open set in X, then by Proposition 2.3.2, f^{-1}(ClV) is a δP_S-open set of X containing x. Similarly, f^{-1}(ClV) is a δP_S-open set in R_2 and hence, a δP_S-open set in X. Since union of two δP_S-open sets is δP_S-open. Therefore, h^{-1}(ClV) = f^{-1}(ClV) ∪ g^{-1}(ClV) is a δP_S-open set in X and it is clear that h (h^{-1}(ClV)) ⊆ ClV. Hence h is weakly δP_S-continuous.

Proposition 6.3.1.8. Let $f:(X,\tau) \to (Y,\sigma)$ be weakly δP_S-continuous surjection and A be a regular semi-open subset of X. If f is an open function, then the function $g: A \to f(A)$, defined by g (x) = f (x) for each x ∈ A, is weakly δP_S- continuous.

Proof. Putting H = f (A). Let x ∈ A and V be any open set in H containing g (x). Since H is open in Y and V is open in H, then V is open in Y. Since f is weakly δP_S-continuous, there exists a δP_S-open set U in X containing x such that f (U) ⊆ClV. Taking W = U ∩ A, since A is either open or a regular semi-open subset of X, then by Proposition 2.3.5, W is a δP_S-open set in A containing x and g (N) ⊆ $Cl_Y V$ ∩ H = $Cl_H V$. Then g (W) ⊆ $Cl_H V$. This shows that g is weakly δP_S- continuous.

Proposition 6.3.1.9. Let $f: (X, \tau) \to (Y, \sigma)$ be a weakly δP_S-continuous function and for each x ∈ X. If Y is any subset of Z containing f (x), then $f: (X, \tau) \to (Z, \eta)$ is weakly δP_S-continuous.

Proof. Let x ∈ X and V be any open set of Z containing f (x). Then V ∩ Y is open in Y containing f (x). Since $f: (X, \tau) \to (Y, \sigma)$ is weakly δP_S- continuous, there exists a δP_S-open set U of X containing x such that f(U) ⊆ Cl(V ∩ Y) and hence f (U) ⊆ ClV. Therefore, $f: (X, \tau) \to (Z, \eta)$ is weakly δP_S-continuous.

We shall obtain some conditions for which the composition of two functions is weakly δP_S- continuous.

Theorem 6.3.1.10. Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be functions. Then the composition function $g \circ f: (X, \tau) \to (Z, \eta)$ is weakly δP_S-continuous if f and g satisfy one of the following conditions:

a) f is δP_S-continuous and g is weakly continuous.
b) f is weakly δP_S-continuous and g is almoststrongly θ-continuous.
c) f is weakly δP_S-continuous and g is θ-continuous.
d) f is weakly δP_S-continuous and g is continuous.
e) f is continuous and open and g is weakly δP_S-continuous.

Proof. a) Let $x \in X$ and W be an open set of Z containing g(f(x)). Since g is weakly continuous, there exists an open set V of Y containing f (x) such that $g(V) \subseteq Cl(W)$(i.e.,) $f(x) \in V \subseteq g^{-1}(Cl(W))$. Hence $g^{-1}(Cl(W))$ is open in Y containing f (x). Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq g^{-1}(Cl(W))$, from Definition 5.2.1. Therefore, we obtain $(g \circ f)$(U) = g(f (U)) ⊆ ClW. Hence $g \circ f$ is weakly δP_S-continuous.

b) Let W be any regular open subset of Z. Since g is almost strongly θ-continuous, g^{-1}(W) is θ-open subset of Y. Since f is weakly δP_S- continuous, then by Proposition 6.3.10, $(g \circ f)^{-1}$(W) = $f^{-1}(g^{-1}$(W)) is a δP_S-open subset in X. Therefore, $g \circ f$ is almost δP_S-continuous, [by Theorem 6.2.10(e) and hence it is weakly δP_S-continuous, by Lemma 6.3.2(a).

c) Let $x \in X$, and W be an open set of Z containing g(f(x)). Since g is θ-continuous, there exists an open set V of Y containing f(x) such that $g(Cl(V)) \subseteq Cl(W)$. Since f is weakly δP_S- continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Hence $g(f(U)) \subseteq g(Cl(V)) \subseteq Cl(W)$. Therefore, $g \circ f$ is weakly δP_S-continuous.

d) Let $x \in X$ and W be an open set of Z containing $g(f(x))$. Since g is continuous, $g^{-1}(W)$ is an open set of Y containing f(x). Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Clg^{-1}(W)$. Also, since g is continuous, then we have $f(U) \subseteq g^{-1}(ClW)$. This implies that $g(f(U)) \subseteq ClW$. Therefore, $g \circ f$ is weakly δP_S-continuous.

e) Let $x \in X$ and W be an open set of Z containing $g(f(x))$. Since g is weakly δP_S-continuous, there exists a δP_S-open set U of Y containing f(x) such that $g(U) \subseteq CL(W)$. It is clear that $g^{-1}(ClW)$ is a δP_S-open set of Y containing f(x). Since f is continuous and open, then by Proposition 5.2.9, $f^{-1}(g^{-1}(ClW)) = f^{-1}(g^{-1}(Cl(W)) = (g \circ f)^{-1}(Cl(W))$ is a δP_S-open set in X containing x and clearly $(g \circ f)((g \circ f)^{-1}(Cl(W))) \subseteq Cl(W)$. Hence f is weakly δP_S-continuous.

Proposition 6.3.1.11. If $f: (X, \tau) \to (Y, \sigma)$ is a weakly δP_S-continuous function and Y is almost regular, then f is almost δP_S-continuous.

Proof. Let $x \in X$ and let V be any open set of Y containing f(x). By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subseteq ClG \subseteq IntClV$. Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq ClG \subseteq IntClV$. Therefore, f is almost δP_S- continuous, from Definition 6.2.1.

Proposition 6.3.1.12. If $f: (X, \tau) \to (Y, \sigma)$ is a weakly δP_S- continuous function and Y is an extremally disconnected space, then f is almost δP_S- continuous.

Proof. Let $x \in X$ and let V be any open set of Y containing f(x). Since f is weakly δP_S-continuous, there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Since Y is extremally disconnected From Definition 1.1.6, $Cl(V)$ is open, (i.e.,) $Cl(V) = Int(Cl(V))$, then $f(U) \subseteq Int(Cl(V))$. Therefore, f is almost δP_S-continuous.

Corollary 6.3.1.13. A function $f: (X, \tau) \to (Y, \sigma)$ is almost δP_S-continuous if and only if f is weakly δP_S-continuous and it satisfies one of the following properties:

a) Y is almost regular.

b) Y is extremally disconnected.

Proof. The proof follows from Proposition 6.3.2(a). The converse is proved in Proposition 6.3.1.11 and Proposition 6.3.1.12.

Corollary 6.3.1.14. Let $f:(X,\tau)\to(Y,\sigma)$ be a function and X is a locally indiscrete space. Then f is weakly δP_S-continuous if and only if f is weakly continuous.

Proof. Follows from Proposition 2.2.31.

Corollary 6.3.1.15. If X is a locally indiscrete space and Y is either almost regular or an extremally disconnected space, the following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$:

a) f is almost δP_S-continuous.

b) f is weakly δP_S-continuous.

c) f is weakly continuous.

d) f is almost continuous.

Proof. (a) $\Rightarrow (b)$ Follows from Proposition 6.3.2

(b) $\Rightarrow (c)$ Follows from Corollary 6.3.1.14

(c) $\Rightarrow (d)$ Follows from Corollary 6.3.1.12

(d) $\Rightarrow (a)$ Since X is locally indiscrete, $\delta P_S O(X)=\tau$. Hence almost continuous function is a almost δP_S-continuous function, from Proposition 6.2.23.

Corollary 6.3.1.16. If Y is a regular space, the following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$:

a) f is δP_S-continuous.

b) f is almost δP_S-continuous.

c) f is weakly δP_S-continuous.

Proof. Follows from Proposition 6.3.1.11 and Proposition 6.2.1.12 and the fact that every regular space is almost regular and semi-regular space.

Corollary 6.3.1.17. If X is a locally indiscrete space and Y is a regular space, the following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$:

a) f is δP_S-continuous.

b) f is almost δP_S-continuous.

c) f is weakly δP_S-continuous.

d) f is weakly continuous.

e) f is almost continuous.

f) f is continuous.

Proof. Follows from Corollary 6.3.1.15, Corollary 6.3.1.16 and Theorem 1.3.19 and the fact that every regular space is almost regular and semi-regular space.

Proposition 6.3.1.18. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and X is a semi-T_1 space. Then f is weakly δP_S- continuous if and only if f is weakly δ-precontinuous.

Proof. Follows from Proposition 2.2.23.

Corollary 6.3.1.19. If X is a semi-T_1 space and Y is either almost regular or an extremally disconnected space, the following statements are equivalent for a function $f: (X, \tau) \rightarrow (Y, \sigma)$

a) f is almost δP_S-continuous.

b) f is weakly δP_S-continuous.

c) f is weakly δ-precontinuous.

d) f is almost δ-precontinuous.

Proof. Follows from Corollary 6.3.1.13, Proposition 6.3.1.18 and Proposition 6.2.23.

Corollary 6.3.1.20. If X is a semi-T_1 space and Y is a regular space, the following statements are equivalent for a function $f: (X, \tau) \rightarrow (Y, \sigma)$:

a) f is δP_S-continuous.

b) f is almost δP_S-continuous.

c) f is weakly δP_S-continuous.

d) f is weakly δ-precontinuous.

e) f is almost δ-precontinuous.

f) f is δ-precontinuous.

Proof. Follows from Corollary 6.3.1.16, Corollary 6.3.1.19 and Theorem 1.3.20 and the fact that every regular space is almost regular and semi-regular space.

Proposition 6.3.1.21. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a semi-continuous function. Then f is weakly continuous if and only if f is weakly δP_S-continuous.

Proof. Necessity. Let V be any open set of Y. Since f is weakly continuous, by Theorem 1.3.21, Clf^{-1}(V) $\subseteq$ f^{-1}(ClV). Since f is semi-continuous, then f^{-1}(V) is a semi-open set in X. Hence by Proposition 2.4.18, δP_SClf^{-1}(V) = Clf^{-1}(V). Therefore, we obtain δP_SClf^{-1}(V) $\subseteq$ f^{-1}(ClV). Thus, by Theorem 6.3.17(h), f is weakly δP_S- continuous.

Sufficiency. Let V be any open set in Y. Since f is weakly δP_S-continuous, by Theorem 6.3.17(h), δP_SClf^{-1}(V) $\subseteq$ f^{-1}(ClV). Since f is semi- continuous, then f^{-1}(V) is semi-open set of X. Hence by Proposition 2.4.18, we have δP_SClf^{-1}(V)= Clf^{-1}(V). Therefore, we obtain Clf^{-1}(V) $\subseteq$ f^{-1}(ClV). Thus, by Theorem 1.3.21, f is weakly continuous.

Corollary 6.3.1.22. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is weakly δP_S-continuous if and only if f is weakly continuous if it satisfies one of the following properties:

a) X is locally indiscrete space.

b) f is semi-continuous.

Proof. Follows from Corollary 6.3.1.14 and Proposition 6.3.1.21.

Proposition 6.3.1.23. If $f:(X,\tau)\to(Y,\sigma)$ is weakly θs-continuous and weakly δ-precontinuous, then f is weakly δP_S-continuous.

Proof. Let $x\in X$ and let V be any open set of Y containing f (x). Since f is weakly θs-continuous and weakly δ-pre-continuous, then there exists a θ-semi-open and a δ-preopen set U of X containing x such that f(U) ⊆ ClV, respectively. Hence by Lemma 2.2.45, U is a δP_S-open set of X containing x such that f (U) ⊆ ClV. Therefore, f is weakly δP_S- continuous.

Proposition 6.3.1.24. Let $f:(X,\tau)\to(Y,\sigma)$ be a function and X be an extremally disconnected space. If f is weakly θs-continuous, then f is weakly δP_S- continuous.

Proof. Follows from Lemma 2.2.46.

Proposition 6.3.1.25. If $f:(X,\tau)\to(Y,\sigma)$ is weakly δ-precontinuous and either S-continuous or a θ- irresolute function, then f is weakly δP_S- continuous.

Proof. Let $x\in X$ and V be any open set of Y containing f (x). Since f is weakly δ-precontinuous, there exists a δ-preopen set U of Y containing f (x) such that f (U) ⊆ ClV. Then f^{-1}(ClV) is a δ-preopen set of Y containing x. Since ClV is a regular closed set of Y and f is either S- continuous or θ-irresolute, then f^{-1}(ClV) is the union of regular closed sets of X and hence is the union of semi-closed sets of X. By Lemma 2.2.2, f^{-1}(ClV) is a δP_S-open set of X containing x and clearly f (f^{-1}(ClV)) ⊆ ClV. Hence f is weakly δP_S-continuous.

Corollary 6.3.1.26. Let $f:(X,\tau)\to(Y,\sigma)$ be either S- continuous or a θ-irresolute function. Then f is weakly δP_S-continuous if and only if f is weakly δ-precontinuous.

Proposition 6.3.1.27. If a function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous and open, then

f (δP_SClV) ⊆ δP_SClf(V) for each open set V of X.

Proof. Let V be any open set of X. Since f is open, then f (V) is an open set in Y. Since f is weakly δP_S-continuous, then by Proposition 6.3.24, we obtain that δP_SClf^{-1}(f (V)) ⊆ f^{-1}(δP_SClf (V)) which implies that f(δP_SClV) ⊆ δP_SClf (V).

Corollary 6.3.1.28. If a function $f:(X,\tau)\to(Y,\sigma)$ is weakly δP_S-continuous and open, then $\delta P_S Intf(F)\subseteq f(\delta P_S Int(F))$ for each closed set F of X.

Proposition 6.3.1.29. If a function $f:(X,\tau)\to(Y,\sigma)$ is semi-continuous and almost open, then f is weakly δP_S- continuous if and only if δP_SClf^{-1}(V) = f^{-1}(δP_SClV) for each open set V of Y.

Proof. Necessity**.** Let V be any open set of Y. Since f is weakly δP_S-continuous, then by Proposition 6.3.24, $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{V}) \subseteq f^{-1}(\delta P_S \mathrm{ClV})$. Since V is open, hence it is semi-open. Then by Proposition 2.4.18, $\delta P_S Cl(V) = Cl(V)$ which implies that $\delta P_S Cl(V) \subseteq Cl(V)$ and hence $f^{-1}(\delta P_S \mathrm{ClV}) \subseteq f^{-1}(\mathrm{ClV})$. Since V is an open set of Y and f is almost open, then by Theorem 1.3.22, $f^{-1}(\mathrm{ClV}) \subseteq \mathrm{Cl} f^{-1}(\mathrm{V})$. Therefore, we have $f^{-1}(\delta P_S \mathrm{ClV}) \subseteq f^{-1}(\mathrm{ClV}) \subseteq \mathrm{Cl} f^{-1}(\mathrm{V})$ and hence $f^{-1}(\delta P_S \mathrm{ClV}) \subseteq \mathrm{Cl} f^{-1}(\mathrm{V})$. Since V is an open set of Y and f is semi-continuous, then $f^{-1}(\mathrm{V})$ is a semi-open set in X. Then by Proposition 2.4.18 we obtain that $f^{-1}(\delta P_S \mathrm{ClV}) \subseteq \delta P_S \mathrm{Cl} f^{-1}(\mathrm{V})$.Therefore, we have $\delta P_S \mathrm{Cl} f^{-1}(\mathrm{V}) = f^{-1}(\delta P_S \mathrm{ClV})$.

Sufficiency**.** Follows from Proposition 6.3.24.

Corollary 6.3.1.30. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, semi-continuous and almost open, then $\delta P_S \mathrm{Int} f^{-1}(\mathrm{F}) = f^{-1}(\delta P_S \mathrm{IntF})$ for each closed set F of Y.

Proof. Follows from Proposition 6.3.1.29.

Corollary 6.3.1.31. If a function $f:(X,\tau) \to (Y,\sigma)$ is weakly δP_S-continuous, semi-continuous and almost open, then $\mathrm{Cl} f^{-1}(\mathrm{V}) = f^{-1}(\mathrm{ClV})$ for each open set V of Y.

Proof. Follows from Proposition 6.3.1.29 and Proposition 2.4.18.

Corollary 6.3.1.32. Let $f:(X,\tau) \to (Y,\sigma)$ be an almost open function. If f is weakly δP_S-continuous and semi- continuous, then f is almost continuous and hence f is weakly continuous.

Proof. Follows from Proposition 6.3.1.31 and Theorem 1.3.23.