CHAPTER 3

CHAPTER – 3

FUZZY GENERALIZED CLOSED SETS

In this chapter fuzzy rw-closed sets, fuzzy rw-open sets due to Benchalli et al. [16] are studied.

Section 3.1

Preliminary Definitions and Results of Fuzzy Generalized Closed Sets.

Definition: 3.1.1

A fuzzy set A in a fts X is said to be fuzzy semiopen if and only if there exists a fuzzy open set V in X such that $V \leq A \leq cl(V)$.

Definition: 3.1.2

A fuzzy set A in a fts X is said to be fuzzy semi-closed if and only if there exists a fuzzy closed set V in X such that $int(V) \leq A \leq V$.

Theorem: 3.1.3

The following are equivalent:

(a) λ is a fuzzy semiclosed set,

(b) λ^c is a fuzzy semiopen set

(c) $int(cl(\lambda)) \leq \lambda$

(d) $cl(int(\lambda)) \geq \lambda^c$

Theorem: 3.1.4

(a)Any union of fuzzy semiopen sets is a fuzzy semiopen set and (b) any intersection of fuzzy closed sets is a fuzzy semi closed.

Remark: 3.1.5

(i)Every fuzzy open set is fuzzy semiopen but not conversely true.

(ii) Every fuzzy closed set is fuzzy semiclosed set but not conversely true.

(iii) The closure of a fuzzy open set is a fuzzy semiopen set.

(iv) The interior of a fuzzy closed set is a fuzzy semi closed set.

Definition: 3.1.6

A fuzzy set λ of a fts X is called a fuzzy regular open set of X if int(cl(λ))=λ

Definition: 3.1.7

A fuzzy set λ of a fts X is called a fuzzy regular closed set of X if cl(int(λ))= λ

Theorem: 3.1.8

A fuzzy set λ of a fts X is a fuzzy regular open if and only if λ^c fuzzy regular closed sets.

Remark: 3.1.9

(i) Every fuzzy regular open set is a fuzzy open set but not conversely.
(ii) Every fuzzy regular closed set is a fuzzy closed set but not conversely.

Theorem: 3.1.10

(i) The closure of a fuzzy open set is a fuzzy regular closed.
(ii) The interior of a fuzzy closed set is a fuzzy regular open set.

Definition: 3.1.11

A fuzzy set α of a fuzzy topological spaces in X is said to be fuzzy regular semi open set(FRSO) in a fts X if there exists a fuzzy regular open set σ in X such that $\sigma \leq \alpha \leq cl(\sigma)$.

Theorem: 3.1.12

(i) Every fuzzy regular semiopen set is a fuzzy semiopen but not conversely.
(ii) Every fuzzy regular closed set is a fuzzy regular semiopen set but not conversely.
(iii) Every fuzzy regular open set is a fuzzy regular semiopen set but not conversely.

Theorem: 3.1.13

A fuzzy set α of fts X is fuzzy regular semi open if and only if α is both fuzzy semi open and fuzzy semi closed.

Theorem: 3.1.14

If α is fuzzy regular semi open in fts X, then 1- α is fuzzy regular semi open in X.

Definition: 3.1.15

A fuzzy set α in fts X is called fuzzy generalized closed(gf-closed) if cl(α)≤ μ whenever α≤μ and μ fuzzy open and α is fuzzy generalized open if 1- α is fuzzy generalized closed.

Section 3.2

Fuzzy rw-Closed Sets

Definition: 3.2.1

Let (X,T) be a fuzzy topological space. A fuzzy set α of X is called fuzzy regular w-closed (briefly, rw-closed) if cl(α)≤σ whenever α≤σ and σ is fuzzy regular semiopen in fts X.

We denote the class of all fuzzy regular w-closed set in fts X by FRWC(X).

Theorem: 3.2.2

Every fuzzy closed set is a fuzzy rw-closed set in a fts X.

Proof:

Let α be a fuzzy closed set in a fts X. Let β be a fuzzy regular semiopen set in X such that α≤ β. Since α is fuzzy closed, cl(α)= α. Therefore cl(α) ≤ β. Hence α is fuzzy rw-closed set in ftsX.

Preposition: 3.2.3

The converse of the above theorem is not true by the following example.

Example: 3.2.4

Let X= { a,b,c}. Define a fuzzy set α in X by

α(x) = { 1 if x =a

0 otherwise

Let T= {1, 0, α}. Then (X,T) is a fuzzy topological space. Define fuzzy set β in X by

β (x) = {1 if x=b

0 otherwise

Then β is a fuzzy rw-closed set but it is not a fuzzy closed set in ftsX.

Corollary: 3.2.5

By Remark 3.1.9 (ii), it has been proved that every fuzzy regular closed set is a fuzzy closed set but not conversely true. By Theorem 3.2.2, every fuzzy closed set is a fuzzy rw-closed set but not conversely and hence every fuzzy regular closed set is a fuzzy rw-closed set but not conversely.

Remark: 3.2.6

Fuzzy generalized closed sets and fuzzy rw-closed sets are independent.

Example: 3.2.7

Let X= { a,b,c,d} and the functions

α,β, γ :X→ [0,1] be defined by

α(x) = { 1 if x=a

0 otherwise

β (x) = {1 if x=b

0 otherwise

γ (x)= {1 if x=a,b

0 otherwise

Consider T= {1,0, α,β, γ }. Then (X, T) is a fuzzy topological space. In this fts X, the fuzzy set γ :X→ [0,1] defined by

γ (x)= {1 if x=c

0 otherwise

Then λ is a fuzzy generalized closed set in a fts X. In this fts, the fuzzy set δ: X→ [0, 1] defined by

δ (x) = {1 if x=a,c

0 otherwise

Then δ is a fuzzy regular semiopen set containing λ, but δ doesnot contain cl(λ) which is γ^c. Therefore δ is not a fuzzy rw-closed set in fts X.

Example: 3.2.8

Let X= I= [0, 1]. Define a fuzzy set λ in X by

ρ (x) = {½ if x= 2/3

0 otherwise

Let T= {1, 0, ρ }. Then (X, T) is a fuzzy topological space.

Then α(x) = {1/3 if x=2/3

0 otherwise

Then α is a fuzzy rw-closed set in fts X. Now cl(α)= ρ^c and ρ is a fuzzy open set containing α but ρ does not contain cl(α) which is ρ^c.Therefore α is not a fuzzy generalized closed.

Remark: 3.2.9

Fuzzy rw-closed sets and fuzzy semi-closed sets are independent.

Example: 3.2.10

Consider the fuzzy topological space (X, T) defined in example 3.2.7. Then the fuzzy set α= {(a, 1), (b, 0), (c, 0)} is a fuzzy rw-closed set but it is not a fuzzy semi-closed set in fts X.

Example: 3.2.11

Consider the fuzzy topological space (X,T) defined in Example 3.2.7. In this fts X, the fuzzy set μ:X→[0,1] is defined by

μ(x)= { 1 if x=a,c

0 otherwise

Then μ is a fuzzy semi-closed in fts X. μ is also fuzzy regular semiopen set containing μ which does not contain cl(μ)= β^c = {(a,1),(b,0),(c,1), (d,1)}. Therefore μ is not a fuzzy rw-closed set in fts X.

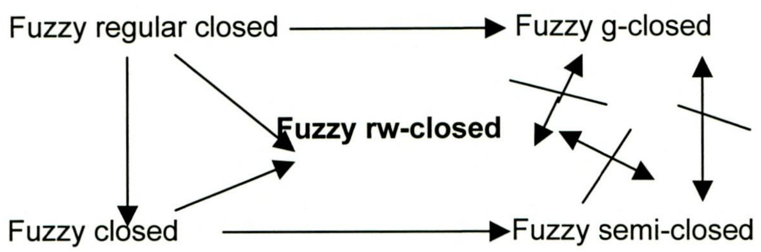
Remark: 3.2.12

From the above discussions and known results we have the following implications

In the following diagram

A→B we mean A implies B but not conversely

A↮B we mean A and B are independent to each other.



Theorem: 3.2.13

If α and β are fuzzy rw-closed sets in fts X, then α ∪ β is a fuzzy rw-closed set in fts X.

Proof:

Let σ be a fuzzy regular semiopen set in a fts X such that α ∪ β ≤σ. Now α≤σ and β≤σ. Since α and β are fuzzy rw-closed sets in fts X, cl(α) ≤ σ and cl(β) ≤ σ. Therefore cl(α) ∪ cl(β) ≤ σ.

But cl(α) ∪ cl(β) ≤ cl(α ∪ β). Thus cl(α ∪ β) ≤ σ. Hence α union β is a fuzzy rw-closed set in fts x.

Remark: 3.2.14

If α and β are fuzzy rw-closed sets in fts X, then α ∩ β need not be a fuzzy rw-closed set in general as seen in the following example.

Example: 3.2.15

Consider the fuzzy topological space (X, T) defined in example 3.2.7.

Let fuzzy sets δ_1, δ_2: X→ [0, 1] be defined by

δ_1 (x) = {1 if x=c,d 0 otherwise

δ_2 (x) = {1 if x=a,b,c 0 otherwise

Then δ_1 and δ_2 are the fuzzy rw-closed sets in fts in X.

Let λ= $\delta_1 \cap \delta_2$. Then λ (x) = {1 if x=c

0 otherwise

Then λ= $\delta_1 \cap \delta_2$ is not a fuzzy rw-closed set in fts X.

Theorem: 3.2.16

If a fuzzy set α of fts X is both fuzzy regular open and fuzzy rw-closed, then α is a fuzzy regular closed set in fts X.

Proof:

Let a fuzzy set α be both fuzzy regular open and fuzzy rw-closed. As every fuzzy regular open set is a fuzzy regular semiopen set cl(α) ≤ α. Therefore cl(α) =α. That is α is fuzzy closed. Since α is fuzzy regular open, int(α)= α. Now cl(int(α))=cl(α)= α. Therefore α is a fuzzy regular closed set in fts X.

Theorem: 3.2.17

If a fuzzy set α of a fts X is both fuzzy regular semiopen and fuzzy rw-closed, then α is a fuzzy closed set in fts X.

Proof:

Let a fuzzy set α of fts X be both fuzzy regular semiopen and fuzzy rw-closed. Then cl(α) ≤ α. Therefore cl(α) =α and hence α is a fuzzy closed set in fts X.

Theorem: 3.2.18

If a fuzzy set α of a fts X is both fuzzy open and fuzzy generalized closed, then α is a fuzzy rw- closed set in fts X.

Proof:

Let a fuzzy set α of fts X be both fuzzy open and fuzzy generalized closed. Then cl (α) ≤ α. Therefore cl(α) =α. That is α is a fuzzy closed set and hence α is a fuzzy rw-closed set in fts X.

Remark: 3.2.19

If a fuzzy set γ is both fuzzy regular open and fuzzy rw-closed set in a fts X then γ need not be a fuzzy generalized closed set in general as seen from the following example.

Example: 3.2.20

Let X= { a,b,c} and the functions α,β, γ :X→ [0,1] be defined as

α(x) = { 1 if x=a
0 otherwise

γ (x)= {1 if x=a,b
0 otherwise

β (x) = {1 if x=b
0 otherwise

Consider T= {1, 0, α,β, γ}. Then (X,T) is a fuzzy topological space. In this fts X, γ is both fuzzy open and fuzzy rw-closed set in fts X but it is not fuzzy generalized closed.

Theorem: 3.2.21

Let α be a fuzzy rw-closed set of a fts X and suppose α≤β≤cl(α). Then β is also a fuzzy rw-closed set in fts X.

Proof:

Let α≤β≤cl(α) and α be a fuzzy rw-closed set of fts X. Let σ be any fuzzy regular semi open set such that β≤ σ. Then α≤ σ and σ is a fuzzy rw-closed we have cl(α) ≤ σ. But cl(β) ≤ cl(α) and thus cl(β) ≤ σ. Hence β fuzzy rw-closed set in fts X.

Theorem: 3.2.22

In a fuzzy topological space X if FRSO(X) = {1, 0}, where FRSO(X) is the family of all fuzzy regular semiopen sets then every fuzzy subset of X is fuzzy rw-closed.

Proof:

Let X be a fuzzy topological space and FRSO(X)= {1,0}. Let α be any fuzzy subset of X. Suppose α=0. Then 0 is a fuzzy rw-closed set in fts X. Suppose α≠0. Then 1 is the only fuzzy regular semiopen set containing α and so cl(α) ≤1. Hence α is a fuzzy rw-closed set in fts X.

Remark: 3.2.23

The converse of the above theorem need not be true in general as seen in the following example.

Example: 3.2.24

Let X= { a,b,c} and the functions α,β:X→ [0,1] be defined as

α(x) = { 1 if x=a	β (x) = {1 if x=b,c
0 otherwise	0 otherwise

Consider T= {1,0, α,β}. Then (X,T) is a fuzzy topological space. In this fts X, every fuzzy subset of X is a fuzzy rw-closed set in fts X, but FRSO= {1, 0, α,β}.

Theorem: 3.2.25

If α is a fuzzy rw-closed set of a fts X and cl(α) ∩ (1-cl(α))=0,then cl(α)- α does not contain any non-zero fuzzy regular semi open set in fts X.

Proof:

Suppose α is a fuzzy rw-closed set of a fts X and cl(α) ∩ (1-cl(α))=0. We prove the result by contradiction. Let β be a fuzzy regular semi open set such that cl(α)- α ≥β and β ≠0. Now β≤cl(α)-α, i.e β≤1-α which implies α≤1-β. Since β is a fuzzy regular semiopen set, by theorem, 1- β is also fuzzy regular semi open set in fts X. Since α is a fuzzy rw-closed set in fts x, by definition cl(α) ≤ 1-β. So β≤1-cl(α). Therefore β≤ (cl(α) ∩ (1-cl(α))=0, by hypothesis. This shows that β=0 which is a contradiction. Hence cl(α)- α does not contain any non-zero fuzzy regular semi open set in a fts X.

Corollary: 3.2.26

If α is a fuzzy rw-closed set in fts X and cl(α) ∩ (1-cl(α))=0, then cl(α) –α does not contain any non-zero fuzzy regular open set in fts X.

Proof:

As the result follows every fuzzy regular open set is a fuzzy regular semi open set.

Corollary: 3.2.27

If α is a fuzzy rw-closed set in fts X and cl(α) ∩ (1-cl(α))=0, then cl(α) –α does not contain any non-zero fuzzy regular closed set in fts X.

Proof:

As the result follows every fuzzy regular closed set is a fuzzy regular semi open set.

Theorem: 3.2.28

Let α be a fuzzy rw-closed set in a fts X and cl(α) ∩ (1-cl(α))=0. Then α is a fuzzy closed set if and only if cl(α)- α is a fuzzy regular semiopen set in fts X.

Proof:

Suppose α is a fuzzy closed set in fts X. Then cl(α)=α and so cl(α)- α=0, which is a fuzzy regular semiopen set in fts X.

Conversely suppose cl(α)- α is a fuzzy regular semi open set in fts X. Since α is fuzzy rw-closed, by theorem 3.2.25 cl(α)- α doesnot contain any non-zero fuzzy regular open set in fts X. Then cl(α)- α=0. That is cl(α)= α and hence α is a fuzzy closed set in fts X.

Section 3.3

Fuzzy rw-Open Sets

Definition: 3.3.1

A fuzzy set α of a fuzzy topological space X is called a fuzzy regular w-open (briefly, fuzzy rw-open) set if its complement α^c is a fuzzy rw-closed set in fts X.

We denote the family of all fuzzy rw-open sets in fts X by FRWO(X).

Theorem: 3.3.2

If a fuzzy set α of a fuzzy topological space X is fuzzy open, then it is fuzzy rw-open but not conversely true.

Proof:

Let α be a fuzzy open set of fts X. Then α^c is fuzzy closed. By theorem 3.2.2, α^c is fuzzy rw-closed. Therefore α is a fuzzy rw-open set in X.

The converse of the above theorem need not be true in general as shown in the following example.

Example: 3.3.3

Let X= { a,b,c}. Define a fuzzy set α in X by

α(x) = { 1 if x=a,b

0 otherwise

Let T= {1, 0, α}. Then (X,T) is a fuzzy topological space. Define a fuzzy set β in X by

β (x) = {1 if x=b

0 otherwise

Then β is a fuzzy rw-open set but it is not a fuzzy open set in fts X.

Corollary: 3.3.4

Every fuzzy regular open set is a fuzzy open set but not conversely and every fuzzy open set is a fuzzy rw-open set but not conversely and hence every regular open set is a fuzzy rw-open set but not conversely.

Theorem: 3.3.5

A fuzzy set α of a fuzzy topological space X is fuzzy rw-open if and only if δ≤ int(α) whenever δ≤ α and δ is a fuzzy regular semi open set in fts X.

Proof:

Suppose that δ≤ int(α) whenever δ≤ α and δ is a fuzzy regular semi open set in fts X. To prove that α is fuzzy rw-open in fts X. Let α^c≤β and β is any fuzzy regular semi open set in fts X. Then β^c≤ α. By theorem 3.1.14, β^c is also a fuzzy regular semiopen set in fts X. By hypothesis, β^c≤ int(α) which implies $(\text{int}(\alpha))^c \leq \beta$. That is $cl(\alpha^c) \leq \beta$, since $cl(\alpha^c) = (\text{int}(\alpha))^c$. Thus α^c is a fuzzy rw-closed and hence α is fuzzy rw-open in fts X.

Conversely, suppose that α is fuzzy rw-open. Let $\beta \leq \alpha$ and β is any fuzzy regular semi open in fts X. Then $\alpha^c \leq \beta^c$ and β^c is also fuzzy regular semi open. Since α^c is fuzzy rw-closed, we have $cl(\alpha^c) \leq \beta^c$ and so $\beta \leq \text{int}(\alpha)$, since $cl(\alpha^c) = (\text{int}(\alpha))^c$.

Theorem: 3.3.6

If α and β are fuzzy rw-open sets in fts X, then α ∩ β is fuzzy rw-open set in fts X.

Proof:

Let α and β are fuzzy rw-open sets in fts X. Then α^c and β^c are fuzzy rw-closed sets in fts X. By theorem 3.2.13, $\alpha^c \cup \beta^c$ is also a fuzzy rw-closed set in fts X. That is $(\alpha^c \cup \beta^c) = (\alpha \cap \beta)^c$ is a fuzzy rw-closed set in X. Therefore $\alpha \cap \beta$ is a fuzzy rw-open set in fts X.

Remark: 3.3.7

The union of two fuzzy rw-open sets in a fts X is generally not a fuzzy rw-open set in fts X.

Example: 3.3.8

Consider the fuzzy topological space (X,T) defined in Example . In this fts X, the fuzzy sets δ_1, δ_2: X→ [0, 1] are defined by

δ_1 (x) = {1 if x=a
0 otherwise

δ_2 (x) = {1 if x=c
0 otherwise

Then δ_1 and δ_2 are the fuzzy rw-open sets in fts in X.

Let λ= $\delta_1 \cup \delta_2$. Then λ (x) = {1 if x=a,c
0 otherwise

Then λ= $\delta_1 \cup \delta_2$ is not a fuzzy rw-open set in fts X.

Theorem: 3.3.9

If int(α) ≤ β≤ α and α is a fuzzy rw-open set of a fts X, then β is also a fuzzy rw-open set in fts X.

Proof:

Suppose int(α) ≤ β ≤ α and α be a fuzzy rw-open set of fts X. To prove β is a fuzzy rw-open set in fts X. Let σ be any fuzzy regular semi open set in a fts x such that σ ≤ β. Now σ≤β≤α. That is σ≤α. Since α is fuzzy rw-open set of fts X, σ≤int(α) by theorem. By hypothesis int(α) ≤β. Then int(int(α) ≤int(β)). That is int(α) ≤int(β). Then σ≤int(β). Again by theorem, β is a fuzzy rw-open set in fts X.

Theorem: 3.3.10

If α is a fuzzy rw-closed set of fts X and cl(α) ∩ (1-cl(α))=0,then cl(α)-α is a fuzzy rw-open set in fts X.

Proof:

Let α be a fuzzy rw-closed set of fts X and cl(α) ∩ (1-cl(α))=0. Let β be a fuzzy regular semi open set of fts x such that β≤ (cl(α)- α). Then by theorem 3.2.25, cl(α)- α does not contain any non-zero fuzzy regular semiopen set and so β=0. Therefore β≤ int(cl(α)- α). By theorem 3.3.5 cl(α)- α is fuzzy rw-open.

Theorem: 3.3.11

Let α and β be two fuzzy subsets of a fts X. If β is a fuzzy rw-open set and α≥int(β), then α ∩ β is a fuzzy rw-open set in fts X.

Proof:

Let β is a fuzzy rw-open set and α≥int(β). That is int(β) ≤ α ∩ β ≤ β and β is a fuzzy rw-open set. By theorem 3.3.9, α ∩ β is also a fuzzy rw-open set in fts X.