

# Chapter 5

## Passivity Analysis of Markovian Jumping Neural Networks with Leakage Time Varying Delay

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### 5.1 Introduction

In the past few decades, neural networks (NNs) have been a hot research topic because of their emerged application in static image processing, pattern recognition, fixed-point computation, associative memory, combinatorial optimization, refer Balasubramaniam et al.(2011), Liu (1997), Meng and Wang (2007), Qiao et al. (2003) and Yu and Li (2007). Because the interactions between neurons are generally asynchronous in biological and artificial neural networks, time delays are usually encountered. Since the existence of time delays is frequently one of the main sources of instability for neural networks, the stability analysis for delayed neural networks had been extensively studied and many papers have been published on various types of neural networks with time delays based on the LMI approach, refer Kwon et al. (2010), Mahmoud and Xia (2011), Ma et al.(2011), Park and Kwon (2009), Wu et al. (2012 b),Zhang et al (2010 c), Zhu et al.(2010) and Zhu and Cao (2011).On the other hand, the main idea of passivity theory is that the passive properties of a system can keep the system internally stable. In addition, passivity theory is frequently used in control systems to prove the stability of systems. The problem of passivity performance analysis has also been extensively applied in many areas such as signal processing, fuzzy control,sliding mode control, see Wu and Zheng (2009), and networked controlas in Gao et al. (2007). The passivity idea is a promising approach to analysis the stability of neural networks, because it can

lead to more general stability results. It is important to investigate the passivity analysis for neural networks with time delays. More recently, dissipativity or passivity performances of neural networks has received increasing attention and many research results have been reported in the literature, Chen et al.(2009 b), Kwon et al.(2011), Lu et al.(2008), Xu et al.(2009) and Zhang et al.(2010 c).

In practice, the RNNs often exhibit the behavior of finite state representations (also called clusters, patterns, or modes) which are referred to as the information latching problems as in Bengio et al. (1993). In this case, the network states may switch (or jump) between different RNN modes according to a Markovian chain, and this gives rise to the so-called Markovian jumping recurrent neural networks. It has been shown that the information latching phenomenon is recognized to exist universally in neural networks as discussed in Lia et al.(2009) and Tino (2004), which can be dealt with extracting finite state representation from a trained network, i.e., a neural network sometimes has finite modes that switch from one to another at different times. The results related to all kinds of Markovian jump neural networks with time delay can also be found in Wang et al. (2009), Zhu and Cao (2010 b) and Zhang and Wang (2008) and the references therein. It should be pointed out that all the above mentioned references assume that the considered transition probabilities in the Markov process or Markov chain are time-invariant, i.e., the considered Markov process or Markov chain is assumed to be homogeneous. It is noted that such kind of assumption is required in most existing results on Markovian jump systems, see Dong et al. (2011) and Shu (2010). The detailed discussion about piecewise homogeneous and nonhomogeneous Markovian jumping parameters have been given in Wu et al (2012) and references therein.

On the other hand, a typical time delay called as Leakage (or “forgetting”) delay may exist in the negative feedback terms of the neural network and it has a great impact on the dynamic behaviors of delayed neural networks and more details are given in Gopalsamy (2007), Li and Huang (2009), Li and Fu (2013), Li et al.(2010 b), Peng (2010) and Song and Cao (2012). In Li and Fu (2013), the author introduced leakage time-varying delay for dynamical systems with nonlinear perturbations and derived leakage-delay-dependent stability conditions via constructing a new type of Lyapunov-Krasovskii functional and LMI approach. Recently, the passivity analysis for neural networks of neutral type with Markovian jumping parameters and time delay in the leakage term have been addressed in Balasubramaniam et al. (2011). With reference to the results above, it has been studied that, many results get to be found out for passivity analysis of Markovian jumping neural networks with leakage

time-varying delays. Thus, the main purpose of this chapter is to shorten such a gap by making the first attempt to deal with the passivity analysis problem for a type of continuous-time neural networks with time-varying transition probabilities and mixed time delays.

In this chapter, the problem of passivity analysis of Markovian jump neural networks with leakage time-varying delay, discrete and distributed time-varying delays is considered. The Markov process in the under lying neural networks is assumed to be finite piecewise homogeneous, which is a special nonhomogeneous (time-varying) Markov chain. Motivated by Wu et al. (2012) a novel Lyapunov–Krasovskii functional is constructed in which the positive definite matrices are dependent on the system mode and a triple-integral term is introduced for deriving the delay-dependent stability conditions. By employing a novel Lyapunov–Krasovskii functional having triple integral terms, new passivity leakage delay-dependent criteria are established to guarantee the passivity performance of the given systems. This performance not only depends on the upper bound of the time-varying leakage delay  $\sigma(t)$  but also depends on the upper bound of the derivative of the time-varying leakage delay  $\sigma_\mu$ . When estimating an upper bound of the derivative of the Lyapunov–Krasovskii functional, we handle the terms related to the discrete and distributed delays appropriately so as to develop less conservative results. Two numerical examples are given to show the validity and potential of the development of the proposed passivity criteria.

**Notations:** Let  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space and the superscript “ T ” denotes the transpose of a matrix or vector.  $I$  denotes the identity matrix with compatible dimensions. For square matrices  $M_1$  and  $M_2$ , the notation  $M_1 > (\geq, <, \leq) M_2$  denotes  $M_1 - M_2$  is a positive-definite (positive-semi-definite, negative, negative-semi-definite) matrix. Let  $(\Omega, \mathfrak{F}, P)$  be a complete probability space with a natural filtration  $\{\mathfrak{F}_t\}_{t \geq 0}$  and  $\mathbf{E}[\cdot]$  stand for the correspondent expectation operator with respect to the given probability measure  $P$ . Also, let  $\tau > 0$  and  $C([-\tau, 0]; \mathbb{R}^n)$  denote the family of continuously differentiable function  $\phi$  from  $[-\tau, 0]$  to  $\mathbb{R}^n$  with the uniform norm  $\|\phi\|_\tau = \max\{\max_{-\tau \leq \theta \leq 0} |\phi(\theta)|, \max_{-\tau \leq \theta \leq 0} |\phi'(\theta)|\}$ .

## 5.2 Problem Description and Preliminaries

Fix a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ ,  $\Omega$  is the sample space,  $\mathcal{F}$  is the  $\sigma$ -algebra of subsets of the sample space and  $\mathcal{P}$  is the probability measure on  $\mathcal{F}$  and consider the

following markov jump neural networks with mixed time-delays:

$$\left\{ \begin{array}{l} \dot{x}(t) = -C(r(t))x(t - \sigma(t)) + A(r(t))g(x(t)) + B(r(t))g(x(t - \tau(t))) \\ \quad + D(r(t)) \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)) \end{array} \right. \quad (5.2.1)$$

where  $x(t - \sigma(t)) = [x_1(t - \sigma(t)) \ x_2(t - \sigma(t)) \ \cdots \ x_n(t - \sigma(t))]^T$  and  $g(x(t)) = [g_1(x_1(t)) \ g_2(x_2(t)) \ \cdots \ g_n(x_n(t))]^T$ ,  $x_i(t - \sigma(t))$  is the state of the  $i$ th neuron at time  $t$  with leakage time varying delay and  $g_i(x_i(t))$  denotes the neuron activation function;

$C(r(t)) = \text{diag}\{C_1(r_1(t)) \ C_2(r_2(t)) \ \cdots \ C_n(r_n(t))\}$ , is a diagonal matrix with positive entries;  $A(r(t)) = (a_{ij}(r(t)))_{n \times n}$ ,  $B(r(t)) = (b_{ij}(r(t)))_{n \times n}$ , and  $D(r(t)) = (d_{ij}(r(t)))_{n \times n}$ , are respectively the connection weight matrix, the discretely delayed connection weight matrix, and the distributively delayed connection weight matrix,  $y(t)$  is the output of the neural network, and  $u(t) \in \mathbb{L}_2[0, \infty)$  is the output,  $\tau(t)$  and  $d(t)$  denote the discrete delay and distributed delay respectively and the time varying delay  $\tau(t)$  satisfy

$$\begin{aligned} 0 \leq \tau(t) \leq \tau, 0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \dot{\tau}(t) \leq \tau_\mu, 0 \leq \sigma(t) \leq \sigma, \dot{\sigma}(t) \leq \sigma_\mu, \\ 0 \leq d(t) \leq d, \end{aligned} \quad (5.2.2)$$

where  $\tau_1, \tau_2, \tau_\mu, \sigma_\mu, \sigma$  and  $d$  are some real constants. By the simple transformation, model (5.2.1) has an equivalent form as follows:

$$\left\{ \begin{array}{l} \frac{d}{dt} \left[ x(t) - C(r(t)) \int_{t-\sigma(t)}^t x(s)ds \right] = -C(r(t))x(t) - C(r(t))x(t - \sigma(t))\dot{\sigma}(t) \\ \quad + A(r(t))g(x(t)) + B(r(t))g(x(t - \tau(t))) \\ \quad + D(r(t)) \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)). \end{array} \right. \quad (5.2.3)$$

Here,  $\{r_t, t \geq 0\}$  is a right continuous Markov chain on the probability space taking values in a finite state space  $\mathcal{S} = \{1, 2, \dots, N\}$  with transition rate matrix  $\Pi^{(\eta_{t+h})} \triangleq \{\pi_{ij}^{(\eta_{t+h})}\}$  given by

$$p_r\{r_{t+h} = j/r_t = i\} = \begin{cases} \pi_{ij}^{(\eta_{t+h})}h + o(h), & j \neq i \\ 1 + \pi_{ii}^{(\eta_{t+h})}h + o(h), & j = i \end{cases} \quad (5.2.4)$$

in which  $h \geq 0$ ,  $\lim_{h \rightarrow 0} o(h)/h = 0$  and  $\pi_{ij}^{(\eta_{t+h})} \geq 0$  for  $j \neq i$ , is the transition rate from

mode  $i$  at time  $t$  to mode  $j$  at time  $t + h$  and  $\pi_{ii}^{(\eta_{t+h})} = - \sum_{j=1, j \neq i}^N \pi_{ij}^{(\eta_{t+h})}$ .

Similarly, the parameter  $\{\eta_t, t \geq 0\}$  is also a right continuous markov chain on the probability space taking values in a finite state space  $\mathcal{M} = \{1, 2, \dots, T\}$  with transition rate matrix  $\Lambda \triangleq \{p_{mn}\}$  given by

$$p_r\{\eta_{t+h} = n/\eta_t = m\} = \begin{cases} p_{mn}h + o(h), & n \neq m \\ 1 + p_{mm}h + o(h), & n = m \end{cases} \quad (5.2.5)$$

in which  $h \geq 0$ ,  $\lim_{h \rightarrow 0} o(h)/h = 0$  and  $p_{mn} \geq 0$  for  $n \neq m$ , is the transition rate from

mode  $m$  at time  $t$  to mode  $n$  at time  $t + h$  and  $p_{mm} = - \sum_{n=m, n \neq m}^T p_{mn}$ .

In this chapter, we make the following assumption, definition and lemmas for deriving the main result.

**Assumption 1:** Each activation function  $f_i(\cdot)$  in (5.2.1) is continuous and bounded, and satisfies

$$F_i^- \leq \frac{g_i(\alpha_1) - g_i(\alpha_2)}{\alpha_1 - \alpha_2} \leq F_i^+, \quad i = 1, 2, \dots, n \quad (5.2.6)$$

where  $g_i(0) = 0$ ,  $\alpha_1, \alpha_2 \in \mathcal{R}$ ,  $\alpha_1 \neq \alpha_2$  and  $F_i^-$  and  $F_i^+$  are known real scalars. It follows from Eqn (6) that the neural activation function satisfies

$$F_i^- \leq \frac{g_i(\alpha)}{\alpha} \leq F_i^+, \quad i = 1, 2, \dots, n. \quad (5.2.7)$$

**Lemma 5.2.1.** (*Jensen Inequality*) For any matrix  $M \geq 0$ , any scalars  $a$  and  $b$  with  $a \leq b$ , and a vector function  $x(t) : [a, b] \rightarrow \mathcal{R}^n$  such that the integrals concerned are well-defined, then the following inequality holds:

$$(b - a) \left[ \int_a^b x(s)^T M x(s) ds \right] \geq \left[ \int_a^b x(s) ds \right]^T M \left[ \int_a^b x(s) ds \right]. \quad (5.2.8)$$

**Lemma 5.2.2.** For any constant matrix  $Z = Z^T > 0$  and scalars  $\sigma > 0, \tau_1 > 0, \tau_2 > 0$  such that the following integrations are well-defined:

$$\begin{aligned} - \int_{-\tau_2}^0 \int_{t+\theta}^t x(s)^T Z x(s) ds d\theta &\leq -\frac{2}{\tau_2^2} \left( \int_{-\tau_2}^0 \int_{t+\theta}^t x(s) ds d\theta \right)^T Z \left( \int_{-\tau_2}^0 \int_{t+\theta}^t x(s) ds d\theta \right) \\ - \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x(s)^T Z x(s) ds d\theta &\leq -\frac{2}{(\tau_2^2 - \tau_1^2)} \left( \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x(s) ds d\theta \right)^T Z \left( \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x(s) ds d\theta \right) \\ - \int_{-\sigma}^0 \int_{t+\theta}^t x(s)^T Z x(s) ds d\theta &\leq -\frac{2}{\sigma^2} \left( \int_{-\sigma}^0 \int_{t+\theta}^t x(s) ds d\theta \right)^T Z \left( \int_{-\sigma}^0 \int_{t+\theta}^t x(s) ds d\theta \right). \end{aligned}$$

The main purpose of this chapter is to establish a delay-dependent sufficient condition to ensure that neural networks (5.2.1) is passive.

**Definition 5.2.1.** The system (5.2.1) is said to be passive, if there exists a scalar  $\nu \geq 0$  such that for all  $t_p \geq 0$  and for all the solutions of (5.2.1), the following inequality

$$2 \int_0^{t_p} E\{y(s)^T u(s)\} ds \geq -\gamma \int_0^{t_p} E\{u(s)^T u(s)\} ds \quad (5.2.9)$$

holds under zero initial conditions.

## 5.3 Main Results

In this section, the author derive a new delay-dependent criterion for passivity of the delayed Markovian jumping neural networks (5.2.1) using the Lyapunov-Krasovskii functional method combining with LMI approach. For presentation convenience, in the following, we denote

$$F_1 = \text{diag}\{F_1^- F_1^+, F_2^- F_2^+, \dots, F_n^- F_n^+\} \quad F_2 = \text{diag}\left\{\frac{F_1^- + F_1^+}{2}, \frac{F_2^- + F_2^+}{2}, \dots, \frac{F_n^- + F_n^+}{2}\right\}.$$

Now, we establish the following passivity condition for the system (5.2.1).

**Theorem 5.3.1.** *The given Markovian jumping Neural Networks (5.2.1) is passive if*

$$\text{there exists } P_{i,m} > 0, Q_{1i,m} = \begin{bmatrix} Q_{1i,m}^1 & Q_{1i,m}^2 \\ Q_{1i,m}^{2T} & Q_{1i,m}^3 \end{bmatrix} > 0, Q_{2i,m} = \begin{bmatrix} Q_{2i,m}^1 & Q_{2i,m}^2 \\ Q_{2i,m}^{2T} & Q_{2i,m}^3 \end{bmatrix} > 0,$$

$$Q_{3i,m} = \begin{bmatrix} Q_{3i,m}^1 & Q_{3i,m}^2 \\ Q_{3i,m}^{2T} & Q_{3i,m}^3 \end{bmatrix} > 0, Q_4 > 0, U = \begin{bmatrix} U^1 & U^2 \\ U^{2T} & U^3 \end{bmatrix} > 0, \text{ positive symmetric}$$

matrices  $S_1 = S_1^T > 0, S_2 = S_2^T > 0, S_3 = S_3^T > 0, T_1 = T_1^T > 0, T_2 = T_2^T > 0, T_3 = T_3^T > 0$ , the positive definite matrices  $W_1 > 0, W_2 > 0$ , and the diagonal matrices  $\Lambda_{i,m}^1 > 0$ ,

$\Lambda_{i,m}^2 > 0, \Lambda_{i,m}^3 > 0, \Lambda_{i,m}^4 > 0, \Lambda_{i,m}^5 > 0$  and a scalar  $\gamma > 0$  such that for any  $(i, m) \in (\mathcal{S}, \mathcal{M})$  the following LMI holds:

$$\Xi = (\Xi_{i,j})_{17 \times 17} < 0 \quad (5.3.1)$$

$$\sum_{n \in \mathcal{M}} p_{mn} Q_{1i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{1j,m} + \sum_{n \in \mathcal{M}} p_{mn} Q_{3i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3j,m} < U \quad (5.3.2)$$

$$\sum_{n \in \mathcal{M}} p_{mn} Q_{2i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{2j,m} < U \quad (5.3.3)$$

$$\sum_{n \in \mathcal{M}} p_{mn} Q_{3i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3j,m} < U \quad (5.3.4)$$

where

$$\begin{aligned} \Xi_{1,1} &= -P_{i,m} C_i - C_i^T P_{i,m} + Q_{1i,m}^1 + Q_{2i,m}^1 + \tau_1 U^1 + Q_3^1 + \tau_2 U_1 + Q_4 - (\tau_2 - \tau_1) S_1 \\ &- S_3 - 4\tau_2^2 T_1 - 4(\tau_2 - \tau_1)^2 T_2 - 4\sigma^2 T_3 - F_1 \Lambda_{i,m}^1 + \left[ \sum_{n \in \mathcal{M}} p_{mn} P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] \\ &+ \sigma^2 W_2, \quad \Xi_{1,2} = P_{i,m} A_i + Q_{1i,m}^2 + Q_{2i,m}^2 + \tau_1 U^2 + Q_{3i,m}^2 + \tau_2 U^2 + F_2 \Lambda_{i,m}^1 \end{aligned}$$

$$\begin{aligned}
\Xi_{1,3} &= 0, & \Xi_{1,4} &= p_{i,m}B_i, & \Xi_{1,5} &= 0, & \Xi_{1,6} &= 0, & \Xi_{1,7} &= (\tau_2 - \tau_1)S_1, & \Xi_{1,8} &= 0 \\
\Xi_{1,9} &= -P_{i,m}C_i\sigma_\mu, & \Xi_{1,10} &= 0, & \Xi_{1,11} &= C_i^T P_{i,m}C_i - \left[ \sum_{n \in \mathcal{M}} p_{mn}P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] C_i \\
\Xi_{1,12} &= S_3, & \Xi_{1,13} &= 4\sigma T_3, & \Xi_{1,14} &= P_{i,m}D_i, & \Xi_{1,15} &= 4\tau_2 T_1, & \Xi_{1,16} &= 4(\tau_2 - \tau_1)T_2 \\
\Xi_{1,17} &= P_{i,m}, & \Xi_{2,2} &= Q_{1,i,m}^3 + Q_{2,i,m}^3 + \tau_1 U^3 + Q_{3,i,m}^3 + \tau_2 U^3 + A_i^T R A_i + d^2 W_1 + A_i^T G A_i \\
&\quad - A_{i,m}^1, & \Xi_{2,3} &= 0, & \Xi_{2,4} &= A_i^T R B_i + A_i^T G B_i, & \Xi_{2,5} &= \Xi_{2,6} = \Xi_{2,7} = \Xi_{2,8} = 0. \\
\Xi_{2,9} &= -A_i^T R C_i - A_i^T G C_i, & \Xi_{2,10} &= 0, & \Xi_{2,11} &= -A_i^T P_{i,m}C_i, & \Xi_{2,12} &= 0, & \Xi_{2,13} &= 0 \\
\Xi_{2,14} &= A_i^T R D_i + A_i^T G D_i, & \Xi_{2,15} &= 0, & \Xi_{2,16} &= 0, & \Xi_{2,17} &= A_i^T R + A_i^T G - I \\
\Xi_{3,3} &= -(1 - \tau_\mu)Q_{1,i,m}^1 - F_1 \Lambda_{i,m}^2, & \Xi_{3,4} &= -(1 - \tau_\mu)Q_{1,i,m}^2 + F_2 \Lambda_{i,m}^2 \\
\Xi_{3,5} &= \Xi_{3,6} = \Xi_{3,7} = \Xi_{3,8} = \Xi_{3,9} = \Xi_{3,10} = \Xi_{3,11} = \Xi_{3,12} = \Xi_{3,13} = \Xi_{3,14} = \Xi_{3,15} = 0 \\
\Xi_{3,16} &= \Xi_{3,17} = 0, & \Xi_{4,4} &= -(1 - \tau_\mu)Q_{1,i,m}^3 + B_i^T R B_i + B_i^T G B_i - \Lambda_{i,m}^2, & \Xi_{4,5} &= 0 \\
\Xi_{4,6} &= \Xi_{4,7} = \Xi_{4,8} = 0, & \Xi_{4,9} &= -B_i^T R C_i - B_i^T G C_i, & \Xi_{4,10} &= 0 \\
\Xi_{4,11} &= -B_i^T P_{i,m}C_i, & \Xi_{4,12} &= \Xi_{4,13} = 0, & \Xi_{4,14} &= B_i^T R D_i + B_i^T G D_i, & \Xi_{4,15} &= \Xi_{4,16} = 0 \\
\Xi_{4,17} &= B_i^T R + B_i^T G, & \Xi_{5,5} &= -Q_{2,i,m}^1 - S_2 - F_1 \Lambda_{i,m}^3, & \Xi_{5,6} &= -Q_{2,i,m}^2 + F_2 \Lambda_{i,m}^3, & \Xi_{5,7} &= S_2 \\
\Xi_{6,6} &= -Q_{2,i,m}^3 - \Lambda_{i,m}^3, & \Xi_{7,7} &= -Q_{3,i,m}^1 - (\tau_2 - \tau_1)S_1 - S_2 - F_1 \Lambda_{i,m}^4 \\
\Xi_{7,8} &= -Q_{3,i,m}^2 + F_2 \Lambda_{i,m}^4, & \Xi_{8,8} &= -Q_{3,i,m}^3 - \Lambda_{i,m}^4, & \Xi_{9,9} &= -(1 - \sigma_\mu)Q_4 + C_i^T R C_i \\
&\quad + C_i^T G C_i - F_1 \Lambda_{i,m}^5, & \Xi_{9,10} &= F_2 \Lambda_{i,m}^5, & \Xi_{9,11} &= C_i^T \sigma_\mu P_{i,m}C_i \\
\Xi_{9,14} &= -C_i^T R D_i - C_i^T G D_i, & \Xi_{9,17} &= -C_i^T R - C_i^T G, & \Xi_{10,10} &= -\Lambda_{i,m}^5 \\
\Xi_{11,11} &= C_i^T \left[ \sum_{n \in \mathcal{M}} p_{mn}P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] C_i - W_2, & \Xi_{11,14} &= -C_i^T P_{i,m}D_i \\
\Xi_{11,17} &= -C_i^T P_{i,m}, & \Xi_{12,12} &= -S_3, & \Xi_{13,13} &= -4T_3, & \Xi_{14,14} &= D_i^T R D_i + D_i^T G D_i - W_1 \\
\Xi_{14,17} &= D_i^T R + D_i^T G, & \Xi_{15,15} &= -4T_1, & \Xi_{16,16} &= -4T_2, & \Xi_{17,17} &= R + G - \gamma I \\
R &= \tau_2^2(\tau_2 - \tau_1)S_1 + (\tau_2 - \tau_1)^2 S_2 + \sigma^2 S_3 \\
G &= \tau_2^4 T_1 + (\tau_2^2 - \tau_1^2)^2 T_2 + \sigma^4 T_3
\end{aligned}$$

and the remaining coefficients are all zero.

*Proof.* Denote  $\zeta = [x(t)^T \quad g(x(t))^T]^T$  and consider the following Lyapunov-Krasovskii functional for neural network (5.2.1):

$$\begin{aligned} V(x_t, r_t, \eta_t) &= V_1(x_t, r_t, \eta_t) + V_2(x_t, r_t, \eta_t) + V_3(x_t, r_t, \eta_t) + V_4(x_t, r_t, \eta_t) \\ &\quad + V_5(x_t, r_t, \eta_t) \end{aligned} \quad (5.3.5)$$

where

$$\begin{aligned} V_1(x_t, r_t, \eta_t) &= \left[ x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \right]^T P_{r(t), \eta(t)} \left[ x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \right] \\ V_2(x_t, r_t, \eta_t) &= \int_{t-\tau(t)}^t \zeta(s)^T Q_{1r(t), \eta(t)} \zeta(s) ds + \int_{t-\tau_1}^t \zeta(s)^T Q_{2r(t), \eta(t)} \zeta(s) ds \\ &\quad + \int_{t-\tau_2}^t \zeta(s)^T Q_{3r(t), \eta(t)} \zeta(s) ds + \int_{t-\sigma(t)}^t x(s)^T Q_4 x(s) ds \\ &\quad + \int_{-\tau_1}^0 \int_{t+\theta}^t \zeta(s)^T U \zeta(s) ds d\theta + \int_{-\tau_2}^0 \int_{t+\theta}^t \zeta(s)^T U \zeta(s) ds d\theta \\ V_3(x_t, r_t, \eta_t) &= \tau_2(\tau_2 - \tau_1) + \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s)^T S_1 \dot{x}(s) ds d\theta \\ &\quad + (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}(s)^T S_2 \dot{x}(s) ds d\theta + \sigma \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}(s)^T S_3 \dot{x}(s) ds d\theta \\ V_4(x_t, r_t, \eta_t) &= \int_{-d}^0 \int_{t+\theta}^t g(x(s))^T W_1 g(x(s)) ds d\theta + \sigma \int_{t-\sigma}^t \int_{\theta}^t x(s)^T W_2 x(s) ds d\theta \\ V_5(x_t, r_t, \eta_t) &= 2\tau_2^2 \int_{-\tau_2}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_1 \dot{x}(s) ds d\lambda d\theta + 2(\tau_2^2 - \tau_1^2) \int_{-\tau_1}^{-\tau_2} \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_2 \dot{x}(s) ds d\lambda d\theta \\ &\quad + 2\sigma^2 \int_{-\sigma}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_3 \dot{x}(s) ds d\lambda d\theta. \end{aligned}$$

Define infinitesimal generator (denoted by  $\mathbb{L}$ ) of the markov process acting on  $V(x_t, r_t, \eta_t)$

defined as follows:

$$\mathbb{L}V(x_t) = \lim_{h \rightarrow 0} \frac{1}{h} \left\{ E \left\{ \frac{V(x_{t+h}, r_{t+h}, \eta_{t+h})}{x_t}, r_t = i, \eta_t = m \right\} - V(x_t, r_t = i, \eta_t = m) \right\}.$$

It can be calculated that

$$\begin{aligned} \mathbb{L}V(x_t, r_t, \eta_t) &= \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \sum_{n \in \mathcal{M}, n \neq m} p_{mn} h \left[ \sum_{j \in \mathcal{S}, j \neq i} \pi_{ij}^n h V(x_{t+h}, j, n) + (1 + \pi_{ii}^n h) V(x_{t+h}, i, n) \right] \right. \\ &\quad \left. + (1 + p_{mm} h) \left[ \sum_{j \in \mathcal{S}, j \neq i} \pi_{ij}^m h V(x_{t+h}, j, m) + (1 + \pi_{ii}^m h) V(x_{t+h}, i, m) \right] - V(x_t, i, m) \right\} \\ &= \lim_{h \rightarrow 0} \left\{ \sum_{n \in \mathcal{M}, n \neq m} p_{mn} V(x_{t+h}, i, n) + p_{mm} V(x_{t+h}, i, m) \right. \\ &\quad \left. + \frac{1}{h} \left[ \sum_{j \in \mathcal{S}, j \neq i} \pi_{ij}^m h V(x_{t+h}, j, m) + (1 + \pi_{ii}^m h) V(x_{t+h}, i, m) \right] - \frac{1}{h} V(x_t, i, m) \right\} \\ &= \lim_{h \rightarrow 0} \left\{ \sum_{n \in \mathcal{M}} p_{mn} V(x_{t+h}, i, n) + \sum_{j \in \mathcal{S}} \pi_{ij}^m V(x_{t+h}, j, m) + \frac{1}{h} [V(x_{t+h}, i, m) - V(x_t, i, m)] \right\} \\ &= \sum_{n \in \mathcal{M}} p_{mn} V(x_t, i, n) + \sum_{j \in \mathcal{S}} \pi_{ij}^m V(x_t, j, m) + \dot{V}(x_t, i, m). \end{aligned}$$

From eqn (5.3.5), it can be seen that

$$\begin{aligned} \mathbb{L}V(x_t, r_t, \eta_t) &= \mathbb{L}V_1(x_t, r_t, \eta_t) + \mathbb{L}V_2(x_t, r_t, \eta_t) + \mathbb{L}V_3(x_t, r_t, \eta_t) + \mathbb{L}V_4(x_t, r_t, \eta_t) \\ &\quad + \mathbb{L}V_5(x_t, r_t, \eta_t). \end{aligned} \tag{5.3.6}$$

Based on the above equation, along the solution of the neural network (5.2.3), we

obtain that for each  $(i, m) \in \mathcal{S} \times \mathcal{M}$

$$\begin{aligned}
\mathbb{L}V_1(x_t, r_t, \eta_t) &= \begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix}^T P_{i,m} \frac{d}{dt} \begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix} \\
&+ \frac{d}{dt} \begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix}^T P_{i,m} \begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix} \\
&+ \begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix}^T \left[ \sum_{n \in \mathcal{M}} p_{mn} P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] \\
&\begin{bmatrix} x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \\ \end{bmatrix}, \tag{5.3.7}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}V_2(x_t, r_t, \eta_t) &\leq \zeta^T(t) Q_{1,i,m} \zeta(t) - \zeta^T(t - \tau(t)) Q_{1,i,m} \zeta(t - \tau(t)) (1 - \tau_\mu) \\
&+ \int_{t-\tau(t)}^t \zeta^T(s) \left[ \sum_{n \in \mathcal{M}} p_{mn} Q_{1,i,m} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{1,j,n} \right] \zeta(s) ds + \zeta^T(t) Q_{2,i,m} \zeta(t) \\
&- \zeta^T(t - \tau_1) Q_{2,i,m} \zeta(t - \tau_1) + \int_{t-\tau_1}^t \zeta^T(s) \left[ \sum_{n \in \mathcal{M}} p_{mn} Q_{2,i,m} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{2,j,n} \right] \zeta(s) ds \\
&+ \zeta^T(t) Q_{3,i,m} \zeta(t) - \zeta^T(t - \tau_2) Q_{3,i,m} \zeta(t - \tau_2) \\
&+ \int_{t-\tau_2}^{t-\tau(t)} \zeta^T(s) \left[ \sum_{n \in \mathcal{M}} p_{mn} Q_{3,i,m} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3,j,n} \right] \zeta(s) ds \\
&+ \int_{t-\tau(t)}^t \zeta^T(s) \left[ \sum_{n \in \mathcal{M}} p_{mn} Q_{3,i,m} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3,j,n} \right] \zeta(s) ds + x^T(t) Q_4 x(t) \\
&- x^T(t - \sigma(t)) Q_4 x(t - \sigma(t)) (1 - \sigma_\mu) + \tau_1 \zeta^T(t) U \zeta(t) - \int_{t-\tau_1}^t \zeta^T(s) U \zeta(s) ds \\
&+ \tau_2 \zeta^T(t) U \zeta(t) - \int_{t-\tau_2}^{t-\tau(t)} \zeta^T(s) U \zeta(s) ds - \int_{t-\tau(t)}^t \zeta^T(s) U \zeta(s) ds, \tag{5.3.8}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}V_3(x_t, r_t, \eta_t) &= \tau_2^2(\tau_2 - \tau_1)\dot{x}(t)^T S_1 \dot{x}(t) - \tau_2(\tau_2 - \tau_1) \int_{-\tau_2}^0 \dot{x}(t + \theta)^T S_1 \dot{x}(t + \theta) d\theta \\
&\quad + (\tau_2 - \tau_1)^2 \dot{x}(t)^T S_2 \dot{x}(t) - (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \dot{x}(t + \theta)^T S_2 \dot{x}(t + \theta) d\theta \\
&\quad + \sigma^2 \dot{x}(t)^T S_3 \dot{x}(t) - \sigma \int_{-\sigma}^0 \dot{x}(t + \theta)^T S_3 \dot{x}(t + \theta) d\theta \\
&= \dot{x}(t)^T [\tau_2^2(\tau_2 - \tau_1)S_1 + (\tau_2 - \tau_1)^2 S_2 + \sigma^2 S_3] \dot{x}(t) \\
&\quad - \tau_2(\tau_2 - \tau_1) \int_{t-\tau_2}^t \dot{x}(s)^T S_1 \dot{x}(s) ds - (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{x}(s)^T S_2 \dot{x}(s) ds \\
&\quad - \sigma \int_{t-\sigma}^t \dot{x}(s)^T S_3 \dot{x}(s) ds, \tag{5.3.9}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}V_4(x_t, r_t, \eta_t) &= d^2 g(x(t))^T W_1 g(x(t)) - d(t) \int_{t-d(t)}^t g(x(s))^T W_1 g(x(s)) ds \\
&\quad + \sigma^2 x(t)^T W_2 x(t) - \sigma(t) \int_{t-\sigma(t)}^t x(s)^T W_1 x(s) ds, \tag{5.3.10}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}V_5(x_t, r_t, \eta_t) &= \dot{x}(t)^T [\tau_2^4 T_1 + (\tau_2^2 - \tau_1^2)^2 T_2 + \sigma^4 T_3] \dot{x}(t) - 2\tau_2^2 \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s)^T T_1 \dot{x}(s) ds d\theta \\
&\quad - (\tau_2^2 - \tau_1^2) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}(s)^T T_2 \dot{x}(s) ds d\theta - 2\sigma^2 \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}(s)^T T_3 \dot{x}(s) ds d\theta. \tag{5.3.11}
\end{aligned}$$

Moreover, based on Lemma 5.2.1 , we can get the following inequalities

$$-(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{x}(s)^T S_2 \dot{x}(s) ds \leq - \left[ \int_{t-\tau_2}^{t-\tau_1} \dot{x}(s) ds \right]^T S_2 \left[ \int_{t-\tau_2}^{t-\tau_1} \dot{x}(s) ds \right] \tag{5.3.12}$$

$$-d \int_{t-d}^t g(x(s))^T W_1 g(x(s)) ds \leq - \left[ \int_{t-d}^t g(x(s)) ds \right]^T W_1 \left[ \int_{t-d}^t g(x(s)) ds \right]. \tag{5.3.13}$$

By using Lemma 5.2.2, we can also get that

$$-2\tau_2^2 \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s)^T Z \dot{x}(s) ds d\theta \leq -4 \left( \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta \right)^T Z \left( \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta \right). \quad (5.3.14)$$

Similarly, we can use Lemma 5.2.1 and Lemma 5.2.2 for other integrals. On the other hand, we have from (5.2.6) that for any  $\lambda = 1, 2, \dots, n$ ,

$$(g_\lambda(x_\lambda(t)) - F_\lambda^- x_\lambda(t))(g_\lambda(x_\lambda(t)) - F_\lambda^+ x_\lambda(t)) \leq 0, \quad (5.3.15)$$

which is equivalent to

$$\zeta^T(t) \begin{bmatrix} F_\lambda^+ F_\lambda^- \hat{e}_\lambda \hat{e}_\lambda^T & -\frac{F_\lambda^+ + F_\lambda^-}{2} \hat{e}_\lambda \hat{e}_\lambda^T \\ -\frac{F_\lambda^+ + F_\lambda^-}{2} \hat{e}_\lambda \hat{e}_\lambda^T & \hat{e}_\lambda \hat{e}_\lambda^T \end{bmatrix} \zeta(t) \leq 0, \quad (5.3.16)$$

where  $\hat{e}_\lambda$  denotes the unit column vector having 1 element on its  $\lambda$ -th row and zeros elsewhere. Thus, for any appropriately dimensioned diagonal matrix  $\Lambda_{i,m}^1 > 0$ , the following inequality holds:

$$0 \leq \zeta^T(t) \begin{bmatrix} -F_1 \Lambda_{i,m}^1 & F_2 \Lambda_{i,m}^1 \\ * & -\Lambda_{i,m}^1 \end{bmatrix} \zeta(t). \quad (5.3.17)$$

Similarly, for any appropriately dimensioned diagonal matrices  $\Lambda_{i,m}^2 > 0, \Lambda_{i,m}^3 > 0, \Lambda_{i,m}^4 > 0$  and  $\Lambda_{i,m}^5 > 0$ , the following inequalities also hold:

$$0 \leq \zeta^T(t - \tau(t)) \begin{bmatrix} -F_1 \Lambda_{i,m}^2 & F_2 \Lambda_{i,m}^2 \\ * & -\Lambda_{i,m}^2 \end{bmatrix} \zeta(t - \tau(t)), \quad (5.3.18)$$

$$0 \leq \zeta^T(t - \tau_1) \begin{bmatrix} -F_1 \Lambda_{i,m}^3 & F_2 \Lambda_{i,m}^3 \\ * & -\Lambda_{i,m}^3 \end{bmatrix} \zeta(t - \tau_1), \quad (5.3.19)$$

$$0 \leq \zeta^T(t - \tau_2) \begin{bmatrix} -F_1 \Lambda_{i,m}^4 & F_2 \Lambda_{i,m}^4 \\ * & -\Lambda_{i,m}^4 \end{bmatrix} \zeta(t - \tau_2), \quad (5.3.20)$$

and

$$0 \leq \zeta^T(t - \sigma(t)) \begin{bmatrix} -F_1 \Lambda_{i,m}^5 & F_2 \Lambda_{i,m}^5 \\ * & -\Lambda_{i,m}^5 \end{bmatrix} \zeta(t - \sigma(t)). \quad (5.3.21)$$

Using inequalities (5.3.7)-(5.3.14) in (5.3.6) and adding (5.3.17)-(5.3.21) in (5.3.6), we get

$$\mathbb{L}V(x_t, r_t, \eta_t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \leq \rho^T(t)\Xi\rho(t) \quad (5.3.22)$$

where  $\rho(t) = [\rho_1^T(t) \quad \rho_2^T(t) \quad \rho_3^T(t)]$  with

$$\begin{aligned} \rho_1(t) &= [x(t) \quad g(x(t)) \quad x(t - \tau(t)) \quad g(x(t - \tau(t))) \quad x(t - \tau_1) \quad g(x(t - \tau_1))] \\ \rho_2(t) &= \left[ x(t - \tau_2) \quad g(x(t - \tau_2)) \quad x(t - \sigma(t)) \quad g(x(t - \sigma(t))) \quad \int_{t-\sigma(t)}^t x(s)ds \quad x(t - \sigma) \right] \\ \rho_3(t) &= \left[ \int_{t-\sigma}^t x(s)ds \quad \int_{t-d(t)}^t g(x(s))ds \quad \int_{t-\tau_2}^t x(s)ds \quad \int_{t-\tau_2}^{t-\tau_1} x(s)ds \quad u(t) \right]. \end{aligned}$$

Hence we can obtain from (5.2.9) that,

$$\mathbb{L}V(x_t, r_t, \eta_t) - 2y(t)^T u(t) - \gamma u(t)^T u(t) \leq 0. \quad (5.3.23)$$

Now, to show the passivity of the delayed neural networks in (5.2.1), we set

$$J(t_p) = \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t)] dt \right\} \quad (5.3.24)$$

where  $t_p \geq 0$ .

Using Dynkin's formula, we have

$$\mathbb{E} \left[ \int_0^{t_p} \mathbb{L}V(x_t, r_t, \eta_t) dt \right] = \mathbb{E} [V(x_{t_p}, r_{t_p}, \eta_{t_p})] - \mathbb{E} [V(x_0, r_0, \eta_0)].$$

Now, we can deduce that

$$\begin{aligned} J(t_p) &= \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t) + \mathbb{L}V(x_t, r_t, \eta_t)] dt \right\} - \mathbb{E} \left[ \int_0^{t_p} \mathbb{L}V(x_t, r_t, \eta_t) dt \right] \\ &= \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t) + \mathbb{L}V(x_t, r_t, \eta_t)] dt \right\} \\ &\quad - \mathbb{E} [V(x_{t_p}, r_{t_p}, \eta_{t_p})] + \mathbb{E} [V(x_0, r_0, \eta_0)]. \end{aligned} \quad (5.3.25)$$

Thus, if (5.3.25) holds, then since  $\mathbb{E}[V(x_{t_p}, r_{t_p}, \eta_{t_p})] \geq 0$  and  $V(x_0, r_0, \eta_0) = 0$  holds under zero initial condition, from (5.3.24) it follows that  $J(t_p) \leq 0$  for any  $t_p \geq 0$ , which implies that (5.3.1) is satisfied and therefore the delayed neural networks (5.2.1) is locally passive. Next we shall prove that  $\mathbb{E}[\|x(t)\|^2] \rightarrow 0$  as  $t \rightarrow \infty$ . Taking expectation on both sides of (5.3.22) and integrating from 0 to  $t$  we have,

$$\int_0^t \mathbb{E}[\mathbb{L}V(x_s, r_s, \eta_s)] ds - 2 \int_0^t \mathbb{E}[y^T(s)u(s)] ds - \gamma \int_0^t \mathbb{E}[u^T(s)u(s)] ds \leq \int_0^t \mathbb{E}[\rho^T(s)\Xi\rho(s)] ds.$$

By using Dynkin's formula, we have

$$\begin{aligned} \mathbb{E}[\mathbb{L}V(x_t, r_t, \eta_t)] - \mathbb{E}[\mathbb{L}V(x_0, r_0, \eta_0)] - 2 \int_0^t \mathbb{E}[y^T(s)u(s)] ds - \gamma \int_0^t \mathbb{E}[u^T(s)u(s)] ds \\ \leq \int_0^t \mathbb{E}[\rho^T(s)\Xi\rho(s)] ds. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[\mathbb{L}V(x_t, r_t, \eta_t)] - \int_0^t \mathbb{E}[\rho^T(s)\Xi\rho(s)] ds &\leq \mathbb{E}[\mathbb{L}V(x_0, r_0, \eta_0)] + 2 \int_0^t \mathbb{E}[y^T(s)u(s)] ds \\ &\quad + \gamma \int_0^t \mathbb{E}[u^T(s)u(s)] ds \\ &< \infty, \quad t \geq 0. \end{aligned} \quad (5.3.26)$$

Using Jenson's inequality and (5.3.26), we have

$$\begin{aligned}
\mathbf{E} \left\| C_i \int_{t-\sigma(t)}^t x(s) ds \right\|^2 &= \mathbf{E} \left[ C_i \int_{t-\sigma(t)}^t x(s) ds \right]^T \left[ C_i \int_{t-\sigma(t)}^t x(s) ds \right] \\
&\leq \lambda_{\max}(C_i^2) \mathbf{E} \left[ \int_{t-\sigma(t)}^t x(s) ds \right]^T \left[ \int_{t-\sigma(t)}^t x(s) ds \right] \\
&\leq \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \left[ \int_{t-\sigma(t)}^t \mathbf{E} x(s) ds \right]^T Q_4 \left[ \int_{t-\sigma(t)}^t \mathbf{E} x(s) ds \right] \\
&\leq \sigma(t) \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \left\{ \int_{t-\sigma(t)}^t \mathbf{E} x^T(s) Q_4 x(s) ds \right\} \\
&\leq \sigma \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \left\{ \int_{t-\sigma}^t \mathbf{E} x^T(s) Q_4 x(s) ds \right\} \\
&\leq \sigma \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \mathbf{E} V_1(x_t, r_t, \eta_t) \\
&\leq \sigma \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \mathbf{E} V(x_t, r_t, \eta_t) \\
&\leq \sigma \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \mathbf{E} V(x_0, r_0, \eta_0), \quad t \geq 0. \tag{5.3.27}
\end{aligned}$$

Similarly, it follows from the definition of  $V_1(x_t, r_t, \eta_t)$  that

$$\begin{aligned}
\mathbf{E} \left\| x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \right\|^2 &= \mathbf{E} \left[ C_i \int_{t-\sigma(t)}^t x(s) ds \right]^T \left[ C_i \int_{t-\sigma(t)}^t x(s) ds \right] \\
&\leq \frac{\mathbf{E} V_1(x_t, r_t, \eta_t)}{\lambda_{\min}(P_{i,\eta(t)})} \\
&\leq \frac{\mathbf{E} V(x_t, r_t, \eta_t)}{\lambda_{\min}(P_{i,\eta(t)})} \\
&\leq \frac{\mathbf{E} V(x_0, r_0, \eta_0)}{\lambda_{\min}(P_{i,\eta(t)})}, \quad t \geq 0.
\end{aligned}$$

Hence, it can be obtained that

$$\begin{aligned}
\mathbf{E} \|x(t)\|^2 &= \mathbf{E} \left\| x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds + C_i \int_{t-\sigma(t)}^t x(s) ds \right\|^2 \\
&\leq 2\mathbf{E} \left\| C_i \int_{t-\sigma(t)}^t x(s) ds \right\|^2 + 2\mathbf{E} \left\| x(t) - C_i \int_{t-\sigma(t)}^t x(s) ds \right\|^2 \\
&\leq 2\sigma \frac{\lambda_{\max}(C_i^2)}{\lambda_{\min}(Q_4)} \mathbf{E} V(x_0, r_0, \eta_0) + 2 \frac{\mathbf{E} V(x_0, r_0, \eta_0)}{\lambda_{\min}(P_{i,\eta(t)})} < \infty, \quad t \geq 0, \tag{5.3.28}
\end{aligned}$$

where

$\mathbf{E}V(x_0, r_0, \eta_0)$

$$\begin{aligned}
&= \mathbf{E} \left[ x(0) - C(r(0)) \int_{-\sigma(0)}^0 x(s) ds \right]^T P_{r(0), \eta(0)} \left[ x(0) - C(r(0)) \int_{-\sigma(0)}^0 x(s) ds \right] \\
&+ \int_{-\tau(0)}^0 \zeta(s)^T Q_{1_{r(0), \eta(0)}} \zeta(s) ds + \int_{-\tau_1}^0 \zeta(s)^T Q_{2_{r(0), \eta(0)}} \zeta(s) ds + \int_{-\tau_2}^0 \zeta(s)^T Q_{3_{r(0), \eta(0)}} \zeta(s) ds \\
&+ \int_{-\sigma(0)}^0 x(s)^T Q_4 x(s) ds + \int_{-\tau_1}^0 \int_{\theta}^0 \zeta(s)^T U \zeta(s) ds d\theta + \int_{-\tau_2}^0 \int_{\theta}^0 \zeta(s)^T U \zeta(s) ds d\theta \\
&+ \tau_2(\tau_2 - \tau_1) \int_{-\tau_2}^0 \int_{\theta}^0 \dot{x}(s)^T S_1 \dot{x}(s) ds d\theta + (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{\theta}^0 \dot{x}(s)^T S_2 \dot{x}(s) ds d\theta \\
&+ \sigma \int_{-\sigma}^0 \int_{\theta}^0 \dot{x}(s)^T S_3 \dot{x}(s) ds d\theta + \int_{-d}^0 \int_{\theta}^0 g(x(s))^T W_1 g(x(s)) ds d\theta + \sigma \int_{-\sigma}^0 \int_{\theta}^t x(s)^T W_2 x(s) ds d\theta \\
&+ 2\tau_2^2 \int_{-\tau_2}^0 \int_{\theta}^0 \int_{\lambda}^0 \dot{x}(s)^T T_1 \dot{x}(s) ds d\lambda d\theta + 2(\tau_2^2 - \tau_1^2) \int_{-\tau_1}^{-\tau_2} \int_{\theta}^0 \int_{\lambda}^0 \dot{x}(s)^T T_2 \dot{x}(s) ds d\lambda d\theta \\
&+ 2\sigma^2 \int_{-\sigma}^0 \int_{\theta}^0 \int_{\lambda}^0 \dot{x}(s)^T T_3 \dot{x}(s) ds d\lambda d\theta \\
&\leq \left\{ 2\lambda_{\max}_{i \in S}(P_{i, \eta(t)}) (1 + \sigma^2 \max_{i \in S} C_i) + \tau \lambda_{\max}(Q_{1_{r(t), \eta(t)}}) + \tau_1 \lambda_{\max}(Q_{2_{r(t), \eta(t)}}) + \tau_2 \lambda_{\max}(Q_{3_{r(t), \eta(t)}}) \right. \\
&+ \sigma \lambda_{\max}(Q_4) + \tau_1^2 \lambda_{\max}(U) \tau_1^2 \lambda_{\max}(U) + \tau_2^3 (\tau_2 - \tau_1) \lambda_{\max}(S_1) + (\tau_2 - \tau_1)^3 \lambda_{\max}(S_2) \\
&+ \sigma^3 \lambda_{\max}(S_3) + d^2 \lambda_{\max}(W_1) + \sigma^3 \lambda_{\max}(W_2) + 2\tau_2^4 \lambda_{\max}(T_1) + 2(\tau_2^2 - \tau_1^2) (\tau_1 - \tau_2)^2 \lambda_{\max}(T_2) \\
&\left. + 2\sigma^4 \lambda_{\max}(T_3) \right\} < \infty. \tag{5.3.29}
\end{aligned}$$

From (5.3.28) and (5.3.29), it can be deduced that the trivial solution of system (5.2.1) is locally passive. Then the solutions  $x(t) = x(t, 0, \phi)$  of system (5.2.1) is bounded on  $[0, \infty)$ . considering (5.2.1), it is known that  $\dot{x}(t)$  is bounded on  $[0, \infty)$ , which leads to the uniform continuity of the solution  $x(t)$  on  $[0, \infty)$ . From (5.3.26),

one can note that the following inequality holds:

$$\begin{aligned}
\lambda_{\min}(\Xi) \int_0^t \mathbf{E}[x^T(s)x(s)]ds &\leq \mathbb{E}[\mathbb{L}V(x_t, r_t, \eta_t)] - \int_0^t \mathbb{E}[\rho^T(s)\Xi\rho(s)]ds \\
&\leq \mathbb{E}[\mathbb{L}V(x_0, r_0, \eta_0)] + 2 \int_0^t \mathbb{E}[y^T(s)u(s)]ds + \gamma \int_0^t \mathbb{E}[u^T(s)u(s)]ds \\
&< \infty, \quad t \geq 0.
\end{aligned}$$

By Barbalats lemma, Gopalsamy (1992), it holds that  $\mathbb{E}[\|x(t)\|^2] \rightarrow 0$  as  $t \rightarrow \infty$  and this completes the proof of the global passivity of the system (5.2.1).

□

**Remark 5.3.1.** When  $\sigma(t) = \sigma$ , the system (5.2.1) becomes

$$\begin{aligned}
\dot{x}(t) &= -C(r(t))x(t - \sigma(t)) + A(r(t))g(x(t)) + B(r(t))g(x(t - \tau(t))) \\
&\quad + D(r(t)) \int_{t-d(t)}^t g(x(s))ds + u(t) \tag{5.3.30} \\
y(t) &= g(x(t)).
\end{aligned}$$

The system (5.3.30) can be written in its equivalent form as follows:

$$\begin{aligned}
\frac{d}{dt} \left[ x(t) - C(r(t)) \int_{t-\sigma}^t x(s)ds \right] &= -C(r(t))x(t) + A(r(t))g(x(t)) \\
&\quad + B(r(t))g(x(t - \tau(t))) + D(r(t)) \int_{t-d(t)}^t g(x(s))ds + u(t) \tag{5.3.31} \\
y(t) &= g(x(t))
\end{aligned}$$

The time varying delay  $\tau(t)$  satisfy,

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \tau_\mu, \quad 0 \leq d(t) \leq d$$

where  $\tau_1, \tau_2, \tau_\mu, d$  are some constants and the leakage delay  $\sigma \geq 0$  is a constant.

Now, the passivity condition for the neural networks (5.3.31) is given in the following corollary and the result follows from Theorem (5.3.1).

**Corollary 5.3.1.** *Neural Networks (5.3.31) is passive if there exists  $P_{i,m} > 0$ ,*

$$Q_{1,i,m} = \begin{bmatrix} Q_{1,i,m}^1 & Q_{1,i,m}^2 \\ Q_{1,i,m}^{2T} & Q_{1,i,m}^3 \end{bmatrix} > 0, \quad Q_{2,i,m} = \begin{bmatrix} Q_{2,i,m}^1 & Q_{2,i,m}^2 \\ Q_{2,i,m}^{2T} & Q_{2,i,m}^3 \end{bmatrix} > 0,$$

$$Q_{3,i,m} = \begin{bmatrix} Q_{3,i,m}^1 & Q_{3,i,m}^2 \\ Q_{3,i,m}^{2T} & Q_{3,i,m}^3 \end{bmatrix} > 0, \quad Q_4 > 0, \quad U = \begin{bmatrix} U^1 & U^2 \\ U^{2T} & U^3 \end{bmatrix} > 0, \text{ positive symmet-}$$

*ric matrices  $S_1 = S_1^T > 0$ ,  $S_2 = S_2^T > 0$ ,  $S_3 = S_3^T > 0$ ,  $T_1 = T_1^T > 0$ ,  $T_2 = T_2^T > 0$ ,  $T_3 = T_3^T > 0$  and the positive definite matrices  $W_1 > 0$ ,  $W_2 > 0$  the diagonal matrices  $\Lambda_{i,m}^1 > 0$ ,  $\Lambda_{i,m}^2 > 0$ ,  $\Lambda_{i,m}^3 > 0$ ,  $\Lambda_{i,m}^4 > 0$ ,  $\Lambda_{i,m}^5 > 0$  and a scalar  $\gamma > 0$  such that for any  $(i, m) \in (\mathcal{S}, \mathcal{M})$  the following LMI holds:*

$$\Xi = (\Xi_{i,j})_{15 \times 15} < 0$$

$$\begin{aligned} \sum_{n \in \mathcal{M}} p_{mn} Q_{1,i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{1,j,m} + \sum_{n \in \mathcal{M}} p_{mn} Q_{3,i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3,j,m} &< U \\ \sum_{n \in \mathcal{M}} p_{mn} Q_{2,i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{2,j,m} &< U \\ \sum_{n \in \mathcal{M}} p_{mn} Q_{3,i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m Q_{3,j,m} &< U \end{aligned}$$

where

$$\begin{aligned} \Xi_{1,1} &= -P_{i,m} C_i - C_i^T P_{i,m} + Q_{1,i,m}^1 + Q_{2,i,m}^1 + \tau_1 U^1 + Q_{3,i,m}^1 + \tau_2 U_1 + Q_4 - (\tau_2 - \tau_1) S_1 - S_3 \\ &\quad - 4\tau_2^2 T_1 - 4(\tau_2 - \tau_1)^2 T_2 - 4\sigma^2 T_3 - F_1 \Lambda_{i,m}^1 + \left[ \sum_{n \in \mathcal{M}} p_{mn} P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] + \sigma^2 W_2 \\ \Xi_{1,2} &= P_{i,m} A_i + Q_{1,i,m}^2 + Q_{2,i,m}^2 + \tau_1 U^2 + Q_{3,i,m}^2 + \tau_2 U^2 + F_2 \Lambda_{i,m}^1, \quad \Xi_{1,4} = P_{i,m} B_i \end{aligned}$$

$$\begin{aligned}
\Xi_{1,7} &= (\tau_2 - \tau_1)S_1, \quad \Xi_{1,9} = S_3, \quad \Xi_{1,11} = C_i^T P_{i,m} C_i - \left[ \sum_{n \in \mathcal{M}} p_{mn} P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] C_i + 4\sigma T_3 \\
\Xi_{1,12} &= P_{i,m} D_i, \quad \Xi_{1,13} = 4\tau_2 T_1, \quad \Xi_{1,14} = 4(\tau_2 - \tau_1)T_2, \quad \Xi_{1,15} = P_{i,m} \\
\Xi_{2,2} &= Q_{1,i,m}^3 + Q_{2,i,m}^3 + \tau_1 U^3 + Q_{3,i,m}^3 + \tau_2 U^3 + A_i^T R A_i + d^2 W_1 + A_i^T G A_i - \Lambda_{i,m}^1 \\
\Xi_{2,4} &= A_i^T R B_i + A_i^T G B_i, \quad \Xi_{2,9} = -A_i^T R C_i - A_i^T G C_i, \quad \Xi_{2,11} = -A_i^T P_{i,m} C_i \\
\Xi_{2,12} &= A_i^T R D_i + A_i^T G D_i, \quad \Xi_{2,15} = A_i^T R + A_i^T G - I, \quad \Xi_{3,3} = -(1 - \tau_\mu) Q_{1,i,m}^1 - F_1 \Lambda_{i,m}^2 \\
\Xi_{3,4} &= -(1 - \tau_\mu) Q_{1,i,m}^2 + F_2 \Lambda_{i,m}^2, \quad \Xi_{4,4} = -(1 - \tau_\mu) Q_{1,i,m}^3 + B_i^T R B_i + B_i^T G B_i - \Lambda_{i,m}^2 \\
\Xi_{4,9} &= -B_i^T R C_i - B_i^T G C_i, \quad \Xi_{4,11} = -B_i^T P_{i,m} C_i, \quad \Xi_{4,12} = B_i^T R D_i + B_i^T G D_i \\
\Xi_{4,15} &= B_i^T R + B_i^T G, \quad \Xi_{5,5} = -Q_{2,i,m}^1 - S_2 - F_1 \Lambda_{i,m}^3, \quad \Xi_{5,6} = -Q_{2,i,m}^2 + F_2 \Lambda_{i,m}^3, \quad \Xi_{5,7} = S_2 \\
\Xi_{6,6} &= -Q_{2,i,m}^3 - \Lambda_{i,m}^3, \quad \Xi_{7,7} = -Q_{3,i,m}^1 - (\tau_2 - \tau_1)S_1 - S_2 - F_1 \Lambda_{i,m}^4, \quad \Xi_{7,8} = -Q_{3,i,m}^2 + F_2 \Lambda_{i,m}^4 \\
\Xi_{8,8} &= -Q_{3,i,m}^3 - \Lambda_{i,m}^4, \quad \Xi_{9,9} = -Q_4 + C_i^T R C_i + C_i^T G C_i - S_3 - F_1 \Lambda_{i,m}^5, \quad \Xi_{9,10} = F_2 \Lambda_{i,m}^5 \\
\Xi_{9,12} &= -C_i^T R D_i - C_i^T G D_i, \quad \Xi_{9,15} = -C_i^T R - C_i^T G, \quad \Xi_{10,10} = -\Lambda_{i,m}^5 \\
\Xi_{11,11} &= C_i^T \left[ \sum_{n \in \mathcal{M}} p_{mn} P_{i,n} + \sum_{j \in \mathcal{S}} \pi_{ij}^m P_{j,m} \right] C_i - W_2 - 4T_3, \quad \Xi_{11,12} = -C_i^T P_{i,m} D_i \\
\Xi_{11,15} &= -C_i^T P_{i,m}, \quad \Xi_{12,12} = -D_i^T R D_i - D_i^T G D_i - W_1, \quad \Xi_{12,15} = D_i^T R + D_i^T G \\
\Xi_{13,13} &= -4T_1, \quad \Xi_{14,14} = -4T_1, \quad \Xi_{15,15} = R + G - \gamma I \\
R &= \tau_2^2 (\tau_2 - \tau_1) S_1 + (\tau_2 - \tau_1)^2 S_2 + \sigma^2 S_3 \\
G &= \tau_2^4 T_1 + (\tau_2^2 - \tau_1^2)^2 T_2 + \sigma^4 T_3
\end{aligned}$$

and the remaining coefficients are all zero.

*Proof.* The Lyapunov functional can be defined for the above neural networks as in Theorem (5.3.1) by replacing  $\sigma(t)$  by  $\sigma$ . The proof is same as that of Theorem (5.3.1), and hence it is omitted.  $\square$

## 5.4 Problems without Switching

### 5.4.1 Description and Preliminaries

In this section, passivity criterion for the delayed neural networks is derived using the Lypapunov-Krasovskii functional without Markovian jumping parameters.

Consider the following neural networks with mixed time-delays:

$$\left\{ \begin{array}{l} \dot{x}(t) = -Cx(t - \sigma(t)) + Ag(x(t)) + Bg(x(t - \tau(t))) + D \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)) \end{array} \right. \quad (5.4.1)$$

Or, it has an equivalent form as follows:

$$\left\{ \begin{array}{l} \frac{d}{dt} \left[ x(t) - C \int_{t-\sigma(t)}^t x(s)ds \right] = -Cx(t) - Cx(t - \sigma(t))\dot{\sigma}(t) + Ag(x(t)) \\ \quad + Bg(x(t - \tau(t))) + D \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)) \end{array} \right. \quad (5.4.2)$$

Now, the passivity condition for the system (5.4.2) is established as follows.

**Theorem 5.4.1.** *Neural Network (5.4.2) is passive if there exists*

$$P > 0, Q_1 = \begin{bmatrix} Q_1^1 & Q_1^2 \\ Q_1^{2T} & Q_1^3 \end{bmatrix} > 0, Q_2 = \begin{bmatrix} Q_2^1 & Q_2^2 \\ Q_2^{2T} & Q_2^3 \end{bmatrix} > 0, Q_3 = \begin{bmatrix} Q_3^1 & Q_3^2 \\ Q_3^{2T} & Q_3^3 \end{bmatrix} > 0, Q_4 > 0, U = \begin{bmatrix} U^1 & U^2 \\ U^{2T} & U^3 \end{bmatrix} > 0, \text{ positive symmetric matrices } S_1 = S_1^T > 0, S_2 =$$

$S_2^T > 0$ ,  $S_3 = S_3^T > 0$ ,  $T_1 = T_1^T > 0$ ,  $T_2 = T_2^T > 0$ ,  $T_3 = T_3^T > 0$ , the positive definite matrices  $W_1 > 0, W_2 > 0$ , and the diagonal matrices  $\Lambda^1 > 0$ ,  $\Lambda^2 > 0$ ,  $\Lambda^3 > 0$ ,  $\Lambda^4 > 0$ ,  $\Lambda^5 > 0$  and a scalar  $\gamma > 0$  such that the following LMI holds:

$$\Xi = (\Xi_{i,j})_{20 \times 20} < 0 \quad (5.4.3)$$

where

$$\begin{aligned} \Xi_{1,1} &= -PC - C^T p + Q_1^1 + Q_2^1 + \tau_1 U^1 + Q_3^1 + \tau_2 U_1 + Q_4 - (\tau_2 - \tau_1)S_1 - S_3 \\ &\quad - 4\tau_2^2 T_1 - 4(\tau_2 - \tau_1)^2 T_2 - 4\sigma^2 T_3 - F_1 \Lambda^1 + \sigma^2 W_2 \\ \Xi_{1,2} &= PA + Q_1^2 + Q_2^2 + \tau_1 U^2 + Q_3^2 + \tau_2 U^2 + F_2 \Lambda^1, \quad \Xi_{1,4} = pB, \\ \Xi_{1,7} &= (\tau_2 - \tau_1)S_1, \quad \Xi_{1,9} = -PC\sigma_\mu, \quad \Xi_{1,11} = C^T PC, \quad \Xi_{1,12} = S_3, \quad \Xi_{1,13} = 4\sigma T_3 \\ \Xi_{1,14} &= PD_i, \quad \Xi_{1,17} = 4\tau_2 T_1, \quad \Xi_{1,19} = 4(\tau_2 - \tau_1)T_2, \quad \Xi_{1,20} = P \\ \Xi_{2,2} &= Q_1^3 + Q_2^3 + \tau_1 U^3 + Q_3^3 + \tau_2 U^3 + A^T RA + d^2 W_1 + A^T GA - \Lambda^1 \\ \Xi_{2,4} &= A^T RB + A^T GB, \quad \Xi_{2,9} = -A^T RC - A^T GC, \quad \Xi_{2,11} = -A^T PC \\ \Xi_{2,14} &= A^T RD + A^T GD, \quad \Xi_{2,20} = A^T R + A^T G - I \\ \Xi_{3,3} &= -(1 - \tau_\mu)Q_1^1 - F_1 \Lambda^2, \quad \Xi_{3,4} = -(1 - \tau_\mu)Q_1^2 + F_2 \Lambda^2 \\ \Xi_{4,4} &= -(1 - \tau_\mu)Q_1^3 + B^T RB + B^T GB - \Lambda^2, \quad \Xi_{4,9} = -B^T RC - B^T GC \\ \Xi_{4,11} &= -B^T PC, \quad \Xi_{4,14} = B^T RD + B^T GD, \quad \Xi_{4,20} = B^T R + B^T G \\ \Xi_{5,5} &= -Q_2^1 - S_2 - F_1 \Lambda^3, \quad \Xi_{5,6} = -Q_2^2 + F_2 \Lambda^3, \quad \Xi_{5,7} = S_2 \\ \Xi_{6,6} &= -Q_2^3 - \Lambda^3, \quad \Xi_{7,7} = -Q_3^1 - (\tau_2 - \tau_1)S_1 - S_2 - F_1 \Lambda^4, \quad \Xi_{7,8} = -Q_3^2 + F_2 \Lambda^4 \\ \Xi_{8,8} &= -Q_3^3 - \Lambda^4, \quad \Xi_{9,9} = -(1 - \sigma_\mu)Q_4 + C^T RC + C^T GC - F_1 \Lambda^5, \quad \Xi_{9,10} = F_2 \Lambda^5 \\ \Xi_{9,11} &= C^T \sigma_\mu PC, \quad \Xi_{9,14} = -C^T RD - C^T GD, \quad \Xi_{9,20} = -C^T R - C^T G \\ \Xi_{10,10} &= -\Lambda^5, \quad \Xi_{11,11} = -W_2, \quad \Xi_{11,14} = -C^T PD, \quad \Xi_{11,20} = -C^T P, \\ \Xi_{12,12} &= -S_3, \quad \Xi_{13,13} = -4T_3, \quad \Xi_{14,14} = D^T RD + D^T GD - W_1 \\ \Xi_{14,20} &= D^T R + D^T G, \quad \Xi_{15,15} = -\frac{1}{\tau_1} U^1, \quad \Xi_{15,16} = -\frac{1}{\tau_1} U^2, \quad \Xi_{16,16} = -\frac{1}{\tau_1} U^3 \end{aligned}$$

$$\begin{aligned}
\Xi_{17,17} &= -\frac{1}{\tau_2}U^1 - 4T_1, & \Xi_{17,18} &= -\frac{1}{\tau_2}U^2, & \Xi_{18,18} &= -\frac{1}{\tau_2}U^3 \\
\Xi_{19,19} &= -4T_2, & \Xi_{20,20} &= R + G - \gamma I \\
R &= \tau_2^2(\tau_2 - \tau_1)S_1 + (\tau_2 - \tau_1)^2S_2 + \sigma^2S_3 \\
G &= \tau_2^4T_1 + (\tau_2^2 - \tau_1^2)^2T_2 + \sigma^4T_3
\end{aligned}$$

and the remaining coefficients are all zero.

*Proof.* Denote  $\zeta = [x(t)^T \quad g(x(t))^T]^T$  and consider the following Lyapunov-Krasovskii functional for neural network (5.4.2):

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t) + V_5(x_t) \quad (5.4.4)$$

where

$$\begin{aligned}
V_1(x_t) &= \left[ x(t) - C_i \int_{t-\sigma(t)}^t x(s)ds \right]^T P \left[ x(t) - C_i \int_{t-\sigma(t)}^t x(s)ds \right] \\
V_2(x_t) &= \int_{t-\tau(t)}^t \zeta(s)^T Q_1 \zeta(s)ds + \int_{t-\tau_1}^t \zeta(s)^T Q_2 \zeta(s)ds + \int_{t-\tau_2}^t \zeta(s)^T Q_3 \zeta(s)ds \\
&\quad + \int_{t-\sigma(t)}^t x(s)^T Q_4 x(s)ds + \int_{-\tau_1}^0 \int_{t+\theta}^t \zeta(s)^T U \zeta(s)dsd\theta + \int_{-\tau_2}^0 \int_{t+\theta}^t \zeta(s)^T U \zeta(s)dsd\theta \\
V_3(x_t) &= \tau_2(\tau_2 - \tau_1) + \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s)^T S_1 \dot{x}(s)dsd\theta \\
&\quad + (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}(s)^T S_2 \dot{x}(s)dsd\theta + \sigma \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}(s)^T S_3 \dot{x}(s)dsd\theta \\
V_4(x_t) &= \int_{-d}^0 \int_{t+\theta}^t g(x(s))^T W_1 g(x(s))dsd\theta + \sigma \int_{t-\sigma}^t \int_{\theta}^t x(s)^T W_2 x(s)dsd\theta \\
V_5(x_t) &= 2\tau_2^2 \int_{-\tau_2}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_1 \dot{x}(s)dsd\lambda d\theta + 2(\tau_2^2 - \tau_1^2) \int_{-\tau_1}^{-\tau_2} \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_2 \dot{x}(s)dsd\lambda d\theta \\
&\quad + 2\sigma^2 \int_{-\sigma}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}(s)^T T_3 \dot{x}(s)dsd\lambda d\theta.
\end{aligned}$$

Taking time derivative acting on  $V(x_t)$  along the neural networks (5.4.2) is defined as follows:

$$\begin{aligned}
\dot{V}_1(x_t) &= \left[ x(t) - C \int_{t-\sigma(t)}^t x(s) ds \right]^T P \frac{d}{dt} \left[ x(t) - C \int_{t-\sigma(t)}^t x(s) ds \right] \\
&+ \frac{d}{dt} \left[ x(t) - C \int_{t-\sigma(t)}^t x(s) ds \right]^T P \left[ x(t) - C \int_{t-\sigma(t)}^t x(s) ds \right], \\
\dot{V}_2(x_t) &\leq \zeta^T(t) Q_1 \zeta(t) - \zeta^T(t - \tau(t)) Q_1 \zeta(t - \tau(t)) (1 - \tau_\mu) + \zeta^T(t) Q_2 \zeta(t) \\
&- \zeta^T(t - \tau_1) Q_2 \zeta(t - \tau_1) + \zeta^T(t) Q_3 \zeta(t) - \zeta^T(t - \tau_2) Q_3 \zeta(t - \tau_2) \\
&+ x^T(t) Q_4 x(t) - x^T(t - \sigma(t)) Q_4 x(t - \sigma(t)) (1 - \sigma_\mu) + \int_{-\tau_1}^0 \zeta^T(t) U \zeta(t) d\theta \\
&- \int_{t-\tau_1}^t \zeta^T(s) U \zeta(s) ds + \int_{-\tau_2}^0 \zeta^T(t) U \zeta(t) d\theta - \int_{t-\tau_2}^t \zeta^T(s) U \zeta(s) ds, \\
\dot{V}_3(x_t) &= \dot{x}(t)^T [\tau_2^2 (\tau_2 - \tau_1) S_1 + (\tau_2 - \tau_1)^2 S_2 + \sigma^2 S_3] \dot{x}(t) \\
&- \tau_2 (\tau_2 - \tau_1) \int_{t-\tau_2}^t \dot{x}(s)^T S_1 \dot{x}(s) ds - (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{x}(s)^T S_2 \dot{x}(s) ds \\
&- \sigma \int_{t-\sigma}^t \dot{x}(s)^T S_3 \dot{x}(s) ds, \\
\dot{V}_4(x_t) &= d^2 g(x(t))^T W_1 g(x(t)) - d(t) \int_{t-d(t)}^t g(x(s))^T W_1 g(x(s)) ds \\
&+ \sigma^2 x(t)^T W_2 x(t) - \sigma(t) \int_{t-\sigma(t)}^t x(s)^T W_1 x(s) ds, \\
\dot{V}_5(x_t) &= \dot{x}(t)^T [\tau_2^4 T_1 + (\tau_2^2 - \tau_1^2)^2 T_2 + \sigma^4 T_3] \dot{x}(t) - 2\tau_2^2 \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}(s)^T T_1 \dot{x}(s) ds d\theta \\
&- (\tau_2^2 - \tau_1^2) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}(s)^T T_2 \dot{x}(s) ds d\theta - 2\sigma^2 \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}(s)^T T_3 \dot{x}(s) ds d\theta. \quad (5.4.5)
\end{aligned}$$

Similarly like Theorem 5.3.1 one can use Lemma 5.2.1 and Lemma 5.2.2 for the integrals. On the other hand, we have from (5.2.5) that for any  $\lambda = 1, 2, \dots, n$ ,

$$(g_\lambda(x_\lambda(t)) - F_\lambda^- x_\lambda(t))(g_\lambda(x_\lambda(t)) - F_\lambda^+ x_\lambda(t)) \leq 0, \quad (5.4.6)$$

which is equivalent to

$$\zeta^T(t) \begin{bmatrix} F_\lambda^+ F_\lambda^- e_\lambda e_\lambda^T & -\frac{F_\lambda^+ + F_\lambda^-}{2} e_\lambda e_\lambda^T \\ -\frac{F_\lambda^+ + F_\lambda^-}{2} e_\lambda e_\lambda^T & e_\lambda e_\lambda^T \end{bmatrix} \zeta(t) \leq 0, \quad (5.4.7)$$

where  $e_\lambda$  denotes the unit column vector having 1 element on its  $\lambda$ -th row and zeros elsewhere. Thus, for any appropriately dimensioned diagonal matrix  $\Lambda^1 > 0$ , the following inequality holds:

$$0 \leq \zeta^T(t) \begin{bmatrix} -F_1 \Lambda^1 & F_2 \Lambda^1 \\ * & -\Lambda^1 \end{bmatrix} \zeta(t). \quad (5.4.8)$$

Similarly, for any appropriately dimensioned diagonal matrices  $\Lambda^2 > 0, \Lambda^3 > 0, \Lambda^4 > 0$  and  $\Lambda^5 > 0$ , the following inequalities also hold:

$$0 \leq \zeta^T(t - \tau(t)) \begin{bmatrix} -F_1 \Lambda^2 & F_2 \Lambda^2 \\ * & -\Lambda^2 \end{bmatrix} \zeta(t - \tau(t)), \quad (5.4.9)$$

$$0 \leq \zeta^T(t - \tau_1) \begin{bmatrix} -F_1 \Lambda^3 & F_2 \Lambda^3 \\ * & -\Lambda^3 \end{bmatrix} \zeta(t - \tau_1), \quad (5.4.10)$$

$$0 \leq \zeta^T(t - \tau_2) \begin{bmatrix} -F_1\Lambda^4 & F_2\Lambda^4 \\ * & -\Lambda^4 \end{bmatrix} \zeta(t - \tau_2), \quad (5.4.11)$$

and

$$0 \leq \zeta^T(t - \sigma(t)) \begin{bmatrix} -F_1\Lambda^5 & F_2\Lambda^5 \\ * & -\Lambda^5 \end{bmatrix} \zeta(t - \sigma(t)). \quad (5.4.12)$$

Using inequalities (5.4.5) and adding (5.4.8)-(5.4.12) to  $\dot{V}(x_t)$ , we get

$$\dot{V}(x_t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \leq \rho^T(t)\Xi\rho(t)$$

where  $\rho(t) = [\rho_1^T(t) \ \rho_2^T(t) \ \rho_3^T(t) \ \rho_4^T(t)]$  with

$$\rho_1(t) = [x(t) \ g(x(t)) \ x(t - \tau(t)) \ g(x(t - \tau(t))) \ x(t - \tau_1) \ g(x(t - \tau_1))]$$

$$\rho_2(t) = \left[ x(t - \tau_2) \ g(x(t - \tau_2)) \ x(t - \sigma(t)) \ g(x(t - \sigma(t))) \ \int_{t-\sigma(t)}^t x(s)ds \right]$$

$$\rho_3(t) = \left[ x(t - \sigma) \ \int_{t-\sigma}^t x(s)ds \ \int_{t-d(t)}^t g(x(s))ds \ \int_{t-\tau_1}^t x(s)ds \ \int_{t-\tau_1}^t g(x(s))ds \right]$$

$$\rho_4(t) = \left[ \int_{t-\tau_2}^t x(s)ds \ \int_{t-\tau_2}^t g(x(s))ds \ \int_{t-\tau_2}^{t-\tau_1} x(s)ds \ u(t) \right].$$

Hence we can obtain from (5.4.3) that,

$$\dot{V}(x_t) - 2y(t)^T u(t) - \gamma u(t)^T u(t) \leq 0. \quad (5.4.13)$$

The remaining part of the proof is same as Theorem 5.3.1 .  $\square$

**Remark 5.4.1.** *In this chapter, Theorem 5.3.1 provides a passivity criteria for the Markovian jumping neural networks with leakage time varying delays. Such stability criterion is derived based on the assumption that the leakage time varying delays are differentiable and the values of  $\sigma_\mu$  are known. A new set of triple integral terms have been introduced in the Lyapunov–Krasovskii functional to derive the leakage delay-dependent passivity conditions via LMI approach. New type of Lyapunov–Krasovskii*

functional is constructed in which the positive definite matrices  $Q_{1i,m}, Q_{2i,m}, Q_{3i,m}$  are dependent on the system mode and a triple-integral term is introduced for deriving the delay-dependent passivity conditions.

## 5.5 Numerical Examples

In this section, two simple examples are provided here in order to illustrate the usefulness of our main results. The aim is to examine the passivity analysis of a given delayed neural networks.

**Example 5.5.1.** Consider the delayed neural networks (5.2.1) with the following parameters and having Markovian jumping parameters as below:

$$\left\{ \begin{array}{l} \dot{x}(t) = -C(r(t))x(t - \sigma(t)) + A(r(t))g(x(t)) + B(r(t))g(x(t - \tau(t))) \\ \quad + D(r(t)) \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)) \end{array} \right. \quad (5.5.1)$$

where

$$C_1 = \begin{bmatrix} 2.4 & 0 \\ 0 & 3.5 \end{bmatrix}, C_2 = \begin{bmatrix} 2.6 & 0 \\ 0 & 3.2 \end{bmatrix}, A_1 = \begin{bmatrix} 0.4 & 1.6 \\ -0.5 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5 & 1.2 \\ -0.5 & 0.8 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.9 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.3 & 0.6 \\ 0.5 & -1.2 \end{bmatrix}, D_1 = \begin{bmatrix} 1.1 & -1.6 \\ 0.4 & 0.9 \end{bmatrix}, D_2 = \begin{bmatrix} 0.6 & 0.4 \\ 1.2 & -0.6 \end{bmatrix}$$

and the activation functions are taken as follows:  $g_1(\alpha) = g_2(\alpha) = \tanh(\alpha)$ . It is

found that,  $F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

Furthermore, the transition probability matrices are  $\Pi^1 = \begin{bmatrix} -0.9 & 0.5 \\ 0.7 & 0.6 \end{bmatrix}$ ,

$\Pi^2 = \begin{bmatrix} -0.5 & 0.5 \\ 0.7 & -0.8 \end{bmatrix}$ ,  $\Pi^3 = \begin{bmatrix} -0.7 & 0.9 \\ 0.5 & -0.8 \end{bmatrix}$  and  $A = \begin{bmatrix} -0.7 & 0.2 & 0.25 \\ 0.5 & -1.2 & 0.3 \\ 0.3 & 0.6 & -0.5 \end{bmatrix}$ .

Since, by making use of the following values of the lower and bounds of the  $\tau(t), \sigma(t), d(t)$  then the solution obtained will becomes the feasible one;

$$\tau_1 = 0.2, \quad \tau_2 = 1.5, \quad \sigma = 0.3, \quad \sigma_\mu = 0.4, \quad \tau_\mu = 0.6, \quad d = 0.5. \text{ By applying,}$$

MATLAB LMI toolbox, the feasible solution is obtained as follows

$$P_{11} = \begin{bmatrix} 6.0138 & 0.2029 \\ 0.2029 & 5.5897 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 5.9453 & 0.2180 \\ 0.2180 & 5.5357 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 6.1753 & 0.5152 \\ 0.5152 & 5.9589 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 9.2206 & -0.4935 \\ -0.4935 & 5.5949 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} 8.3958 & -0.3461 \\ -0.3461 & 5.2798 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 8.4741 & -0.2735 \\ -0.2735 & 5.2809 \end{bmatrix},$$

$$\begin{aligned}
Q_{1111} &= \begin{bmatrix} 96.5032 & 0.8993 \\ 0.8993 & 101.4537 \end{bmatrix}, & Q_{1112} &= \begin{bmatrix} -20.1310 & -0.5306 \\ -0.5306 & -22.4636 \end{bmatrix}, \\
Q_{1113} &= \begin{bmatrix} 63.6503 & -0.1830 \\ -0.1830 & 58.2996 \end{bmatrix}, & Q_{1121} &= \begin{bmatrix} 109.9724 & 1.7802 \\ 1.7802 & 115.7604 \end{bmatrix}, \\
Q_{1122} &= \begin{bmatrix} -25.8818 & -2.1403 \\ -2.1403 & -27.4127 \end{bmatrix}, & Q_{1123} &= \begin{bmatrix} 69.9294 & 0.2372 \\ 0.2372 & 63.8244 \end{bmatrix}, \\
Q_{1211} &= \begin{bmatrix} 121.1754 & -0.3055 \\ -0.3055 & 116.3565 \end{bmatrix}, & \dots, & U_1 &= \begin{bmatrix} 141.1052 & 4.9561 \\ 4.9561 & 148.6923 \end{bmatrix}, \\
U_2 &= \begin{bmatrix} -60.7043 & -2.3397 \\ -2.3397 & -67.6228 \end{bmatrix}, & U_3 &= \begin{bmatrix} 44.3204 & 0.2613 \\ 0.2613 & 46.7241 \end{bmatrix}, & S_1 &= \begin{bmatrix} 0.1771 & -0.0191 \\ -0.0191 & -0.1934 \end{bmatrix}, \\
S_2 &= \begin{bmatrix} 0.2957 & -0.0315 \\ -0.0315 & 0.3223 \end{bmatrix}, & S_3 &= \begin{bmatrix} 30.0136 & -0.8922 \\ -0.8922 & -31.0764 \end{bmatrix}, & W_1 &= \begin{bmatrix} 152.5941 & -29.2867 \\ -28.2867 & 162.9678 \end{bmatrix}, \\
W_2 &= \begin{bmatrix} 172.8581 & -2.3076 \\ -2.3076 & 174.9177 \end{bmatrix}, & T_1 &= \begin{bmatrix} 0.6547 & -0.0596 \\ -0.0596 & 0.7113 \end{bmatrix}, & T_2 &= \begin{bmatrix} 0.6766 & -0.0639 \\ -0.0639 & 0.7368 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
T_3 &= \begin{bmatrix} 32.2540 & -0.0557 \\ -0.0557 & 32.3317 \end{bmatrix}, \quad L_{111} = \begin{bmatrix} 358.6106 & 0 \\ 0 & 408.4828 \end{bmatrix}, \quad L_{112} = \begin{bmatrix} 365.0550 & 0 \\ 0 & 414.4924 \end{bmatrix}, \\
L_{113} &= \begin{bmatrix} 485.6460 & 0 \\ 0 & 535.5544 \end{bmatrix}, \quad L_{121} = \begin{bmatrix} 380.7375 & 0 \\ 0 & 419.7776 \end{bmatrix}, \quad L_{122} = \begin{bmatrix} 374.8279 & 0 \\ 0 & 416.9314 \end{bmatrix}, \\
L_{123} &= \begin{bmatrix} 484.0639 & 0 \\ 0 & 529.9711 \end{bmatrix}, \quad L_{211} = \begin{bmatrix} 99.3611 & 0 \\ 0 & 90.0987 \end{bmatrix}, \quad L_{212} = \begin{bmatrix} 96.8689 & 0 \\ 0 & 90.9087 \end{bmatrix}, \\
L_{213} &= \begin{bmatrix} 87.2398 & 0 \\ 0 & 80.2453 \end{bmatrix}, \quad L_{221} = \begin{bmatrix} 88.4163 & 0 \\ 0 & 110.4032 \end{bmatrix}, \quad L_{222} = \begin{bmatrix} 93.5395 & 0 \\ 0 & 115.5929 \end{bmatrix}, \dots, \\
L_{521} &= \begin{bmatrix} 111.7177 & 0 \\ 0 & 134.5697 \end{bmatrix}, \quad L_{522} = \begin{bmatrix} 110.6615 & 0 \\ 0 & 132.6384 \end{bmatrix}, \\
L_{523} &= \begin{bmatrix} 111.4636 & 0 \\ 0 & 133.5637 \end{bmatrix}, \quad \gamma = 166.0447.
\end{aligned}$$

*This shows that the given Markovian jumping neural networks (5.2.1) or (5.2.3) is globally passive with respect to the passive control.*

**Example 5.5.2.** Consider the delayed neural network (5.4.2) with the following parameters and without Markovian jumping parameters as below:

$$\begin{cases} \dot{x}(t) = -Cx(t - \sigma(t)) + Ag(x(t)) + Bg(x(t - \tau(t))) + D \int_{t-d(t)}^t g(x(s))ds + u(t) \\ y(t) = g(x(t)) \end{cases} \quad (5.5.2)$$

where

$$C = \begin{bmatrix} 2.2 & 0 \\ 0 & 2.5 \end{bmatrix}, \quad A = \begin{bmatrix} 1.2 & -1.5 \\ -1.7 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 0.8 \end{bmatrix}, \quad D = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}.$$

$$\text{Further, we have the matrices, } F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since, by making use of the following values of the lower and bounds of the  $\tau(t), \sigma(t), d(t)$  then the solution obtained will becomes the feasible one  $\tau_1 = 0.5, \tau_2 = 1, \sigma = 0.1, \sigma_\mu = 0.1, \tau_\mu = 0.2, d = 0.5$ . By applying MATLAB LMI toolbox, the feasible solution is obtained as follows:

$$P = \begin{bmatrix} 0.2899 & 0.0371 \\ 0.0371 & 0.2688 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} 4.9833 & -0.0919 \\ -0.0919 & 4.8960 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} -1.3685 & 0.0391 \\ 0.0391 & -1.3516 \end{bmatrix},$$

$$Q_{13} = \begin{bmatrix} 2.7359 & 0.7977 \\ 0.7977 & 2.6571 \end{bmatrix}, \quad Q_{21} = \begin{bmatrix} 4.6114 & -0.0801 \\ -0.0801 & 4.4621 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} -1.2020 & 0.0433 \\ 0.0433 & -1.1716 \end{bmatrix},$$

$$\begin{aligned}
Q_{23} &= \begin{bmatrix} 2.0159 & 0.1757 \\ 0.1757 & 2.0771 \end{bmatrix}, & Q_{31} &= \begin{bmatrix} 4.5805 & -0.0773 \\ -0.0773 & 4.4345 \end{bmatrix}, & Q_{32} &= \begin{bmatrix} -1.1979 & 0.0404 \\ 0.0424 & -1.1685 \end{bmatrix}, \\
Q_{33} &= \begin{bmatrix} 2.0158 & 0.1770 \\ 0.1770 & 2.0774 \end{bmatrix}, & Q_4 &= \begin{bmatrix} 5.2009 & 0.7431 \\ 0.7431 & 5.4021 \end{bmatrix}, & U_1 &= \begin{bmatrix} 5.7038 & -0.0258 \\ -0.0258 & 5.5568 \end{bmatrix}, \\
U_2 &= \begin{bmatrix} -1.4144 & 0.0649 \\ 0.0649 & -1.3884 \end{bmatrix}, & U_3 &= \begin{bmatrix} 2.9386 & 0.2022 \\ 0.2022 & 3.0092 \end{bmatrix}, & S_1 &= \begin{bmatrix} 0.2360 & 0.0515 \\ 0.0515 & 0.2458 \end{bmatrix}, \\
S_2 &= \begin{bmatrix} 0.4259 & 0.0798 \\ 0.0798 & 0.4379 \end{bmatrix}, & S_3 &= \begin{bmatrix} 2.9380 & 0.0495 \\ 0.0495 & 2.9462 \end{bmatrix}, & W_1 &= \begin{bmatrix} 4.8673 & 0.5358 \\ 0.5358 & 4.1789 \end{bmatrix}, \\
W_2 &= \begin{bmatrix} 5.7616 & 0.4026 \\ 0.4026 & 5.7553 \end{bmatrix}, & T_1 &= \begin{bmatrix} 0.0807 & 0.0460 \\ 0.0460 & 0.0859 \end{bmatrix}, & T_2 &= \begin{bmatrix} 0.3553 & 0.0632 \\ 0.0632 & 0.3633 \end{bmatrix}, \\
T_3 &= \begin{bmatrix} 1.9454 & -0.0007 \\ -0.0007 & 1.9459 \end{bmatrix}, & L_1 &= \begin{bmatrix} 17.3673 & 0 \\ 0 & 16.8346 \end{bmatrix}, & L_2 &= \begin{bmatrix} 3.5738 & 0 \\ 0 & 3.2436 \end{bmatrix},
\end{aligned}$$

$$L_3 = \begin{bmatrix} 2.5698 & 0 \\ 0 & 2.5662 \end{bmatrix}, L_4 = \begin{bmatrix} 2.5591 & 0 \\ 0 & 2.5552 \end{bmatrix}, L_5 = \begin{bmatrix} 4.0154 & 0 \\ 0 & 4.2815 \end{bmatrix},$$

$\gamma = 6.5268$ . This shows that the given Markovian jumping neural networks (5.4.2) is globally passive with respect to the passive control.

