CHAPTER V

Vertex Corona product of Fan graph families with Cycle, Path, Star and Wheel graph

In this Chapter, the b-chromatic number of vertex corona product of double fan with cycle, fan with star, fan with wheel, double fan with path, path with barbell are accomplished.

5.1 Introduction

Fan Graph [Vernold, J et al., 2014]

A fan graph $F_{m,n}$ is defined as the graph join $\overline{K}_m + P_n$, where $\overline{K}_m$ is the empty graph on m vertices and p_n is the path graph on n vertices .The case $m = 1$ corresponds to the usual fan graphs, while $m = 2$ corresponds to the double fan, etc,. Pre-computed properties of fan graphs are implemented in the wolfram Language as Graph Data $[\{Fan, \{m, n\}\}]$. The $(r,2)$-fan graph is isomorphic to the complete tripartite graph $k_1, 1_r$ and the $(r,3)$-fan graph to $k_1, 2_r$.

5.2 The b-coloring of vertex corona product of $\varphi[F_{2,n} \circ C_n]$

Theorem 5.2.1: Let $F_{2,n}$ and C_n be the double fan graph and cycle graph with n vertices, $n \geq 1$, then the b-chromatic number is $\varphi[F_{2,n} \circ C_n] = n + 2$

Proof:

Let the vertex set of double fan graph be $V(F_{2,n}) = \{u_i : 1 \leq i \leq n + 2\}$ and

Let the vertex set of cycle graph be $V(C_n) = \{y_i : 1 \leq i \leq n\}$

By the definition of corona graph, each vertex of $F_{2,n}$ is adjacent to every vertex of number of copies of C_n.

i.e., every vertex $u_i \in V(F_{2,n})$ is adjacent to every vertex from the set

$\{y^i{}_j : 1 \leq i \leq n + 2, 1 \leq j \leq n\} \in V(C_n)$.

Then the vertex set of the corona product is,

$V(F_{2,n} \circ C_n) = \{u_i : 1 \leq i \leq n + 2\} \cup \{y^i{}_j : 1 \leq i \leq n + 2, 1 \leq j \leq n\}$.

Assign a proper coloring to $V(F_{2,n} \circ C_n)$ as follows

- For $u_i : 1 \le i \le n+2$, assign the color c_i
- For $y^i{}_j : 1 \le i \le n+2, i \le j \le n$ assign the colors $c_k : 1 \le k \le n+2$

By this coloring procedure, we have that $\varphi[F_{2,n} \circ C_n] \ge n+2$

To prove the lower bound, let us assume that b-chromatic number of corona product of double fan graph $F_{2,n}$ with cycle graph C_n is greater than $n+2$.

That is the b-chromatic number of $(F_{2,n} \circ C_n) = n+3$

We can assign $n+3$ colours is only possible when the graph $F_{2,n} \circ C_n$ contains $n+3$ vertices with degree $n+2$ .But the graph $F_{2,n} \circ C_n$ contains only $n+2$ vertices with maximum degree $n+3$,which makes the maximization of $n+3$ colors impossible.

$\therefore \varphi[F_{2,n} \circ C_n] \le n+2$

Hence $\varphi[F_{2,n} \circ C_n] = n+2$.

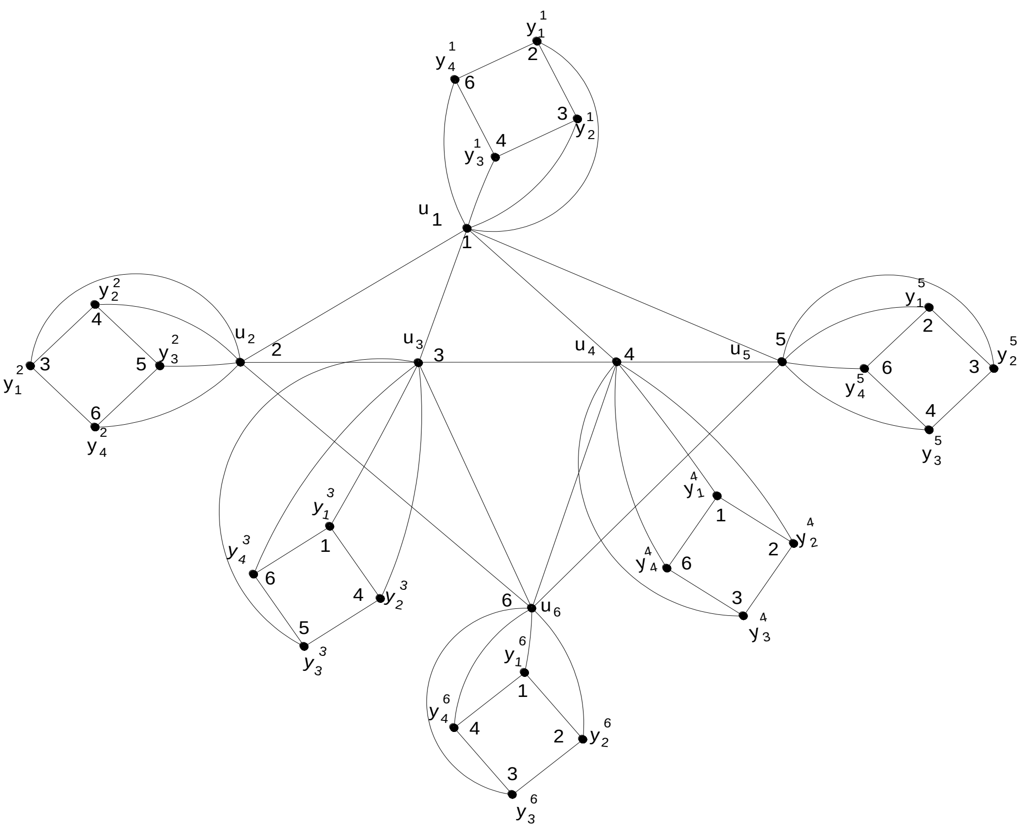

Fig 5.1: $\varphi[F_{2,4} \circ C_4] = 6$

5.3 The b-coloring of vertex corona product of $\varphi[F_{1,n} \circ K_{1,n}]$

Algorithm 5.3.1:

Input: The number "n" of $F_{1,n} \circ K_{1,n}$

Output: Assigning b- coloring to the vertices of $F_{1,n} \circ K_{1,n}$

begin

for $i = 1$ to $n + 1$

{

$V_1 = \{f_i\};$

$C(f_i) = i;$

}

for $i = 1$ to $n + 1$, $j = 1$ to $n + 1$

{

$V_2 = \{s^i{}_j\};$

if $i = 1$

$C(f^i{}_j) = n + 2;$

Else

$C(f^i{}_j) = k, k = 1 \text{ to } n;$

}

$V = V_1 \cup V_2;$

end.

Theorem 5.3.1: For a fan graph $F_{1,n}$ and a star graph $K_{1,n}$, the b-chromatic number of the corona product of $F_{1,n} \circ K_{1,n}$ is given by, $\varphi[F_{1,n} \circ K_{1,n}] = n + 2, n \geq 3.$

Proof:

Let $V(F_{1,n}) = \{f_i : 1 \leq i \leq n + 1\}$ and $V(K_{1,n}) = \{s_i : 1 \leq i \leq n + 1\}$

By the definition of corona product each vertex of $F_{1,n}$ is adjacent to every vertex of number of copies of $K_{1,n}$.

i.e., every vertex $\{f_i : 1 \le i \le n+1\} \in V(F_{1,n})$ is adjacent to every vertex from the set $\{s^i{}_j : 1 \le i \le n+1, 1 \le j \le n+1\}$

Assign a proper coloring to $V(F_{1,n} \circ K_{1,n})$ by using the above algorithm.

From this coloring procedure we have that the b– chromatic number of corona graph of fan graph $F_{1,n}$ with star graph $K_{1,n}$ is $n+2$.

i.e., $\varphi[F_{1,n} \circ K_{1,n}] = n+2, n \ge 3.$

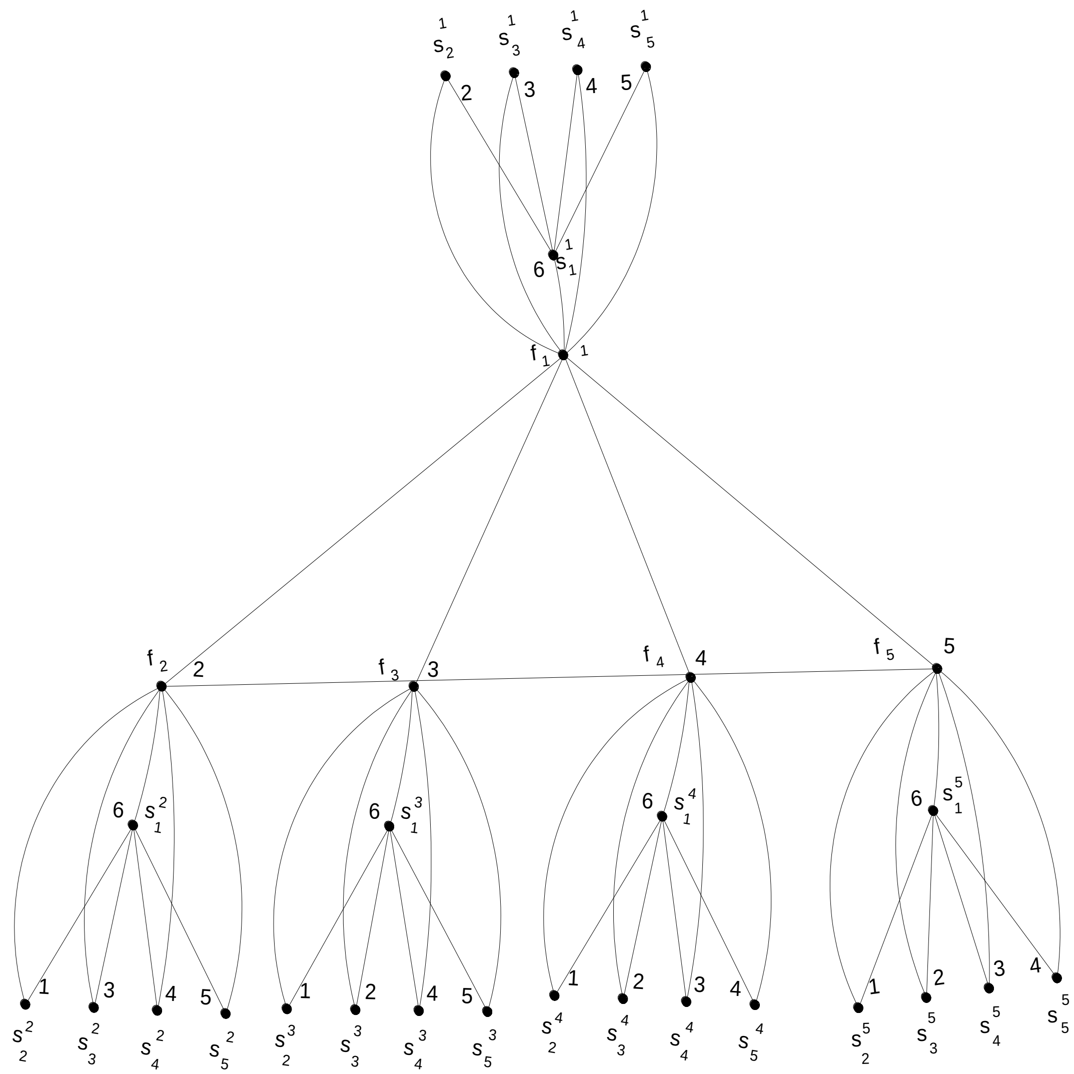

Fig 5.2: $\varphi[F_{1,4} \circ K_{1,4}] = 6$

5.4 The b-coloring of vertex corona product of $\varphi[F_{1,n} \circ W_n]$

Algorithm 5.4.1:

Input: The number "n" of $F_{1,n} \circ W_n$

Output: Assigning b coloring to the vertices of $F_{1,n} \circ W_n$

begin

for $i = 1 \quad 1$ to n

{

$V_1 = \{v_i\};$

$C(v_i) = i;$

}

for $i = 1$ to $n - 1$, $j = 1$ to n, $k = 1$ to $n - 1$

{

$V_2 = \{w^i{}_j\};$

$C(w^i{}_j) = k;$

}

for $i = 1$ to n, $j = 1$ to $n - 1$

{

$V_3 = \{w^i{}_j\};$

$C(w^i{}_j) = n + 1;$

}

for $i = n$, $j = n$

{

$V_4 = \{w^i{}_j\}$;

$C(w_j^i)$=k, k= 1to n;

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4$;

end.

Theorem 5.4.1: For a fan graph $F_{1,n}$ and a wheel graph W_n, the b-chromatic number of the corona product $F_{1,n} \circ W_n$ is given by $\varphi[F_{1,n} \circ W_n] = n+1, n \geq 4.$

Proof:

Let the vertex set of the fan graph

$$V(F_{1,n}) = \{v_i : 1 \leq i \leq n+1\}$$

and the vertex set of the wheel graph be

$$V(W_n) = \{w_i : 1 \leq i \leq n\}.$$

By the definition of corona product each vertex of $F_{1,n}$

is adjacent to every vertex of number of copies of W_n.

i.e., every vertex $v_i \in V(F_{1,n})$ is adjacent to every vertex from the set

$\{w^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n+1\}$.

Assign a proper coloring to $F_{1,n} \circ W_n$ by using the above algorithm.

From this coloring procedure we have that the b -chromatic number of corona graph of fan graph $F_{1,n}$ with wheel graph W_n is n+1.

i.e., $\varphi[F_{1,n} \circ W_n] = n+1, n \geq 4.$

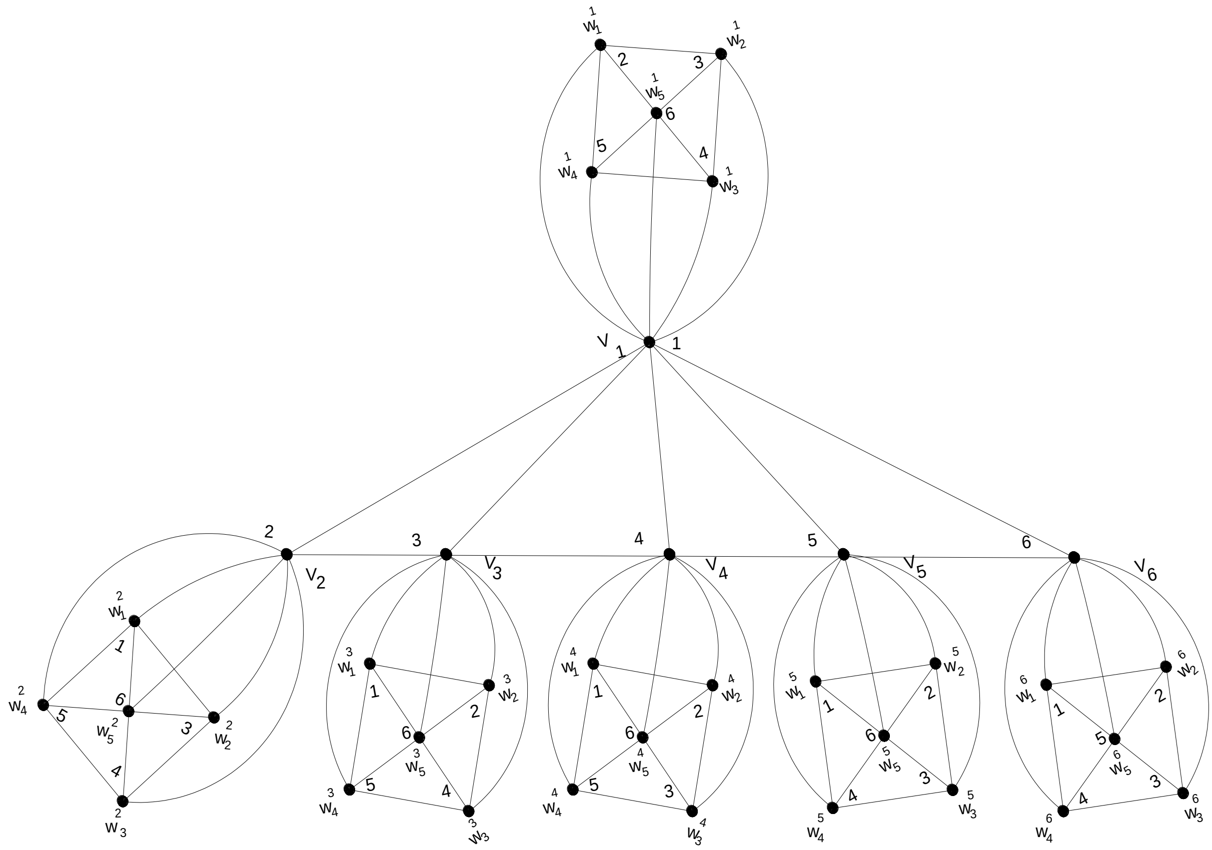

Fig 5.3: $\varphi[F_{1,5} \circ W_5] = 6$

5.5 The b-chromatic number of the corona product $\varphi[F_{2,n} \circ P_n]$

Algorithm 5.5.1:

Input: The number "n" of $F_{2,n} \circ P_n$.

Output: Assigning b coloring to the vertices of $F_{2,n} \circ P_n$.

begin

for $i = 1$ to n

{

$V_1 = \{u_i\};$

$C(u_i) = i;$

}

{

$V_2 = \{p^i{}_j\};$

$C(p^i{}_j) = m, 1 \le m \le i+1;$

}

$V = V_1 \cup V_2;$

end.

Theorem 5.5.1: For a double Fan graph $F_{2,n}$ and a path graph P_n , $n \ge 2$ the b-chromatic number of the corona product of $F_{2,n} \circ P_n$ is given by,

$\varphi[F_{2,n} \circ P_n] = n+2.$

Proof:

Let the vertex set of double Fan graph be

$V(F_{2,n}) = \{u_i : 1 \le i \le n\}$ and $V(P_n) = \{p^i{}_j : 1 \le i \le n, 1 \le j \le n\}$

By the definition of corona graph each vertex of $F_{2,n}$ is adjacent to every vertex of number of copies of P_n.

i.e., every vertex $u_i \in V(F_{2,n})$ is adjacent to every vertex from the set $\{p^i{}_j : 1 \le i \le n, 1 \le j \le n\}$

Then the vertex set of $V(F_{2,n} \circ P_n)$ is,

$$V(F_{2,n} \circ P_n) = \{u_i : 1 \le i \le n+2\} \cup \{p^i{}_j : 1 \le i \le n, 1 \le j \le n\}.$$

Assign a proper coloring to $F_{2,n} \circ P_n$ by using the above algorithm.

From this coloring procedure we have that,

$\varphi[F_{2,n} \circ P_n] \ge n+2, n \ge 3.$

To prove: $\varphi[F_{2,n} \circ P_n] \le n+2$

Let us assume that $\varphi[F_{2,n} \circ P_n]$ is greater than n+2.

i.e., $\varphi[F_{2,n} \circ P_n] = n+3$.

There must be atleast $n+3$ vertices of degree $n+2$ in $F_{2,n}\circ P_n$ all with distinct colors and each adjacent to vertices all of the other colors. But there is only $n+2$ vertices with the maximum degree, this is the contradiction, b-coloring with $n+3$ colors is impossible.

Thus we have $\varphi[F_{2,n}\circ P_n] \le n+2$.

Hence $\varphi[F_{2,n}\circ P_n] = n+2$.

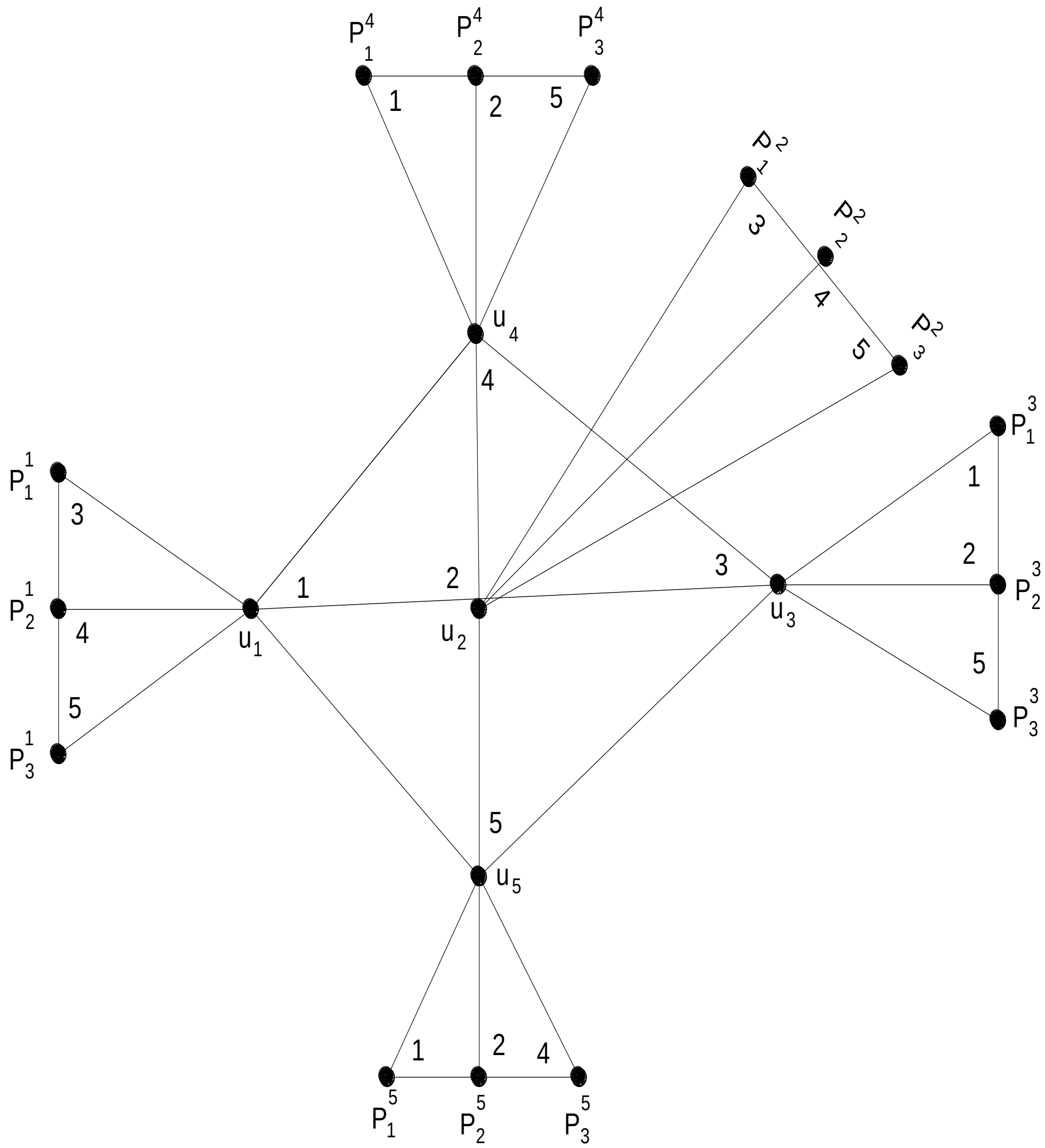

Fig 5.4: $\varphi[F_{2,3}\circ P_3] = 5$

5.6 The b-coloring of vertex corona product of $\varphi[P_n \circ B(K_n, K_n)]$

Algorithm 5.6.1:

Input: $P_n \circ B(K_n, K_n), n \geq 3$.

$V \leftarrow \{p_1, p_2, p_3, ..., p_n, x^1{}_1, x^1{}_2, ..., x^n{}_n, y^1{}_1, y^1{}_2, ..., y^n{}_n\}$.

for $i = 1$ to n

$p_i \leftarrow i;$

end for

for $i = 1$ to n, $j = 1$ to n-1, $k = 1$ to n

$x^i{}_j \leftarrow k;$

end for

for j=n

$x^i{}_j \leftarrow n + 1;$

end for

for $i = 1$ to n, $j = 1$ to n-1, $k = 1$ to n

$y^i{}_j \leftarrow k;$

end for

for $j = 1$

$y^i{}_j \leftarrow n + 2;$

end for

end procedure

output: vertex colored $P_n \circ B(K_n, K_n)$.

Theorem 5.6.1: For any path graph P_n and barbell graph $B(K_n, K_n)$, the b-chromatic number is $\varphi[P_n \circ B(K_n, K_n)] = n+2, n \geq 3.$

Proof:

Let $V(P_n) = \{p_i : 1 \leq i \leq n\}$ and

$V[B(K_n, K_n)] = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$

By the definition of corona product each vertex of P_n is adjacent to every vertex of number of copies of $B(K_n, K_n)$, then the vertex set of the corona product $P_n \circ B(K_n, K_n)$ is,

$$V[P_n \circ B(K_n, K_n)] = \{p_i : 1 \leq i \leq n\} \cup \{x^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{y^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Assign the colors as per the algorithm.

By this coloring procedure, we have that $\varphi[P_n \circ B(K_n, K_n)] \geq n+2.$

To prove the lower bound, let us assume that, the b-chromatic number of corona product of path graph P_n with barbell graph $B(K_n, K_n)$ is greater than n+2.

That is the b-chromatic number of $P_n \circ B(K_n, K_n)$ is equal to $n+3$.

We can assign $n+3$ colors only if the graph having $n+3$ vertices assigned with $n+3$ distinct colors which are adjacent to each other.

Here the colour class c_{n+3} is not adjacent to the color class $c_{n+2}, c_{n+1} \& c_n$, which is the contradiction.

Therefore assigning $n+3$ colors is not possible.

$$\therefore \varphi[P_n \circ B(K_n, K_n)] \leq n+2$$

Hence $\varphi[P_n \circ B(K_n, K_n)] = n+2$.

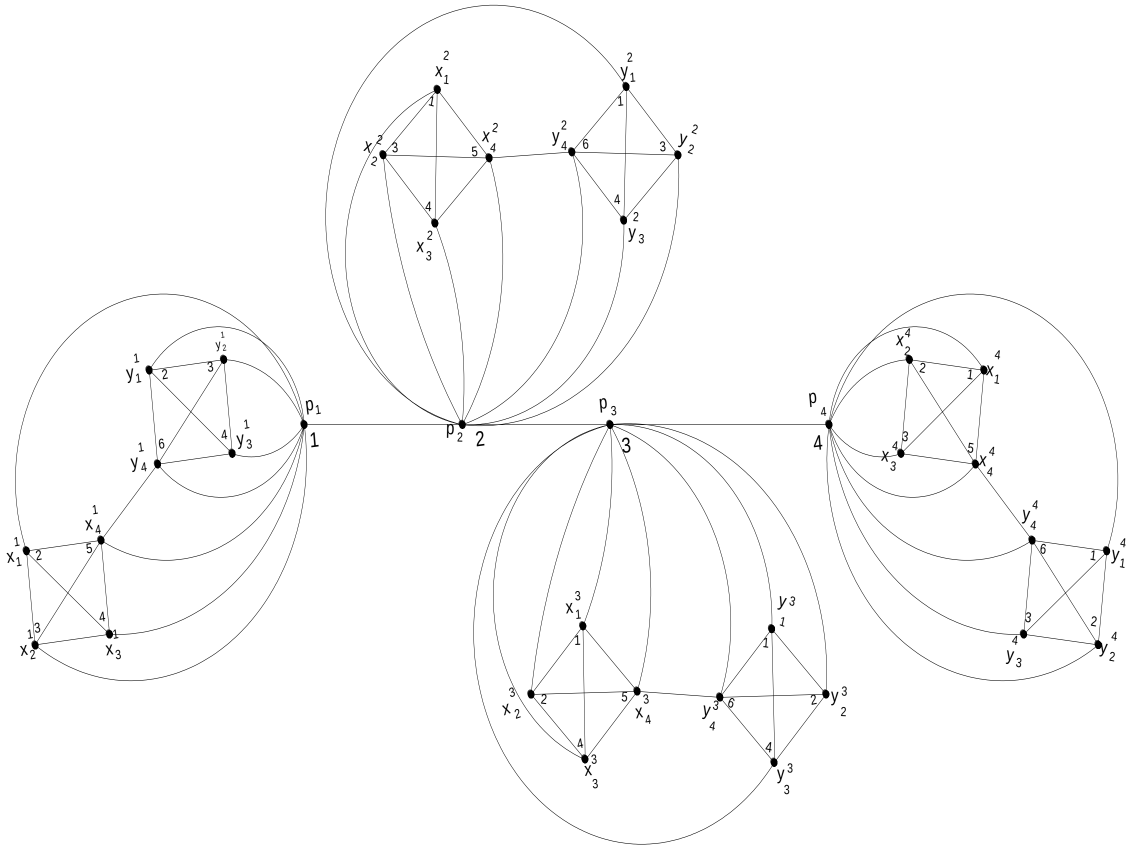

Fig 5.5: $\varphi[P_4 \, oB(K_4, K_4)] = 6$