Transient Analysis Of Markovian Queue With Additional Service

By

N. Mala

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN - DEEMED UNIVERSITY, COIMBATORE - 641 043

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Introduction

INTRODUCTION

Reading furnishes the mind only with materials of knowledge,

it is thinking that makes what we read ours.

- **JOHN LOCKE**

Queueing theory is a branch of applied mathematics utilizing concepts from the field of stochastic processes. The queueing theory had its origin in 1909. When A.K. Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic. The study of queueing systems finds application in a variety of real life situation like business, industry, engineering, transportation, communication, computer and consumer activities.

QUEUEING SYSTEM :

A Queue is a waiting line of " customers" (units) requiring service from one or more "servers" (Service facility). In our day-today life we have been a "Unit" in one or other of the following waiting lines: registration for the school term, ticket booth ,at a movie theatre, teller window at a bank and so on. Queueing models also are applicable to the arrival of rainfall to dams via rivers, arrivals of fire calls to fire departments and money into and out of bank accounts.

CHARACTERSISTICS OF QUEUEING SYSTEMS :

The basic characteristics of a queueing system are:

1. Arrival Process
2. Service Mechanism
3. Queue Discipline
4. System Capacity
5. Service Channels

ARRIVAL PROCESS :

The arrival process characterises the arrival of units into the queueing system. The calling source can consists of a finite or infinite number of units. Arrival of unit can occur either singly or in bulk. Total, partial or no control of arrivals can be exercised by the queueing system. The arrival pattern is measured in terms of the mean arrival rate or mean inter arrival time.

SERVICE MECHANISM :

Service mechanism describe the manner in which service is rendered. Customer may be served either singly or in batches. The queueing system, where the service is done in batches, is called bulk service queueing system. The time required for servicing a unit (or group) is called service time.

Sometimes, the service rate may depend on the number of customers waiting for service. When the queue becomes longer, a server may work faster or conversely, he may become less efficient. The situation in which service depends on the number of waiting customers is known as state dependent service.

QUEUE DISCIPLINE :

Queue discipline refers to the behaviour of arriving units both in the selection (or rejection) of a waiting line and in the act of waiting. The most common queue discipline is first-in-first-out **(FIFO)** or first-come-first-served **(FIFS)** is often dictated by "fairness", as in the ticket numbering system or bakeries, butcher shops, or delicatessens.

Another queue discipline is last-in-first-out **(LIFO)** or last-come-first-served **(LCFS)** can be illustrated by an elevator queueing system whereby the last customers to enter are the first to exit in any given floor. Customers may also be served randomly irrespective of their arrivals into the system as in the selection of Bingo numbers. This type of queue discipline is called service in random order **(SIRO).** Another discipline is priority queue discipline, which allows service to be offered to customers depending on their priority in relation to other customers. There are two types in priority discipline viz., Preemptive priority and non-preemptiive priority. In the preemptive case, the customer with high priority is allowed to enter service immediately, suspending the service in progress to a customer with lower priority. In non-preemptive case, the higher priority goes to the head of the queue but gets into service only after the completion of service in progress to the customer with lower priority.

SYSTEM CAPACITY :

Some of the queueing processes admit the physical limitation to the amount of waiting room, so that when the waiting line reaches the maximum room capacity, no further customer is allowed to enter until space becomes available by a service completion.

SERVICE CHANNELS :

The service facility can have one or multiple channels. Multiple channels can be parallel, in series or both. Channels in parallel can be co-operative (idle servers help busy servers) or un-cooperative. As in the arrival process, service times can be deterministic or probabilistic. State dependent service parameters refer to cases where the mean, standard deviation, or other parameters of a service - time distribution are affected or changed by the state of the system that is number of units in the system. Breakdowns amoung servers also can be considered.

NOTATION :

Kendall proposed a convenient notation to denote queueing system. A queueing process is described by the notation A/B/C/X/Y.

A : Represents the interarrival time distribution of the customers from the source population.

B: The service time distribution of the given service facility .

C: The number of parallel service channels.

X: The capacity of the system and

Y: The type of queue discipline.

A and B usually take one of the following symbols.

M: for exponential (Markovian) distribution

E_k: for Erlang - k distribution

G: for arbitrary (General) distribution

D: for fixed (deterministic) distribution

For example, M / M / C / ∞/ FIFO represents a queueing system having Poisson input, exponential service, C service channels and there is no limit on the system capacity while the customers are served following first-in - first - out queue discipline. In practice, this system is represented as M/M/C. That is, if the system capacity is infinite and the queue discipline is FIFO then the corresponding symbols can be omitted from the system representation.

Synopsis

SYNOPSIS

In this disseration,a study has been made to analyse the transisent behaviour of some Markovian queueing system.

The first chapter deals with the preliminary definitions of Markovian, Non-Markovian, transisent and steady state. Some relevant literatures are also presented.

In the second chapter an attempt has been made to analyse finite capacity queueing model with state dependent service. Laplace transforms of the difference differential equations are expressed in the matrix form . Applying Cramer's rule the time dependent probabilities are derived.Further the steady state probabilities are deduced.

In the third chapter we derive the transient probabilities of queueing system having a variable number of channels whose minimum number is one which increases with the length of the queue. When the queue size increases to some pre-assigned fixed number k, then with each arriving unit a new channel is made available.Here the number of additional channel is limited to r and the room capacity is limited to N.The steady state probabilities are also derived.

Chapter I

CHAPTER - 1

SOME PRELIMINARY DEFINITIONS

MARKOVIAN AND NON-MARKOVIAN :

Queueing models are classified into Markovian queueing models and Non-Markovian queueing models. A queueing model is called Markovian, if both arrival process and service process follow Poisson distribution. Models in which arrivals and / or departures do not follow the Poisson process are called Non - Markovian.

The differential - difference equation method and matrix - geometric method are used to solve Markovian queues. Phase technique, the technique of imbedded Markov-chain and the supplementary variable technique are usually employed in studying Non-Markovian queues.

TRANSIENT AND STEADY STATE QUEUEING SYSTEM:

A queueing system is said to be in transient state, when its operating characteristics (such as input,output, mean queue length etc.,) are dependent on time. Otherwise the system is said to be in steady state or equilibrium state. Solution of a queueing system depending upon time is called a transient solution and independent of time, is called steady state solution.

Many applications of queueing theory involve queues which are emptied and restarted periodically (for example, banks, barber shops and traffic signals). These queues will never reach equilibrium state. Hence steady state solutions are not always adequate, and it is desirable to have time-dependent solution.

Most of the analysis of queueing models are confined to steady state results. Very little seems to have been done to evaluate the corresponding transient results.This is because the transient solution are, in general, mathematically more involved.

Some of the methods used to study the transient behaviour of queues are given below.

(1) Spectral method of Ledermann and Reuter (13)

(2) Combinational method of Champernowne (4)

(3) Difference equation technique of Conolly(5)

(4) Method of Parthasarathy (17) by defining generating function in a special way and using properties of Bessel functions.

(5) Method of Sharma(22) by using the properties of real, symmetric and tridiagonal matrix.

RELEVANT LITERATURE SURVEY:

Queueing theory had its origin in 1909, when Danish telephone engineer, Agner Krarup Erlang published his paper ' The theory of probabilities and telephone conversations '. It is interesting and also important to note that this field is continuously presenting challenging problems to the many capable investigators working in the field throughout the world. One of the most fruitful areas of applied probability theory for computer science application is that of queueing theory.

QUEUES WITH ADDITIONAL SERVERS:

In many situations, when there are too many people waiting to be infront of service facility, the system opens another service facility to reduce congestion. This happens normally in banks, booking offices and supermarkets.

Singh (20) has discussed a Markovian queue with numbers of servers depending upon queue length. Garg and Khanna(6) have considered the steady state behaviour of a queueing system with queue dependent additional serves facility,where in arrivals occur in batches of variable size. Bindhi Singh(3) and Murari(14, 15) has discussed queue with additional servers. There are many queueing situations where the number of servers depends upon the queue length. Whenever the queue in front of a server exceeds a pre-assigned number, an additional server may be employed. The additional server may be withdrawn after the completion of service on hand, either if the queue length decreases to the pre-assigned number or if the queue length becomes zero.

In real situations, it is common that the executive concerned is obliged to offer special service facility to his customers for fear of loosing their goodwill. Systems with the number of servers depending on the queue length may be observed in the banks during the peak days of transaction every month, in tax offices for a period before the last date for submission of the returns or payments, at transport facilities during the festival seasons and in many other places.

Murari(14) has considered the queueing problem with service in batches of variable size, Poisson input and exponential service time distribution, when the queue length increases to some pre-assigned fixed number N, a special service channel capable of serving a group of N units is made available and is cancelled at the termination of service, if the queue length becomes less than N. The room capacity is limited to k. The author has obtained the Laplace transforms of various probability generating function and the coressponding results for steady state.

Bindhi Singh(3) has analysed a limited waiting space queueing system, wherein there is a regular service facility serving the units one by one. A search for an additional service facility for serving a group of unit starts, when the queue length reaches the maximum size. The search continues till the queue length reduces to some tolerable fixed size. The author has obtained the Laplace transform of the transient probabilities.

Singh(20) has presented a paper on queue dependent servers. He has obtained the steady state results for the system with homogeneous servers and also for the system with heterogeneous servers.

Romani(19)has derived the steady state probabilities of queueing system having a variable number of channels whose minimum number is one and which increases with the length of the queue. When the queue size increases to some pre-assigned fixed number N, then with each arriving unit, a new channel is made available. Channels are cancelled at the termination of service if there are no units waiting , with exception of one channel which remains open at all times.

Philips(18)has modified the Romani's model, by limiting the number of new channels to M.When M channels are operating,the queue is allowed to grow without limit.He has obtained the steady state probabilities.

Murari(15)has considered the M/M/1/K queueing system with c-additional homogeneous servers.When the queue length increases to come pre-assigned number m (1), then another channel is added. Inspite of two service channels operating,if the queue length increases to the number m(2) > m(1), the third channel is called and so on.The author has derived the Laplace transforms of transisent probablities and also the steady state probabilities.

Krishna Reddy, Nadarajan and Kandasamy (12) have analysed the Markovian general bulk service queueing systems with vacation and additional server.They have obtained steady state results using matrix geometric method.

Chapter II

CHAPTER - II

MARKOVIAN QUEUE WITH STATE DEPENDENT SERVICE

In this chapter we discuss the Markovian queue with state dependent servers. The transient and steady state probabilities are derived.

MODEL DESCRIPTION

Assume that the customers arrive individually at a service facility in accordance with a Poisson process having rate λ. The queue discipline is FIFO. The service facility consists of single server. The service time follows exponential distribution with mean μ_n

$$\mu_n = \begin{cases} \mu \ \ if \ 1 \le n \le M-1 \\ \mu_1 \ \ if \ M \le n \le N \end{cases}$$

The server starts the service with rate μ. If the system size increases to M, then the server switch over to the rate μ_1.

GOVERNING EQUATIONS

Let Pn (t) be the probability that there are n - units in the system at time t = 0.

The governing differential difference equations are as follows:-

$$p_o'(t) = -\lambda p_o(t) + \mu p_1(t)$$

$$p_n'(t) = -(\lambda + \mu) p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \quad ; 1 \le n \le M-2$$

$$p_{M-1}'(t) = -(\lambda + \mu) p_{M-1}(t) + \lambda p_{M-2}(t) + \mu_1 p_M(t)$$

$$p_n'(t) = -(\lambda + \mu_1) p_n(t) + \lambda p_{n-1}(t) + \mu_1 p_{n+1}(t) \quad ; M \le n \le N-1$$

$$p_N'(t) = -\mu_1 p_N(t) + \lambda p_{N-1}(t)$$

Taking Laplace transform of the above equations we get,

$$(s+\lambda) p_0(s) = \mu p_1(s) + \delta_{i,o} \quad \rightarrow (1)$$

$$(s + \lambda + \mu) p_n(s) = \lambda p_{n-1}(s) + \mu p_{n+1}(s) + \delta_{i,n} \quad ; 1 \le n \le M-2 \quad \rightarrow (2)$$

$$(s + \lambda + \mu) p_{M-1}(s) = \lambda p_{M-2}(s) + \mu_1 p_M(s) + \delta_{i,M-1} \quad \rightarrow (3)$$

$$(s + \lambda + \mu_1) p_n(s) = \lambda p_{n-1}(s) + \mu_1 p_{n+1}(s) + \delta_{i,n} \quad ; M \le n \le N-1 \rightarrow (4)$$

$$(s + \mu_1) p_N(s) = \lambda p_{N-1}(s) + \delta_{i,N} \quad \rightarrow (5)$$

where $\delta_{i,j}$ in the Kronecker delta and

$$p_n(s) = \int_0^\infty e^{-st} p_n(t)\, dt.$$

	0	1	2	3	. . .	M − 1	M	M + 1	. . .	N-1	N
0	$s + \lambda$	$-\mu$									
1	$-\lambda$	$s+\lambda+\mu$	$-\mu$								
2		$-\lambda$	$s+\lambda+\mu$	$-\mu$							
3			$-\lambda$	$s+\lambda+\mu$							
⋮											
M - 1						$s+\lambda+\mu$	$-\mu$				
M						$-\lambda$	$s+\lambda+\mu_1$	$-\mu_1$			
M + 1							$-\lambda$	$s+\lambda+\mu_1$			
⋮											
N-1										$s+\lambda+\mu_1$	$-\mu_1$
N										$-\lambda$	$s+\mu_1$

Co- efficient matrix A(s)
fig. 1

	1	2	3	…	M −1	M	M +1	…	N-1	N
1	$s+\lambda+\mu$	$-\sqrt{\lambda\mu}$								
2	$-\sqrt{\lambda\mu}$	$s+\lambda+\mu$	$-\sqrt{\lambda\mu}$							
3		$-\sqrt{\lambda\mu}$	$s+\lambda+\mu$							
⋮										
M −1					$s+\lambda+\mu$	$-\sqrt{\lambda\mu}$				
M					$-\sqrt{\lambda\mu}$	$s+\lambda+\mu_1$	$-\sqrt{\lambda\mu_1}$			
M+1						$-\sqrt{\lambda\mu_1}$	$s+\lambda+\mu_1$			
⋮										
N-1									$s+\lambda+\mu_1$	$-\sqrt{\lambda\mu_1}$
N									$-\sqrt{\lambda\mu_1}$	$s+\mu_1$

Symmetric matrix D (s)
fig.2

TRANSIENT PROBABILITIES :

Equations (1) through (5) can be expressed in the matrix form as

$$A(s).\ P(s) = P(0) \rightarrow (6)$$

where $P(s) = (p_o(s)\ ,\ p_1(s)\ ,\ \ldots\ldots\ldots\ p_N(s))$

$$P(0) = (\delta_{i,0},\ \delta_{i,0},\ \ldots\ldots\ldots\ \delta_{i,N})$$

and the co-efficient matrix A(s) is presented in fig.1. Applying some elementary row and column transformation on $|A(s)|$ we obtain $|A(s)| = s\ |D(s)|$ where D(s) is the symmetric tridiagonal matrix of order N with negative off diagonal elements as presented in fig.2. The roots of the polynomial $|D(s)|$ are the negatives of the eigenvalues of the matrix D obtained by putting s = 0 in the matrix D(s). Since the matrix D is positive, definite, real and symmetric, the eigenvalues of D are real, distinct and positive. Hence the roots of the polynomial $|A(s)|$ are real; distinct and negative.

Let the roots be $s_o (= 0),\ s_1,\ s_2,\ \ldots\ldots\ s_N$

Then $A(s) = s(s- s_1)\ (s - s_2)\ldots\ldots\ldots\ (s - s_N)$ and

$$p_n(s) = \frac{|A_{n+1}(s)|}{s(s-s_1)\ldots..(s-s_N)} \rightarrow (7)$$

where $|A_{n+1}(s)|$ is the determinant of the matrix obtained by replacing the $(n+1)^{th}$ column of A(s) by the column of A(s) by the column vector P(0).

Let Tr(s) and Br(s) be the determinants of the left and bottom right $r \times r$ matrices obtained from the matrix A(s). Without loss of generality we can take

To(s) = Bo (s) = 1.

Now we express $|A_{n+1}(s)|$ in terms of Tr(s) and Br(s) as follows.

For $0 \leq i \leq M-1$, we have

$$|A_{n+1}(s)| = \mu^{i-n} \; B_{N-i}(s) \; T_n(s) \quad ; \; 0 \leq n \leq i$$

$$= \lambda^{n-i} \; B_{N-n}(s) \; T_i(s) \quad ; \; i+1 \leq n \leq N$$

$\rightarrow$ (8)

For $M \leq i \leq N$, we have

$$|A_{n+1}(s)| = \mu^{M-1-n} \; \mu_1^{i-(M-1)} \; B_{N-i}(s) \; T_n(s) \; ; \; 0 \leq n \leq M-1$$

$$= \mu_1^{i-n} \; B_{N-i}(s) \; T_n(s) \quad ; \; M \leq n \leq i$$

$$= \lambda^{n-i} \; B_{N-n}(s) \; T_i(s) \quad ; \; i+1 \leq n \leq N$$

$\rightarrow$ (9)

Now we express $|A_1(s)|$ as follows

$$|A_1(s)| = \mu^{i} \; B_{N-i} \; (s) \quad ; \; 0 \leq i \leq M-1$$

$$= \mu^{M-1} \; \mu_1^{\,i-(M-1)} \; B_{N-i} \; (s) \quad ; \; M \leq i \leq N$$

From the expressions given by (8) and (9) we see that the degree of $|A_{n+1}(s)|$ is strictly less than the degree of $|A(s)|$.

Hence the right handside of (7) can be revolved into partial fractions as

$$p_n(s) \quad = \quad \frac{c_n}{s} + \sum_{k=1}^{N} \frac{c_{n,k}}{s-s_k} \qquad \rightarrow (10)$$

where $c_n \quad = \quad \lim_{s\to 0} \; s\, p_n(s)$ and

$$c_{n,k} \quad = \quad \lim_{s \to s_k} (s - s_k)\, p_n(s)$$

Applying Laplace inverse transform on (10) we obtain

$$p_n(t) \quad = \quad c_n + \sum_{k=1}^{N} c_{n,k} \exp\{s_k (t)\}, \; n = 0, 1, \ldots\ldots N \qquad \rightarrow (11)$$

STEADY STATE PROBABILITIES :

Denoting by p_n, the probability of n - customers being in the system in the equilibrium state, we have

$$p_n \quad = \quad \lim_{t\to\infty} p_n(t)$$

$$= \quad \lim_{s\to o} s\, p_n(s)$$

Using equations (7) and (10) we obtain

$$p_n \quad = \quad \lim_{s\to\infty} s\, p_n(s)$$

$$= \quad \frac{|A_{n+1}(0)|}{\underset{i=1}{\overset{N}{\pi}} (-si)}$$

Therefore $p_o = \dfrac{|A_1(0)|}{\overset{N}{\underset{i=1}{\pi}} (-s_i)}$

Hence $p_n = \dfrac{|A_{n+1}(0)|}{|A_1(0)|} p_o$

Using the definitions of Tr(s) and Br(s), we have the following expressions for

Tr(0) and Br(0) and $|A_1(0)|$

$$Tr(0) = \lambda^r \quad \text{for} \quad r = 0, 1, 2, \ldots\ldots$$

$$B_{N-n}(0) = \mu^{M-1-n} \mu_1^{N-M+1} \quad ; \ 0 \le n \le M-1$$

$$= \mu_1^{N-n} \quad ; \ M \le n \le N$$

$$|A_1(0)| = \mu^i B_{N-i}(0) \quad ; \ 0 \le i \le M-1$$

$$= \mu^{M-1} \mu_1^{i-(M-1)} B_{N-i}(0) \quad ; \ M \le i \le N$$

We now derive the steady state probabilities p_n for all values of n in terms of p_o.

There are two cases namely.

(i) $0 \leq n \leq M-1$

(ii) $M \leq n \leq N$

Case : (i) $0 \leq n \leq M-1$

Then the value of i may be in any one of the following intervals.

$$0 \leq i \leq n \;;\; n \leq i \leq M-1 \;;\; M \leq i \leq N$$

Now assume $0 \leq i \leq n$

$$\text{Then } p_n = \frac{|A_{n+1}(0)|}{|A_1(0)|} p_o$$

$$= \frac{\lambda^{n-i} \; T_i(0) \; B_{N-n}(0)}{\mu^i \; B_{N-i}(0)} p_o$$

$$= \frac{\lambda^{n-i} \; \lambda^i \; \mu^{M-1-n} \; \mu_1^{N-M+1}}{\mu^i \; \mu^{M-1-i} \; \mu_1^{N-M+1}} p_o$$

$$= \frac{\lambda^n}{\mu^n} p_o$$

It is easy to prove that the expression for p_n ; $0 \leq n \leq M-1$ is the same for all the remaining values of i.

Case : (ii) . $M \le n \le N$

Then the value of i may be in any one of the following intervals.

$0 \le i \le M-1$; $M \le i \le n$; $n \le i \le N$

Let us derive the expression of p_n by assuming $0 \le i \le M-1$

$$\text{Then } p_n = \frac{|A_{n+1}(0)|}{|A_1(0)|} p_o$$

$$= \frac{\lambda^{n-i} \quad B_{N-n}(0) \; Ti(0)}{\mu^{M-1} \mu_1^{\,i-(M-1)} \; B_{N-i}\;(0)} p_o$$

$$= \frac{\lambda^{n-i} \lambda^{i} \mu_1^{\,N-n}}{\mu^{M-1} \mu_1^{\,i-(M-1)} \; \mu_1^{\,N-i}} p_o$$

$$= \frac{\lambda^{n}}{\mu^{M-1} \mu_1^{\,i-M+1-i+n}} p_o$$

$$= \frac{\lambda^{n}}{\mu^{M-1} \mu_1^{\,n-M+1}} p_o$$

It is easy to prove that the expression for p_n ; $M \le n \le N$ is the same for all the remaining values of i.

Accordingly we have the steady state probabilities p_n as

$$p_n = \frac{\lambda^n}{\mu^n} p_o \qquad ; 0 \le n \le M-1$$

$$= \frac{\lambda^{n}}{\mu^{M-1} \mu_1^{\,n-M+1}} p_o \qquad ; M \le n \le N.$$

Chapter III

CHAPTER III

MARKOVIAN QUEUE WITH r - ADDITIONAL SERVERS

In this chapter M/M/1/N queueing system with r - additional homogenous service channels is studied. It is assumed that there are i - units in the system at time t = 0. The transient and steady state probabilities are derived.

MODEL DESCRIPTION :

Assume that the customers arrive individually at a service facility in accordance with a Poisson process having rate λ. The queue discipline in FIFO. The service facility consists of one regular service channel and r - additional service channels.

The system starts with regular channel having service rate μ. The regular channel is always open irrespective of queue length. If the system is empty, then the regular channel will be idle. If the number of customers in the system increases to one then the regular channel will become busy. The number of additional service channels at any instant is dependent on the queue length. If the number of customers in the system increases to k+1, then one additional service channel with service rate μ_1 is started, which will be dropped at the termination of service, if the system size becomes k. When two channels are operating, the number of units in the system increases to k+2, the second additional service channel with service rate μ_1 is opened, which will be dropped at the termination of service, if the system size becomes k +1 and so on. The number of additional channel is limited to N. It is assumed that initially there are i - customers in the system.

GOVERNING EQUATIONS

Let Pn(t) be the probability that at time t, the system size in n.

The governing differential difference equations are as follows:

$$p_o'(t) = -\lambda p_o(t) + \mu p_1(t)$$

$$p_n'(t) = -(\lambda + \mu) p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \quad ; \quad 1 \leq n \leq k-1$$

$$p_n'(t) = -(\lambda + \mu + \overline{n-1}\,\mu_1)\, p_n(t) + \lambda p_{n-1}(t) + (\mu + \overline{n-k+1}\,\mu_1)\, p_{n+1}(t)$$

$$; \quad k \leq n \leq k+r-1$$

$$p_n'(t) = -(\lambda + \mu + r\mu_1)\, p_n(t) + \lambda p_{n-1}(t) + (\mu + r\mu_1)\, p_{n+1}(t)$$

$$; \quad k+r \leq n \leq N-1$$

$$p_N'(t) = -(\mu + r\mu_1) p_N + \lambda p_{N-1}$$

Laplace transform of the above set of equations are given below.

$$(s+\lambda)p_0(s) = \mu p_1(s) + \delta_{i,o} \quad \rightarrow (1)$$

$$(s+\lambda+\mu)\, p_n(s) = \lambda p_{n-1}(s) + \mu p_{n+1}(s) + \delta_{i,n}$$

$$;\ 1 \le n \le k-1 \quad \rightarrow (2)$$

$$\left(s+\lambda+\mu+\overline{n-k}\,\mu_1\right) p_n(s) = \lambda p_{n-1}(s) + (\mu + \overline{n-k+1}\,\mu_1) p_{n+1}(s) + \delta_{i,n}$$

$$;\ k \le n \le k+r-1 \quad \rightarrow (3)$$

$$(s+\lambda+\mu+r\mu_1)\, p_n(s) = \lambda\, p_{n-1}(s) + (\mu + r\mu_1) p_{n+1}(s) + \delta_{i,n}$$

$$;\ k+r \le n \le N-1 \quad \rightarrow (4)$$

$$(s+\lambda+\mu+r\mu_1)\, p_N(s) = \lambda p_{N-1}(s) + \delta_{i,N} \quad \rightarrow (5)$$

where $\delta_{i,j}$ is the Kronecker delta and

$$p_n(s) = \int_0^\infty e^{-st}\, p_n(t)\, dt$$

	0	1	2	. . .	k	k +1	. . .	k+r	k+r+1	. . .	N-1	N
0	$s+\lambda$	$-\mu$										
1	$-\lambda$	$s+\lambda+\mu$	$-\mu$									
2		$-\lambda$	$s+\lambda+\mu$									
⋮												
k					$s+\lambda+\mu$	$-(\mu+\mu_1)$						
k+1					$-\lambda$	$s+\lambda+\mu+\mu_1$						
⋮												
k+r								$s+\lambda+\mu+r\mu_1$	$-(\mu+r\mu_1)$			
k+r+1								$-\lambda$	$s+\lambda+\mu+r\mu_1$			
⋮												
N-1											$s+\lambda+\mu+r\mu_1$	$-(\mu+r\mu_1)$
N											$-\lambda$	$s+\mu+r\mu_1$

Co - efficient matrix A(s)
fig.3

	1	2	...	k	k +1	...	k+r	k+r+1	...	N-1	N
1	$s+\lambda+\mu$	$-\sqrt{\lambda\mu}$									
2	$-\sqrt{\lambda\mu}$	$s+\lambda+\mu$									
⋮											
k				$s+\lambda+\mu$	$-\sqrt{\lambda(\mu+\mu_1)}$						
k+1				$-\sqrt{\lambda(\mu+\mu_1)}$	$s+\lambda+\mu+\mu_1$						
⋮											
k+r							$s+\lambda+\mu+r\mu_1$	$-\sqrt{\lambda(\mu+r\mu_1)}$			
k+r+1							$-\sqrt{\lambda(\mu+r\mu_1)}$	$s+\lambda+\mu+r\mu_1$			
⋮											
N-1										$s+\lambda+\mu+r\mu_1$	$-\sqrt{\lambda(\mu+r\mu_1)}$
N										$-\sqrt{\lambda(\mu+r\mu_1)}$	$s+\mu+r\mu_1$

Symmetric matrix D(s)

fig. 4

TRANSIENT PROBABILITIES

Equations (1) through (5) can be expressed in the matrix form as

$$A(s)\,P(s) \quad = \quad P(0) \qquad \rightarrow (6)$$

where $P(s) = (p_0(s), p_1(s), \dots\dots\dots\, p_N(s))$

$P(o) = (\delta_{i,o}, \delta_{i,1}, \dots\dots\dots\, \delta_{i,N})$

and the co-efficient matrix A(s) is presented in fig . 3.

Applying some elementary row and column transformation on $|A(s)|$ we obtain $|A(s)| = s\,|D(s)|$ where D(s) is the symmetric tridiagonal matrix of order N with negative off diagonal elements as presented in fig.4. The roots of the polynomial $|D(s)|$ are the negatives of the eigenvalues of the matrix D obtained by putting s = 0 in the matrix D(s). Since the matrix D is positive, definite, real and symmetric, the eigenvalues of D are real, distinct and positive. Hence the roots of the polynomial $|A(s)|$ are real, distinct and negative.

Let the roots be s_0 (= 0), $s_1, s_2, \dots\dots s_N$

Then $A(s) = s(s-s_1)(s-s_2)\dots\dots\dots\dots(s-s_N)$ and

$$p_n(s) = \frac{|A_{n+1}(s)|}{s(s-s_1)\,\dots\dots\,(s-s_N)} \qquad \rightarrow (7)$$

where $|A_{n+1}(s)|$ is the determinant of the matrix by replacing the $(n+1)^{th}$ column of A(s) by the column of A(s) by the column vector P(0).

Let $T_r(s)$ and $B_r(s)$ be the determinants of the left and bottom right $r \times r$ matrices obtained from the matrix A(s). Without loss of generality, we can take $T_0(s) = B_0(s) = 1$.

Now, we express $|A_{n+1}(s)|$ in terms of $T_r(s)$ and $B_r(s)$ as follows.

When $0 \leq i \leq k$ we have

$$|A_{n+1}(s)| = \mu^{i-n} B_{N-i}(s) \; T_n(s) \quad ; \quad 0 \leq n \leq i$$

$$= \lambda^{n-i} B_{N-n}(s) \; T_i(s) \quad : \quad i+1 \leq n \leq N$$

$\rightarrow (8)$

When $k+1 \leq i \leq k+r$ we have

$$|A_{n+1}(s)| = \mu^{k-n} (\mu+\mu_1)(\mu+2\mu_1)\ldots\ldots(\mu+\overline{i-k}\mu_1) B_{N-i}(s)\, T_n(s) \quad ; \quad 0 \leq n \leq k$$

$$= (\mu+\overline{n-k+1}\mu_1)(\mu+\overline{i-k}\mu_1) B_{N-i}(s)\, Tn(s) \quad ; \quad k+1 \leq n \leq i$$

$$= \lambda^{n-i} B_{N-n}(s)\, Ti(s) \quad ; \quad i+1 \leq n \leq N$$

$\rightarrow (9)$

When $k + r + 1 \leq i \leq N$ we have

$$|A_{n+1}(s)| = \mu^{k-n}(\mu + \mu_1)(\mu + 2\mu_1) ..... (\mu + r\mu_1)(\mu + r\mu_1)^{\overline{i-k+r}} B_{N-i}(s) T_n(s) \quad ; \quad 0 \leq n \leq k$$

$$= (\mu + \overline{n-k+1}\,\mu_1) ..... (\mu + r\mu_1)(\mu + r\mu_1)^{\overline{i-k+r}} B_{N-i}(s)\, T_n(s) \quad ; \quad k+1 \leq n \leq i$$

$$= \lambda^{n-i} B_{N-n}(s)\ T_i(s) \quad ; \quad i+1 \leq n \leq N$$

$\rightarrow$ (10)

Now we express $|A_1(s)|$ as follows :

$$|A_1(s)| = \mu^i B_{N-i}(s) \quad ; \quad 0 \leq i \leq k$$

$$= \mu^k(\mu + \mu_1)(\mu + 2\mu_1) ..... (\mu + \overline{i-k}\,\mu_1) B_{N-i}(s) \quad ; \quad k+1 \leq i \leq k + r$$

$$= \mu^k(\mu + \mu_1)(\mu + 2\mu_1) ...... (\mu + r\mu_1)(\mu + r\mu_1)^{\overline{i-k+r}} B_{N-i}(s) \quad ; \quad k + r + 1 \leq i \leq N$$

From the expressions given by (8) through (10) we see that the degree of

$|A_{n+1}(s)|$ is strictly less than the degree of $|A_{n+1}(s)|$ is strictly less than

degree of $|A(s)|$.

Hence the right hand side of (7) can be resolved into partial fractions as

$$p_n(s) = \frac{c_n}{s} + \sum_{k=1}^{N} \frac{c_{n,k}}{s-s_k} \qquad \rightarrow (11)$$

where $c_n = \lim_{s \to 0} s\, p_n(s)$ and

$$c_{n,k} = \lim_{s \to s_k} (s - s_k)\, p_n(s)$$

Applying Laplace inverse transform on (11) we obtain

$$p_n(t) = c_n + \sum_{k=1}^{N} c_{n,k} \exp\{s_k (t)\}, \quad n = 0, 1, \ldots\ldots N \qquad \rightarrow (12)$$

STEADY STATE PROBABILITIES

Denoting by p_n, the probability of n - customers being in the system in the equilibrium state, we have

$$p_n = \lim_{t \to \infty} p_n(t)$$

$$= \lim_{s \to 0} s\, p_n(s)$$

Using equations (7) and (11) we obtain

$$p_n = \lim_{s \to \infty} s\, p_n(s)$$

$$= \frac{|A_{n+1}(0)|}{\underset{i=1}{\overset{N}{\pi}} (-si)}$$

Therefore $p_0 = \dfrac{|A_1(0)|}{\underset{i=1}{\overset{N}{\pi}} (-si)}$

Hence $p_n = \dfrac{|A_{n+1}(0)|}{|A_1(0)|} p_0$

Using the definitions of T_r (s) and B_r (s), we have the following expressions for T_r (0), B_r (0) and $|A_1(0)|$

$$T_r(0) = \lambda^r \quad \text{for} \quad r = 0, 1, 2, \ldots\ldots$$

$$B_{N-n}(0) = (\mu + r\mu_1)^{N-\overline{k+r-1}}(\mu+\mu_1)\ldots\ldots(\mu+\overline{r-1}\mu_1)\mu^{k-n}$$

$$; 0 \le n \le k$$

$$= (\mu + r\mu_1)^{N-\overline{k+r-1}}(\mu+\overline{r-1}\mu_1)\ldots.$$

$$(\mu+\overline{n-k+1}\mu_1)$$

$$; k+1 \le n \le k+r$$

$$= (\mu+r\mu_1)^{N-n} \qquad ; k+r+1 \le n \le N.$$

$$|A_1(0)| = \mu^i B_{N-i}(s) \qquad 0 \le i \le k$$

$$= \mu^k (\mu+\mu_1)(\mu+2\mu_1)\ldots..(\mu+\overline{i-k}\mu_1)B_{N-i}(s)$$

$$; k+1 \le i \le k+r$$

$$= \mu^k(\mu+\mu_1)(\mu+2\mu_1)\ldots\ldots(\mu+r\mu_1)$$

$$(\mu+r\mu_1)^{i-\overline{k+r}} B_{N-i}(s)$$

$$; k+r+1 \le i \le N$$

We now derive the steady state probabilities p_n for all values of n in terms of p_0.

There are three cases namely.

(i) $0 \leq n \leq k$

(ii) $k+1 \leq n \leq k+r$

(iii) $k+r+1 \leq n \leq N$

Case : (i) $0 \leq n \leq k$

Then the value of i may be in any one of the following intervals.

$$0 \leq i \leq n, \quad n \leq i \leq k, \quad k+1 \leq i \leq k+r, \quad k+r \leq i \leq N$$

Now assume $0 \leq i \leq n$

$$\text{Then } p_n = \frac{|A_{n+1}(0)|}{|A_1(0)|} p_0$$

$$= \frac{\lambda^{n-i}\, T_i(0)\, B_{N-n}(0)}{\mu^i\, B_{N-i}(0)} \quad p_0$$

$$= \frac{\lambda^{n-i}\,\lambda^{i}\,B_{N-n}(0)}{\mu^{i}\,B_{N-i}(0)}\quad p_0$$

$$= \frac{\lambda^{n}(\mu+\mu_1)(\mu+2\mu_1)......(\mu+\overline{r-1}\mu_1)(\mu+\mu_1)^{N-\overline{k+r-1}}\mu^{k-n}}{\mu^{k-i}\mu^{i}(\mu+\mu_1)....(\mu+\overline{r-1}\mu_1)(\mu+r\mu_1)^{N-\overline{k+r-1}}}\quad p_0$$

$$= \frac{\lambda^{n}\mu^{k-n}}{\mu^{k}}\quad p_0$$

$$= \frac{\lambda^{n}}{\mu^{n}}\quad p_0$$

It is easy to prove that the expression for p_n ; $0 \le n \le k$ is the same for all the remaining values of i.

Case : (ii) $k+1 \le n \le k+r$

Then the value of i may be in any one of he following intervals.

$$0 \le i \le k, \quad k+1 \le i \le n, \quad n \le i \le k+r, \quad k+r+1 \le N.$$

Assume $0 \le i \le k$

Then p_n $$= \frac{|A_{n+1}(0)|}{|A_1(0)|}\quad p_0$$

$$= \frac{\lambda^{n-i}B_{N-n}(0)\,T_i(0)}{\mu^{i}B_{N-i}\,(0)}$$

$$= \frac{\lambda^{n-i}\, B_{N-n}(0)\, \lambda^{i}}{\mu^{i}\, B_{N-i}(0)} \quad p_0$$

$$= \frac{\lambda^{n}\, (\mu+r\mu_1)^{\overline{N-k+r-1}}\, (\mu+\overline{r-1}\mu_1) \;......\; (\mu+\overline{n-k+1}\mu_1)}{(\mu+r\mu_1)^{\overline{N-k+r-1}}\, (\mu+\mu_1) \;......\; (\mu+\overline{r-1}\mu_1)\, \mu^{k-i}} \quad p_0$$

$$= \frac{\lambda^{n}}{(\mu+\mu_1)(\mu+2\mu_1).....(\mu+\overline{n-k}\mu_1)\mu^{k}} \quad p_0$$

It is easy to prove that the expression for p_n ; $k+1 \le n \le k+r$ is the same for all the remaining values of i.

Case (iii) : $k+r+1 \le n \le N$

Then the value of i may be in any one of the following intervals

$0 \le i \le k$; $k+1 \le i \le k+r$; $k+r+1 \le i \le n$;

$n \le i \le N$;

Assume $0 \le i \le k$

Then $p_n \;=\; \dfrac{|A_{n+1}(0)|}{|A_1(0)|} \; p_0$

$$= \frac{\lambda^{n-i}\, \lambda^{i} B_{N-n}(0)}{\mu^{i} B_{N-i}(0)} \quad p_0$$

$$= \frac{\lambda^n(\mu+r\mu_1)^{N-n}}{\mu^i(\mu+r\mu_1)^{N-\overline{k+r-1}}(\mu+\mu_1)(\mu+2\mu_1)........(\mu+\overline{r-1}\mu_1)\mu^{k-i}}\, p_0$$

$$= \frac{\lambda^n}{\mu^k(\mu+\mu_1)(\mu+2\mu_1)........(\mu+\overline{r-1}\mu_1)(\mu+r\mu_1)^{N-\overline{k+r-1}}}\, p_0$$

It is easy to prove that the expression for p_n ; $k+r+1 \le n \le N$ is the same for all the remaining values of i.

Accordingly we have the steady state probabilities p_n as follows:

$$p_n = \frac{\lambda^n}{\mu^n}\, p_0 \qquad ; \; 0 \le n \le k$$

$$= \frac{\lambda^n}{(\mu+\mu_1)(\mu+2\mu_1).........(\mu+\overline{n-k}\mu_1)\mu^{k-i}}\, p_0$$

$$; \; k+1 \le n \le k+r$$

$$= \frac{\lambda^n}{\mu^k(\mu+\mu_1)(\mu+2\mu_1).......(\mu+\overline{r-1}\mu_1)(\mu+r\mu_1)^{N-\overline{k+r-1}}}\, p_0$$

$$; \; k+r+1 \le n \le N$$

Summary and Conclusion

SUMMARY AND CONCLUSION

For the two model in this disseration, differential difference equations are obtained. Expressing the Laplace transform of the differential difference equations in the matrix form and using the properties of symmetric tridiagonal matrix, the transient and steady state probabilities are derived.

Bibliography

BIBLIOGRAPHY

"Of making many books there is no end ; and

much study in a weariness of the flesh"

1. **Barrer, D.Y (1957) :** Queueing with impatient customers and indifferent clerks, Oper, Res, 5.5.

2. **Bhat, U.N (1969) :** Sixty years of queueing theory, Mgmt. Sci.15.B, 280 - 292.

3. **Bindhi Singh, K (1978) :** A queueing system with random additional service facility, RAIRO, Oper, Res.12, No.3, 311 - 318

4. **Champernowne D.G (1956) :** An elementary method of solution of the queueing problem with a single server and constant parameter, J.Roy, Statist. Soc B 18, 125 - 128.

5. **Conolly, B.W(1958) :** A difference equation technique applied to the simple queue with arbitrary arrival interval distribution, J.Roy Statist Soc. B 21, 168 - 175.

6. **Garg, R.L., and Khanna, S.K (1983) :** Queue dependent additional server queueing problem with arrivals, RAIRO, oper, Res.17, 391 - 398.

7. **Gross.B and Harris.C.M (1974) :** Fundamentals of queueing theory, John Wiley, NewYork..

8. **Homma. T.(1955) :** On a certain queueing process, Rept, Statist, Application Research, Union Japanese Scientists and Engrs 4, No.1.

9. **Homma.T (1956) :** On the many server queueing process with a particular type of queue discipline, Rept, Statist, Application Research Union Japanese Scientists and Engrs 4, 20 - 31.

10. **Kendall, D.G (1951) :** Some problems in the theory of queues, J.Roy statist, Soc B13, 151 - 185.

11. **Kendall, D.G (1953) :** Stochastic processes occuring in the theory of queues and their analysis by the method of embedded Markov chain, Ann. Math. Statist 24. 338 - 354.

12. **Krishna Reddy, G.v., Nadarajan, R. and Kandasamy P.R. (1991) :** Markovian general bulk service queueing system with vacation and additional server, APJOR 8, 80 - 89.

13. **Ledermann, W., and Reuter G.E (1956) :** Spectral theory for the differential equation of simple birth and death processes, Philos. Trans, Raj, Soc. London, A 246, 321 - 369.

14. **Murari, K.(1967) :** An additional special channel, limited spare queueing problem with service in batches of variable size, ZAMM 51, 315 - 318.

15. **Murari, K (1971) :** Some c - additional service channels limited space queueing problem, ZAMM 51, 315 - 318.

16. **Palm, C.(1957) :** Research on Telephone traffic carried by full availability groups, Tele (English ed) No.1.

17. **Parthasarathy, P.R and Sharafali, M (1989) :** Transient solution to the many server Poisson queue. A simple approach, J.Appl. Prob.26, 584 - 594.

18. **Phillips, C.R (1960) :** A variable channel queueing model with a limited number of channels, Thesis, Georgia Institute of Technology.

19. **Romani, J. (1957) :** A queueing model with a variable number of channels, Tarbajos de Estadistica 8, No.3 175 - 189.

20. **Singh V.P (1973) :** Queue dependent servers, Journal of Eng. Math 7, No.2, 123 - 126.

21. **Sharma, O.P and Dass. J (1988) :** Multiserver Markovian queue with finite waiting space, The Indian journal of statistic, volume 50, series B, pt.3, 423 - 431.

22. **Sharma O.P (1990) :** Markovian queues, Ellis Horwood, London.