Erlangian Bulk Service Queueing Model Under Server's Vacation

By

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CONTENTS

INTRODUCTION

CHAPTER - I

Queueing theory is a branch of Operations Research utilizing concepts from the field of Stochastic Processes. The study of queueing systems has increasingly occupied the attention of researchers, since, the annoyance to wait in line (queue) is becoming more and more prevalent in our congested society. The formation of queues occurs whenever the current demand for a service exceeds the current capacity to provide that service.

1.1 Queueing System

The queueing system can be described as customers arriving for service, waiting for service if it is not immediate. The waiting units take their turn according to a pre-assigned rule and leave the system after being served. The ultimate goal of queueing theory is to achieve an economic balance between the cost of service and the cost associated with waiting for that service. The queueing theory contributes vital information required for minimizing cost by predicting various characteristics of the waiting line such as the average waiting time. These predictions help us to anticipate situations and to take appropriate measures to shorten the queues.

The study of queues is mainly applied in the fields such as

Business - Booking Offices, Super Market, Post Offices, Banks ...

Industries - Storage, Serving of automatic machines, Product lines ...

Transportation - Airplanes waiting in stack, Ships at sea waiting for free berths, Traffic jams of cars ...

Engineering - Processing in computers, Communication networks ...

Everyday life - Barber shop,Telephone booth, Theatres, Grocery shops ...

1.2 Characteristics Of Queueing Models

The basic characteristics of a queueing system are the following

1. The arrival process
2. The service process
3. The queue discipline
4. System capacity
5. Service channels

1.2.1 The Arrival Process

The arrival process or input describes the manner in which customers arrive and join the system. The arrival of customers may be deterministic, or as in most of the cases be stochastic. The source from which the customers come may be finite or infinite.

A customer may arrive either singly or in a group (or) batches. In the bulk arrival situation,not only the time between successive arrivals of the batches be probabilistic but also the number of customers in the batch.

If the probabilistic structure of the arrival process does not vary with time , then it is said to be stationary; otherwise, it is said to be non-stationary.

1.2.2 **The Service Process**

The service mechanism describes the manner in which service is rendered. Service may be deterministic (or) probabilistic. Customers may be served singly or in batches. In case of batch service the service system is termed as 'Bulk Service System'. In bulk service system, the batches may be of fixed size or variable size.

There could be number of policies or rules according to which batches for bulk service may be formed. Nuets [21] introduced the **General Bulk Service Rule.** Under such a rule a server starts service only if atleast ' a ' customers are present in the queue, the maximum capacity being ' b' customers. In otherwords , if there are ' n ' customers waiting at the completion of a service , the following rule is followed.

For $0 \leq x < a$	No service.
For $a \leq x \leq b$	Service is done for a batch of x customers.
For $x > b$	Service is done for a batch of b customers and the remaining (x - b) customers continue to wait in the queue.

In particular, if $a = 1$, the above rule will be usual bulk service rule and if $a = b = k$, the rule is then called fixed size bulk service rule.

1.2.3 Queue Discipline

Queue discipline is the rule according to which customers are selected for service when a queue has been formed.

1. The most common queue discipline observed in every day life is First-In-First-Out(**FIFO**)

2. Another queue discipline is the Last-In-First-Out (**LIFO**) which is applicable to some inventory systems.

3. Yet another discipline is the Service-In-Random -Order (**SIRO**) wherein customers are selected at random for service.

4. Sometimes, there are priority queue disciplines where service is offered depending upon the priorities of the customers. In this, we have the "**PRE-EMPTIVE** " situation wherein the customer with a higher priority is allowed to enter service immediately even if a customer of lower priority is already in service, when the higher priority customers enter the system. That is, the service to the lower priority customer is pre-empted, the service stopped, to be resumed again after the higher priority customer is served.

5. In the second situation called "**NON-PREEMPTIVE**", the higher priority customer goes to the head of the queue but can enter service only after the service of the customer under service is completed.

1.2.4 System Capacity

The number of customers, both in the queue and service put together, is called the system capacity. It may be finite or infinite. A system with finite capacity can be viewed as one with forced balking of a customer arriving when the system is to its full capacity.

1.2.5 Service Channels

The number of servers or service channels in a queueing model may be finite or infinite. Depending on the model, if the number of servers is more than one, the customers may form a single queue waiting for service or separate queues in front of each server. In some multiserver queueing system, the service pattern may be parallel and in some others it may be in series.

1. In parallel channels, a number of channels provide identical service facilities, so that several customers may be served simultaneously.

2. In case of series channels, a customer must pass necessarily through the ordered channels, before service is completed.

Queueing system is called single server model, when the system has one server only and when the system has two or more parallel servers, it is known is multiserver model.

1.3 Kendall's Notation

Kendall [15] proposed a convenient notation A/B/C/X/Y, where

`A' represents the interarrival time distribution of the customers,

`B' the service time distribution of the given service facility,

`C' the number of parallel service channels,

`X' the capacity of the system and

`Y' the type of queue discipline.

But in general only the first three symbols are used commonly, unless otherwise stated the queue discipline can be considered as FIFO and the system capacity can be considered infinite. Some of the familiar notations are the following:

M - Exponential (Markovian) distribution

E_k - Erlang - K - distribution

GI - General Input distribution

G - General service time distribution.

Thus $E_k/M_{(a,b)}/C$ means, that the inter-arrival time follows an Erlangian distribution, the service time Exponentially distributed following the general bulk service rule with 'c' servers in the system.

1.4 Queueing Models with Servers Vacation

In some queueing systems a situation may arise where the server is required to be given some rest due to the heavy rush of the arriving customers and continuous serving. In some other cases, such as bulk service models, the required number of customers may not be available for extending service and the server may utilise such situation for taking rest. From practical consideration, it may not be always be worthwile to keep servers unnecessarily idle. In such situations, the server may utilise his idle time in an useful and optimal way to perform additional jobs or for preventive maintenane owrk, if the server is a machine. This is termed as the server's vacation. There are different types of vacation available in the Queueing Theory. Some of them are the following :

1.4.1 Repeated Vacation

In case of bulk service models, a server on completion of a service will start servicing again, if the system has the minimum number of customers required to start the service. Otherwise, the server will withdraw from the system for a vacation. On returning from a vacation, if the server finds less than the required number of customers, he may immediately take another vacation. He will continue in this manner, until he finds, upon returning from a vacation, the required number of waiting customers.

1.4.2 Single Vacation

The assumptions are same, as those of repeated vacations except that, even if the server finds less than the minimum number of customers required for service, when he returns from a vacation, he stays in the system awaiting the queue length to reach the minimum number for starting his next service.

1.4.3 Exceptional First Vacation

In repeated vacations, the duration of the first vacation and the subsequent vacations have the same distributions. In exceptional first vacation, the first vacation is differently distributed from the subsequent vacations.

1.4.4. Gated Vacation

When the server returns from vacation, he accepts only those customers who were waiting, differing the service of subsequent arrivals until after the next vacation. One can imagine that when the server returns from a vacation, a gate closes behind the last waiting customer and the server will serve only those customers in front of the gate before opening it and departing for another vacation.

1.4.5 Random Vacation

The random failure of a server to serve, independent of the status of the queue is regarded as server's vacation. Alternatively, this system can be viewed as a two priority single server queue with break downs (vacations) having a pre-emptive priority over the regular productive work.

1.4.6 Limited Service Vacation

Sometimes, the servermay not be allowed to work more than a specified duration of time or to produce more than specific number of items in case the server is a machine and the server may be allowed to take a vacation or the preventive maintenance work of the machine may be attended to for a limited time. This limited time of non-availability of service is called limited service vacation.

1.5 Some Bulk Service Policies

There could be a number of policies or rules according to which batches for bulk service may be formed. The following are the types of bulk service policies or rules frequently discussed in the literature.

Bailey [4] considers that units are served in batches of not more than \`b', the capacity of the server. If, immediately after the completion of a service, the server finds more than \`b' units waiting, he takes a batch of \`b' for service while others wait; if he finds \`r' units ($0 \leq r \leq b$), he takes all the \`r' units in, a batch for service. Jaiswal [12], Neuts [21] considers the same rule with the restriction that $r \neq 0$ ($1 \leq r \leq b$), (ie) the service facility stops until a unit arrives. This rule is called the usual bulk service rule, while Bailey's rule is called the bulk service rule with intermittently available server. Jaiswal [12] points out that the distribution of the queue length for the modified rule can be obtained from that of the usual rule.

The rule with a fixed batch size \`K' has been considered by Fabens, Takacs and Foster. In this case, the server waits until there are \`K' units.

Neuts [21] and Medhi [19] considers the following bulk service rule: If immediately after the completion of a service, the server finds less than \`a' units present in the queue, he waits until there are \`a' units for service ; if he finds \`a' (or) more but almost \`b', he takes them all in the batch and if he finds more than b, he takes \`b' units in a batch for service, while others wait. The batch takes a minimum of \`a' units and a maximum of \`b' units. This rule is called the general bulk service rule.

The size of a batch may be a variable, that is customers may be served in batches of variable capcity Y, where Y is a random variable, Bhat, Tegham, Cohen examine this rule.

Further, bulk service may be with accessible batches. If a batch being served does not utilise its full capacity for service, it may remain accessible for units arriving during the service time of the batch until its full capacity is attaied, the total service time is not altered by inclusion of such joining units in course of ongoing service. Newell considers such queues as models for traffic light queues.

1.6 Review of Literature

The theory of queues originated in 1909 with the publication of "The theory of probabilities and telephone conversation" by the Darish Mathematician A.K.Erlang (1878-1929). During the last 86 years the literature of the subject has grown tremendously. Being the pioneer researcher, Erlang is called the father of queueing theory.

1.6.1 Bulk Service Queueing Models

The theory of batch service queues originated with the work of Bailey [4]. He considered a queue with Poisson arrivals and fixed queue service.

Jaiswal [12] has analysed the system $M/E_{R(1,b)}/1$, where E_R denotes modified Erlangian distribution having a random number R of exponential phases.Using a modified Erlangian technique, he has obtained the transient and steady state distribution of the queue size and the waiting time distribution in steady state.

Borthakur[5] has obtained the steady state probabilities of the number of customers in the queue and Medhi[19] has derived the waiting time distribution for M/M(a,b)/1 queueing model.

Sim and Templeton [23] have discussed the steady state results for $M/M_{(a,b)}/c$ batch service system.

Shankaranarayanan and Nadarajan [22] have studied the general bulk service queueing model with Erlang input and obtained the steady state probability distribution for the model.

Easton and Chaudhry [9] have obtained steady state results for the system $E_k/M_{(a,b)}/1$ including the waiting time distribution and the busy period distribution.

Afthab Begum[1] has considered $E_k/M_{(a,b)}/1$ queueing model with servers single and multiple vacation.The expression for the average queue length and waiting time distribution are presented in such a way that the computations are easily done making the model potentially useful. Numerical results are also given to compare this model with that of the model without vacation.

Madill and Chaudhry [18] have considered the waiting time moments associated with the more general $GI/M_{(a,b)}/1$ system in the steady state and derived expressions that will permit them to be evaluated efficiently and stably for high values of quorum and capacity.

Chaudhry , Medhi, Sim and Templeton [7] together have considered a two heterogeneous servers Markovian queue with General Bulk Service Rule such that the channel i (i=1,2) will serve customers in batches of size atleast a_i and atmost$_i$ b . They have derived queue length distribution and busy period distribution which are the generalisation of those given by Arora [2], Medhi[19],Borthakur[5].

1.6.2 Vacation Models

During the last two decades, there has been considerable attention paid to the analysis of queueing systems with servers vacation. Recently the interest in such systems has been further enhanced by its applicability in communication, computer and production systems.
Single server queues with vacation periods have been studied by a number of authors. Notable among them are Miller , Cooper, Levy and Yechiali[17], Heyman and Sobel, Shanthikumar,Scholl and Kleinrock, Lee , Keilson and Servi , Fuhrman.

Most of the complicated bulk service queueing model with servers vacation have been analysed using Matrix-Geometric Approach. We here list some of the recent work.

Mohana Dhas [3], has dicussed

Multiserver General Bulk Service Queue With Vacation

Two Heterogeneous Servers With General Bulk Service And Vacation.

Multiserver General Bulk Service Queue With Bulk Arrivals And Vacation

Sitrarasu [24], has discussed a Heterogeneous Multiserver Markovian Queue With General Bulk Service And Vacation.

Jayaraman [13], has discussed

Multiserver General Bulk Servicing Queueing Model With Servers Vacation And Phase Type Inter-Arrival Time Distribution.

Phase Type Distributed Single Server General Bulk Service Queue With Servers Vacations.

Series Queue With General Bulk Service, Random Breakdown And Vacation.

In all these research work they have obtained the stability conditions under which the solution exists and have calculated the numerical values for the probability vectors.

1.7 Methods for solving Queueing Models

Queueing systems are classified into Markovian Queueing Models and Non-Markovian Models. A queueing model is said to be Markovian, if both, inter arrival time of customers and the service time of customers follow exponential distributions; otherwise, it is said to be non-Markovian.

1.7.1 Markovian Queueing Models

Markovian queueing models can be solved by

Differential-Difference equation method using Rouche's Theorem and generating functions*

Neuts Matrix-Geometric Algorithm**

** The Differential- Difference equation method*

This analytic method is used to solve Markovian queueing models in transient (time dependent) and steady (time independent) states and discussed in detail by Gross and Harris [11], Kleinrock [16] . Based on the distributions of inter - arrival and service times, the differential - difference equations are derived. For solutions of these equations, a number of methods have been put forward. In the transient case, we have the method of generating functions (using Rouche's theorem) of Bailey [4] ; the combinatorial method of Champernowne [6]; the difference equation technique of Conolly [8].In the steady state case one has the methods using the Rouche's theorem and iteration; the Rouche's theorem and generating functions; the Laplace Transforms.

*** The Matrix- Geometric Algorithmic Method*

Most of the queueing models require the application of Rouche's theorem. Neuts [21a] broke completely new grounds and developed the Matrix-Geometric method not using Rouche's theorem, for the solution of Markovian and Non-Markovian systems. This computational method uses matrix method as an alternative to closed form analytic methods in solving steady state problems. The approach involves only real arithmetic and avoids the calculation of complex roots based on Rouche's theorem. A queueing problem is to be mathematically formulated. Then, by lexicographically or otherwise ordering the states, a rate matrix of infinite order is to be formed and partitioned in some convenient way into submatrices. Capitalising the rich reservoir of results developed by Neuts and his associates, the stationary probability vectors, the waiting time distribution and other perfomance measures of interest are computed. The results are verified by showing that the total probability is nearly equal to unity.

1.7.2 Non-Markovian Models

Queues in which inter arrival and / or service time distributions are other than exponential, are known as Non-Markovian queues. Since these systems do not have memory-less property, the study of Non-Markovian systems is much more complicated. The following methods are generally used to study non-Markovian models.

Imbedded Markov Chain Technique

Non-Markovian queues are reduced to Markovian by this technique which was introduced by Kendall [15].

Supplementary Variable Technique

The technique of introducing one or more random variables to convert a Non-Markovian process into a Markovian process is called the Supplementary variable technique.

Diffusion Process Approximation

In the queueing systems where the departure process depends on the arrival, Diffusion Process Approximation makes use of justifiable criteria, namely, if the number of customers in the system is not zero, the departure and arrival process are independent of each other.

1.8 Synopsis

Most of the general bulk service queueing models with servers vacation have been analysed by many authors using the Matrix-Geometric Approach.The notable authors among them are Audsin Mohana Dhas [3], Nadarajan and Subramanian [20], Neuts [21a], P.R.Kandasamy [14], Jayaraman [13], Sitrarasu [24], Tian, Zhang and Cao [25]. In all these literature the authors have derived the stability condition for the existence of the steady state probabilities and has presented the numerical values for the steady state probability vector of number of customers in the queue and verified the normalising condition.

Here our object is to understand the Matrix-Geometric Approach and apply the method to $E_k/M_{(a,b)}/c$ queueing model. To analyse the $E_k/M_{(a,b)}/c$ model with servers vacation in detail we first study the $E_k/M_{(a,b)}/1$ queueing model with servers vacation using the analytical approach discussed by Afthab Begum [1]. This we have presented in Chapter2. Differential - Difference equations are formulated for the $E_k/M_{(a,b)}/1$ queueing model and the steady state probabilities are obtained by solving the differential-difference equations using the Rouche's theorem and generating functions techniques. The expressions for the average queue length and waiting time distributions are presented in such a way that the computations can be easily done. Numerical results for the expected queue length are also presented.

In chapter 3 our object is to discuss the multiserver queueing model $E_k/M_{(a,b)}/c$ with servers vacation using the Matrix-Geometric Approach. The state space is identified and the steady state probability vector of the number of customers in the queue and the stability condition are obtained. Numerical results are also presented for the mean queue length and the normalising condition is verified numerically. Particular cases are also derived.

The final chapter contains conclusions and some suggestions for further work on related problems.

CHAPTER II

$E_k/M(a,b)/1$ REPEATED VACATION

It is difficult to obtain analytic situation for bulk service queueing models with servers vacation. Audsin Mohana Dhas [3] has made an attempt to get the analytical solution for the single bulk service queueing model $M/M_{(a,b)}/1$ with servers repeated vacation.Afthab Begum [1] has discussed $M/M_{(a,b)}/1$ queueing model under two bulk service rules along with servers vacation.Steady state queue size probabilities and waiting time distributions are expressed in a closed form so that numerical computations can be easily obtained. Our objective is to discuss the $E_k/M_{(a,b)}/1$ queueing model with servers repeated vacation[1].

2.1 PROBLEM DESCRIPTION

It is assumed that the inter-arrival times have an Erlang distribution with a mean of $1/k\lambda$ for each phase, such that the overall mean interarrival time is $1/\lambda$. Service time is exponentially distributed with parameter $1/\mu$. Service is governed by a general bulk service rule. Repeated vacation is considered.

2.1.1 General Bulk Service Rule with Repeated Vacation

Service begins only when 'a' units are present. If the queue length is greater than 'a' or less than 'b' the entire queue is taken up. If there are more than 'b' units the server accepts the first 'b' unitsand the remaining units will wait in the queue for their turn. After completion of their service, if the server finds less than 'a' units in the system he leaves the system for a random period of time called vacation which is exponentially distirbute with a mean of $1/\theta$. On returning from the vacation if the server finds more than 'a' units he starts his service according to the above genreal bulk service rule. On the otherhand if he finds less than 'a' units he takes immediately another vacation. He continues his vacation until he finds atleast 'a' waiting customers upon returning from a vacation.

2.2 Mathematical Model

Let $Q_{n\ r}(t)$ denote the probability that at time t there are n units waiting in the queue, the arrival is in phase 'r' and the server is busy. $(0 \leq n < \infty)$, $(1 \leq r \leq k)$. $P_{n\ r}(T)$ refers to the situation when the server is on vacation $(0 \leq n < \infty)$, $(1 \leq r \leq k)$.

Let $Q_{n\ r} = \underset{t\to\infty}{\mathrm{lt}}\ Q_{n\ r}(t)$ and $P_{n\ r} = \underset{t\to\infty}{\mathrm{lt}}\ P_{n\ r}(t)$ be the steady-state probabilities. The steady-state equations satisfied by $Q_{n\ r}$ and $P_{n\ r}$ are the following.

$$K\lambda P_{0\ 1} = \mu\ Q_{0\ 1} \tag{2.2.1}$$

$$K\lambda P_{n\ 1} = \mu\ Q_{n\ 1} + K\lambda P_{n-1\ k} \quad (1 \leq n \leq a-1) \tag{2.2.2}$$

$$K\lambda P_{n\ r} = \mu\ Q_{n\ r} + K\lambda P_{n\ r-1} \quad (0 \leq n \leq a-1);(2 \leq r \leq k\ ) \tag{2.2.3}$$

$$(K\lambda + \theta)\ P_{n\ 1} = K\lambda P_{n-1\ k} \quad (n \geq a) \tag{2.2.4}$$

$$(K\lambda + \theta)\ P_{n\ r} = K\lambda P_{n\ r-1} \quad (n \geq a);(2 \leq r \leq k) \tag{2.2.5}$$

$$(K\lambda + \mu)\ Q_{0\ 1} = \sum_{j=a}^{b}(\mu\ Q_{j\ 1} + \theta\ P_{j\ 1}) \tag{2.2.6}$$

$$(K\lambda + \mu)\ Q_{0\ r} = \sum_{j=a}^{b}(\mu\ Q_{j\ r} + \theta\ P_{j\ r}) + K\lambda\ Q_{0\ r-1} \quad (2 \leq r \leq k) \tag{2.2.7}$$

$$(K\lambda + \mu)\ Q_{n\ 1} = K\lambda Q_{n-1\ k} + \mu Q_{n+b\ 1} + \theta P_{n+b\ 1} \quad (n \geq 1) \tag{2.2.8}$$

$$(K\lambda + \mu)\ Q_{n\ r} = K\lambda Q_{n\ r-1} + \mu Q_{n+b\ r} + \theta P_{n+b\ r} \quad (n \geq 1);\ (2 \leq r \leq k) \tag{2.2.9}$$

From equation (2.2.5), we have

$$P_{n\,r} = R^{r-1} P_{n\,1} \qquad (1 \leq r \leq k);(n \geq a)$$

where

$$R = (\frac{k\lambda}{k\lambda + \theta})$$

Equations (2.2.4) and (2.2.5) imply

$$P_{n\,1} = R^k P_{n-1\,1} = R^{k+1} P_{n-2\,k} \qquad (n \geq a)$$

Repeating this process, we have

$$P_{n\,r} = R^{(n-a)k+r} P_{a-1\,k} \qquad (n \geq a)\ (1 \leq r \leq k)\ (2.2.10)$$

To obtain the remaining probabilities, we define the following generating functions :

$$Q_r(z) = \sum_{n=0}^{\infty} Q_{n\,r}\, z^n \qquad (1 \leq r \leq k)$$

$$Q(x,z) = \sum_{r=1}^{k} Q_r(z)\, x^r$$

Multiplying the n^{th} equation of (2.2.9) by z^n, summing over all n and adding (2.2.7) yield,

$$(k\lambda + \mu - \frac{\mu}{z^b})\, Q_r(z)$$

$$= k\lambda Q_{r-1}(z) + \mu \sum_{n=a}^{b-1} Q_{n\,r} - \frac{\mu}{z^b} \sum_{n=0}^{b-1} Q_{n\,r}\, z^n$$

$$+ \theta R^r \left(\frac{R^{(b-a+1)k}\, z}{(1-R^k z)} + \sum_{n=a}^{b} R^{(n-a)k}\right) P_{a-1\,k}$$

$$(2 \leq r \leq k) \qquad (2.2.11)$$

Similarly, from equations (2.2.6) and (2.2.8), we get

$$(K\lambda + \mu - \frac{\mu}{z^b}) Q_1(z)$$

$$= K\lambda Q_k(z) + \mu \sum_{n=a}^{b-1} Q_{n\,1} - \frac{\mu}{z^b} \sum_{n=0}^{b-1} Q_{n\,1} z^n$$

$$+ \theta R \left(\frac{R^{(b-a+1)k} z}{(1-R^k z)} + \sum_{n=a}^{b} R^{(n-a)k}\right) P_{a-1\,k} \qquad (2.2.12)$$

Multiplying the r^{th} equation of (2.2.11) by x^r, (2.2.12) by x and adding, we get

$$(k\lambda + \mu - \frac{\mu}{z_b} - k\lambda x) Q(x, z)$$

$$= \mu \sum_{r=1}^{k} \left(\sum_{n=a}^{b-1} Q_{n\,r} - \sum_{n=0}^{b-1} Q_{n\,r} z^{n-b}\right) x^r + K\lambda (Q_k(z)) x (z-x^k)$$

$$+ \theta P_{a-1\,k} \left(\sum_{n=a}^{b} R^{(n-a)k} + \frac{R^{(b-a+1)k} z}{(1-R^k z)}\right) \sum_{r=1}^{k} R^r x^r \qquad (2.2.13)$$

Substituting $x = \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z^b}\right)$ in (2.2.13), we get

$$Q_k(z) =$$

$$\frac{\frac{\mu}{k\lambda}(1-R^k z)\sum_{r=1}^{k}\left(\sum_{n=0}^{b-1} Q_{n\ r}\, z^{n-b} - \sum_{n=a}^{b-1} Q_{n\ r}\right)\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b}\right)^{r-1} - \frac{\theta}{k\lambda}P_{a-1\ k}\left(\sum_{n=a}^{b} R^{(n-a)k}(1-R^k z)+R^{(b-a+1)k}z\right)\left(\sum_{r=1}^{k} R^r\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b}\right)^{r-1}\right)}{(1-R^k z)\left(z - \left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b}\right)^{k}\right)} \qquad (2.2.14)$$

The zeros of the denominator of (2.2.14) are given by the solution of equations

$$(1-R^k z) = 0 \text{ and } \left(z - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z^b}\right)^k\right) = 0 \qquad (2.2.15)$$

Applying Rouche's theorem, it is found that if ρ $(=\lambda/b\mu)$ is less than 1, there are $(bk-1)$ roots inside, one on the boundary and one outside of $|z| = 1$, for the equation

$$\left(Z-\left(1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z^b}\right)^k\right)=0.$$

The root Z_0 that lies outside the unit circle is real. Thus, $z= 1/R^k$ and $z=z_0$ are the roots of the denominator of (2.2.14) that lie outside of the unit circle $|z| = 1$. Since the numerator of (2.2.14) is a polynomial of degree $(bk+1)$ and $Q_k(z)$ exists for $|z| \leq 1$, the zeros of the denominator of (2.2.14) that lie inside and on $|z| = 1$ are also the zeros of the numerator. Cancelling the zeros of the numerator and denominator, we can write

$$Q_k(z) = \frac{A}{(z-z_o)} + \frac{B}{(1-R^k z)}$$

i.e., $\sum_{n=0}^{\infty} Q_{n\ k}\, z^n = (-\frac{A}{z_o}) \sum_{n=0}^{\infty} (\frac{z}{z_0})^n + B \sum_{n=0}^{\infty} (R^k z)^n$

Equating the like coefficients, we get

$$B = (Q_{0\ k} + \frac{A}{z_0}) \qquad (n=0)$$

and $Q_{n\ k} = (C' R^{nk} - \frac{1}{z_0^n})\, C\, Q_{o\ k} \qquad (n \geq 0) \qquad (2.2.16)$

where $C' = (1 + \frac{1}{C})$ and

$$\frac{A}{z_o} = C\, Q_{0\ k}$$

using (2.2.16) in (2.2.9), we get recursively

$$Q_{n\ r} = C\, Q_{0\ k}\, (C' R^{nk} g_R^{k-r} - \frac{g_{z0}^{k-r}}{z_o^n}$$

$$- E\, R^{(n-1)k+r}\, (1-(Rg_R)^{k-r}) \qquad (2.2.17)$$

$$(n \geq 1), \ (1 \leq r \leq k)$$

where

$$g_{zo} = (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_o^{b}})$$

$$g_{R} = (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda} R^{bk}) \qquad (2.2.18)$$

and $$E = \frac{\theta R^{(b-a+1)k+1}}{k\lambda \ (1-Rg_R)} P_{a-1\ k}$$

Since Z_0 is the root of $(z - (1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z^b})^{k}) = 0$, $g_{z0}{}^{k} = z_0$.

Substituting (2.1.17) in (2.1.8), we obtain after simplification

$$C\ Q_{0\ k} = - \frac{E}{C'}$$

Using this in equation (2.1.17), the expression for $Q_{n\ r}$ becomes

$$Q_{n\ r} = E\ (\frac{g_{z0}{}^{-r}}{C'\ z_0^{n-1}} - R^{(n-1)k+r}\ ) \qquad (2.2.19)$$

$(n \geq 1), (1 \leq r \leq k)$

From equation (2.1.7), $Q_{0\ r}$ ($1 \leq r \leq k$) is obtained as

$$Q_{or} = E \frac{1}{C'\ z_0^{a-1}\ (z_0-1)} ((1+\frac{\mu}{k\lambda})^{k-r}(z_0^{a}-z_0^{b})+(z_0^{b+1}-z_0^{a})g_{z0}{}^{-r})$$

$$+ \frac{1}{R^{bk}(1-R^{k})} ((1+\frac{\mu}{k\lambda})^{k-r}(R^{ak}-R^{bk})+(R^{(b+1)k}-R^{ak})R^{r-k})$$

$(1 \leq r \leq k)$ (2.2.20)

Finally, we have to calculate $P_{n\ r}$ for $(0 \leq n \leq a-1)$ and $(1 \leq r \leq k)$. Adding equations (2.2.3) over 2 to r, we get

$$K\lambda P_{n\ r} = (k\lambda P_{n\ 1} + \mu \sum_{j=2}^{r} Q_{n\ j}) \qquad (2.2.21)$$

$$(0 \leq n \leq a-1)\ ;\ (2 \leq r \leq k)$$

Substituting for $P_{n\ 1}$ from equation (2.2.2), we have

$$K\lambda\ P_{n\ r} = (k\lambda\ P_{n-1k} + \mu \sum_{j=1}^{r} Q_{n\ j}) \qquad (2.2.22)$$

$$(1 \leq r \leq k)\ ;\ (0 \leq n \leq a-1)$$

When r=k, (4.2.22) becomes

$$K\lambda P_{n\ k} = K\lambda P_{n-1\ k} + \mu \sum_{j=1}^{k} Q_{n\ j}) \qquad (2.2.23)$$

$$(0 \leq n \leq a-1)$$

Adding equation (2.2.23) over n to a-1, we get

$$K\lambda P_{a-1\ k} = (\ K\lambda P_{n-1\ k} + \mu \sum_{s=n}^{a-1} \sum_{j=1}^{k} Q_{s\ j}) \qquad (2.2.24)$$

Substituting for $P_{n-1\ k}$ in (2.2.22) from equation (2.2.24), we get

$$K\lambda P_{n\ r} = K\lambda P_{a-1\ k} - \mu \sum_{s=n}^{a=1} \sum_{j=1}^{k} Q_{s\ j} + \mu \sum_{j=1}^{r} Q_{n\ j})$$

$$(0 \leq n \leq a-1);(1 \leq r \leq k)$$

Using (2.1.19) and simplifying, we find

$$P_{n\ r} = P_{a-1\ k} + E \left(\frac{1}{C'(1-z_0^{-b})}\right) (z_0^{1-a} - z_0^{1-n}\ g_{z0}^{-r}) + \frac{\mu}{\theta} (R^{(n-1)k+r} - R^{(a-1)k}) \qquad (2.2.25)$$

$$(0 \leq n \leq a-1);(1 \leq r \leq k)$$

Expressing $P_{a-1\ k}$ in terms of E from (2.2.18) and substituting it in (2.2.25), we get

$$P_{n\ r} = E \frac{1}{C'(1-z_0^{-b})}(z_0^{1-a}-z_0^{1-n}\ g_{z0}^{-r}) + \frac{\mu}{\theta} R^{(n-1)k+r} - \left(\frac{\mu}{\theta} -1\right)R^{(a-b-1)k}) \qquad (2.2.26)$$

$$(1 \leq n \leq a-1);(1 \leq r \leq k)$$

Substituting for $Q_{j\ 1}$ $(0 \leq n \leq a-1)$ and $P_{j\ 1}$ $(a \leq j \leq b)$ from (2.2.19), (2.2.20) and (2.2.10) in equation (2.2.6), C′ is obtained as

$$C' = \frac{f\left(\frac{1}{z_0}\right)}{f(R^k)} \qquad (2.2.27)$$

where

$$f(x) = \frac{[(1+\mu/k\lambda)^k\ (x^b - x^a) + (x^{a-1} - x^b)]}{x^b(1-x)}$$

Finally

$$P_{0\ r} = \frac{\mu}{k\lambda} \sum_{s=1}^{r} Q_{0\ s} \text{ (from 2.2.3)}$$

Substituting for $Q_{0\ s}$ from (2.2.20) and simplifying, we get

$$P_{0\ r}=E\frac{1}{C'(1-z_0^{-1})}\left(\frac{z_0^{1-a}-z_0^{-b}}{1-z_0^{-b}}(1-z_0^{b}\ g_{z0}^{-r})-\frac{z_0^{-b}-z_0^{-a}}{z_0^{-b}}(1+\frac{\mu}{k\lambda})^{k-r}\right)$$

$$+\frac{1}{1-R^k}\left(\frac{R^{(a-1)k}-R^{bk}}{R^{bk}}(1-\frac{\mu}{\theta}\genfrac{}{}{0pt}{}{(1-R^r)}{k\lambda})-\frac{R^{ak}-R^{bk}}{R^{bk}}(1+\frac{\mu}{k\lambda})^{k-r}\right)$$

$$(1 \leq r \leq k)$$

Thus, the steady-state probabilities are expressed in terms of E and E can be evaluated from the normalizing equation

$$\sum_{n=0}^{\infty}\sum_{r=1}^{k} P_{n\ r} + \sum_{n=0}^{\infty}\sum_{r=1}^{k} Q_{n\ r} =1$$

After simplification, E is given by

$$E^{-1}=k(\frac{1}{C'(1-z_0^{-b})}(\frac{z_0^{1-a}-z_0^{-b}}{1-z_0^{-1}}+(a-1)Z_0^{1-a})$$

$$+(1-\frac{\mu}{\theta})(\frac{R^{(1-b-1)k}-1}{(1-R^k)}+(a-1)R^{(a-b-1)k})] \qquad (2.2.28)$$

2.3 MEAN QUEUE LENGTH

Let L_q denote the expected number of units in the queue. Then

$$L_q = \sum_{r=1}^{k} \left(\sum_{n=1}^{a-1} nP_{n\ r} + \sum_{n=1}^{\infty} n\ Q_{n\ r} \right) \qquad (2.2.29)$$

Using the equations (2.2.10), (2.2.19) and (2.2.26), we find

$$\sum_{r=1}^{k} \sum_{n=a}^{\infty} n\ P_{n\ r} = \frac{k\lambda}{\theta} \left(a + \frac{1}{1-R^{k}} \right) P_{a-1\ k} \qquad (2.2.30a)$$

$$\sum_{r=1}^{k} \sum_{n=1}^{\infty} n\ Q_{n\ r} = E\left(\frac{1}{\mu C' (1-z_0^{-b})(1-z_0^{-1})} - \frac{1}{\theta(1-R^{k})} \right) k\lambda \qquad (2.2.30b)$$

and

$$\sum_{r=1}^{k} \sum_{n=1}^{a-1} n\ P_{n\ r} = E \frac{a(a-1)}{2} \frac{1}{C' z_0^{a-1} (1-z_0^{-b})} + R^{(a-b-1)k} \left(1 - \frac{\mu}{\theta}\right) - \frac{k\lambda(1-z_0^{-1})}{\mu C' (1-z_0^{-b})2} \sum_{n=1}^{a-1} n \left(\frac{1}{z_0}\right) n-1 + \frac{\mu R(1-R^{k})}{\theta(1-R)} \sum_{n=1}^{a-1} n(R^{k})^{n-1} \qquad (2.2.30c)$$

Adding the equations (2.2.30a, b, and c) and using the relation

$$E\ R^{(a-b-1)k} \frac{(\mu-\theta)}{\mu} = (R^{(a-1)k} E - \frac{\theta}{\mu} P_{a-1\ k})$$

we have

$$L_q = Ek\ [\frac{H_1(1/z_0)}{C'} - H_2(R^k)] + k\frac{\lambda}{\theta}(a + \frac{R^k}{(1-R^k)} + \frac{a(a-1)}{2}]\ P_{a-1\ k}$$

$$\text{where } H_i(x) = A_i[\frac{1}{1-x} + \frac{\mu}{\lambda}(\frac{a(a-1)}{2} x^{a-1} + \frac{A_i}{1-x}[ax^{a-1}(1-x)-(1-x^a))] \qquad (2.2.31)$$

$$(i=1,2)$$

$$A_1 = \frac{\lambda}{\mu(1-z_0^{-b})} \quad \text{and} \quad A_2 = \frac{\lambda}{\theta}$$

2.4 WAITING TIME DISTRIBUTION

Imbedded Markov Chain Results

It is possible to consider the relationship of the random point probabilities to those of the arrival epoch. Defining

Q^-_n = Probability of n in queue at arrival epoch and the server is busy

and using the standard procedure, we get

$$Q^-_n = K\ Q_{n\ k} \qquad (n \geq 0)$$

Similarly, if one considers the vacation case, then

$$P^-_n = k\ P_{n\ k} \qquad (n \geq 0)$$

Thus, the normalizing condition becomes,

$$\left(\sum_{n=0}^{\infty} Q_{n\ k} + \sum_{n=0}^{\infty} P_{n\ k} \right) = \frac{1}{k}$$

Substituting for $Q_{n\ k}$ and $P_{n\ k}$, we have

$$E \frac{1}{C'(1-z_0^{-b})} \left[\frac{z_0^{1-a} - z_0^{-b}}{(1-z_0^{-1})} + (a-1) z_0^{1-a} \right]$$

$$+ \left(1 - \frac{\mu}{\theta}\right) \left(\frac{R^{(a-b-1)k} - 1}{(1-R^k)} + (a-1)\ R^{(a-b-1)k} \right) = \frac{1}{k} \quad (2.2.32)$$

which is same as (2.2.28)

Here, we consider the actual waiting time distributionof a customer, as opposed to the virtual waiting time distribution. Since we know the relations between random and arrival epoch probabilities, one may consider the virtual waiting time distribution along similar lines.

An arriving unit whch must necessarily be in phase k of the arrival channel, will find the system in one of the following six states :

(i) (2, 1b+m) ; $1 \geq 0$, $(a-1) \leq m \leq (b-1)$

(ii) (1, m) ; $(a-1) \leq m \leq (b-1)$

(iii) (1, 1b+m) ; $1 \geq 1$, $(a-1) \leq m \leq (b-1)$

(iv) (2, 1b+m) ; $1 \geq 0$, $0 \leq m \leq (a-2)$

(v) (1, m) ; $0 \leq m \leq (a-2)$

(vi) (1, 1b+m) ; $1 \geq 1$, $0 \leq m \leq (a-2)$

In state (i,n), i refers to server state (1-vacation, 2-busy) and n is the number of units in the queue.

In case (I), the unit has to wait for (1+1) service completions, whoe density function is

$$\frac{\mu\ e^{-\mu t}(\mu t)^{1}}{\underline{\lfloor 1}}$$

In case (ii), the unit has to wait for the return of the server from vacation, which has density function θ e-θt.

In case (iii), the unit has to wait for server to return from vacation and complete 1 services. The density function in this case is given by

$$S_m(t) = \int_0^t \theta\ e^{\theta-(t-h)}\ k\lambda\ e^{-k\lambda h} \frac{k\lambda h)^{k(a-1-m)-1}}{\underline{|k(a-m-1) - 1}} dh \quad (II)$$

The situations in cases (iv) and (vi) are somewhat different, as there are two criteria which must be satisfied prior to an arriving unit proceed to the service channel. Thus, in case (iv), the unit has to wait for either the completion of (1+1) services or for the arrival of (a-m-1) units to complete the quorum and the server returns from vacation, whichever occurs later.

The density function in this case is

$$h(t) = \frac{\mu e^{-\mu t}(\mu t)_1}{\underline{|1}} Qm(t) + Sm(t) \int_0^t \frac{\mu e^{-\mu h}(\mu h)^{1}}{\underline{|1}} dh \quad (III)$$

where $Q_m(t) = \int_0^t S_m(h)\ dh \quad (IV)$

In case (vi), the arriving unit has to wait for either the server to return from vacation and complete l services or the arrival of (a-l-m) customers to complete the quorum and the server to return from vaction, whichever occurs later. Thus, the density function in case (vi) is

$$g(t) = [f(\mu,\theta,l-1,t)\, Q_m(t) + S_m(t) \int_0^t f(\mu,\theta,l-1,h)dh \quad (V)$$

If $\nu(t)$ denotes the probability density function of the waiting time distribution of an arriving customer at phase k, then

$$\nu(t) = \sum_{l=0}^{\infty} \sum_{m=a-1}^{b-1} Q_{lb+m\ k} \frac{\mu e^{-\mu t}(\mu t)^{l}}{\underline{|l}} + \sum_{m=a-1}^{b-1} P_{m\ k}\, \theta\, e^{-\theta t}$$

$$+ \sum_{l=1}^{\infty} \sum_{m=a-1}^{b-1} P_{lb+m\ k}\, f(\mu,\theta,l-1,t) + \sum_{m=0}^{a-2} P_{m\ k}\, S_m(t)$$

$$+ \sum_{l=0}^{\infty} \sum_{m=0}^{l-2} Q_{lb+m}\, h(t) + \sum_{l=1}^{\infty} \sum_{m=0}^{a-2} P_{\ lb+m\ k}\, g(t) \qquad (VI)$$

For the computation of $\nu(t)$, we note that

$$\sum x^{lb} f(\mu,\ \theta,\ l-1,\ t) = \frac{x^b\ \mu\theta(e^{-\theta t} - e^{-\mu(1-Rkb)t})}{(\mu(1-R^{kb})-\theta)} \qquad (VII)$$

The sum of the first three terms of equation (VI) is

$$\nu_1(t) = E \sum_{l=0}^{\infty} \sum_{m=a-1}^{b-1} \left(\frac{1}{C' z_0^{\,lb+m}} - R^{(lb+m)k}\right) \mu\, e^{-\mu t} \frac{\mu t)^{l}}{\underline{|l}}$$

$$+ \sum_{m=a-1}^{b-1} R^{(m-a+1)k}\ \theta\ e^{-\theta t}\ P_{a-1\ k}$$

$$+\sum_{l=1}^{\infty} \sum_{m=a-1}^{b-1} R^{(lb+m-a+1)k} f(\mu, \theta, l-1, t) P_{a-1\ k}$$

Using (VII) and simplifying, we get

$$\nu_1(t) = \frac{E}{C'} \frac{(z_0^{1-a}-z_0^{-b}}{(1-z_0^{-1})} \mu e^{-\mu(1-z0-b)t}$$

$$- \frac{E}{R^{(b-a+1)k}} \frac{1-R^{(b-a+1)k}}{1-R^{k}} (\mu-\theta) e^{-\theta t} \qquad (VIII)$$

The fourth term of (VI) is

$$\nu_2(t) = \frac{E}{C'(1-z_0^{-b})} (z_0^{1-a} \sum_{m=0}^{a-2} S_m(t) - \sum_{m=0}^{a-2} z_0^{-m} S_m(t)$$

$$+ E(\frac{\mu}{\theta} \sum_{m=0}^{a-2} R^{mk} S_m(t) + (1-\frac{\mu}{\theta}) R^{(a-b-1)k} \sum_{m=0}^{a-2} S_m(t) \qquad (IX)$$

The fifth term of (VI) is

$$\nu_3(t) = E \sum_{l=0}^{\infty} \sum_{m=0}^{a-2} (\frac{1}{C' Z_0^{lb+m}} - R^{(lb+m)k}) h(t)$$

Substituting for h(t) and simplifying, we get

$$\nu_3(t) = \frac{E}{C'} \mu e^{-\mu t}(1-z_0^{-b}) \sum_{m=0}^{a-2} z_0^{-m} (Q_m(t) - \frac{S_m(t)}{\mu(1-z_0^{-b})})$$

$$+ \frac{E}{C'(1-z-b0)} \sum_{m=0}^{a-2} z_0^{-m} S_m(t) - \frac{E}{1-R^{kb}} \sum_{m=0}^{a-2} R^{mk} S_m(t)$$

$$- E \mu e^{-\mu t}(1-R^{kb}) \sum_{m=0}^{a-2} R^{mk} (A_m(t) - \frac{S_m(t)}{\mu(1-R^{kb})}$$

Writing

$$\sum_{m=0}^{a-2} Z_0^{-m} \left(Q_m(t) - \frac{S_m(t)}{\mu(1-z_0^{-b})}\right) e^{-\mu(1-z0-b)t} = W_1(t)$$

$$\sum_{m=0}^{a-2} R^{mk} \left(Q_m(t) - \frac{S_m(t)}{\theta}\right) e^{-\theta t} = W_2(t)$$

$$\sum_{m=0}^{a-2} S_m(t) = W_3(t)$$

$$\sum_{m=0}^{a-2} R^{mk} \left(Q_m(t) - \frac{S_m(t)}{\mu(1-R^{bk})}\right) e^{-\mu(1-Rkb)t} = W_4(t)$$

We have

$$\nu_3(t) = E \frac{1}{C'}(\mu\, w_1(t) + \frac{1}{(1-z_0^{-b})} \sum_{m=0}^{a-2} z_0^{-m} S_m(t)$$

$$- [\mu w_4(t) + \frac{1}{1-R^{bk}} \sum_{m=0}^{a-2} R^{mk} S_m(t)]$$

The last term of (VI) is

$$\nu_4(t) = \sum_{1=1}^{\infty} \sum_{m=0}^{a-2} R^{(1b+m-a+1)k} P^{a-1\ k} [f(\mu,\ \theta,\ 1-1,\ t)\ Q^m(t)$$

$$+ S_m(t) \int_0^t f(\mu,\ \theta, 1-1, h) dh]$$

Integrating equation (VII) and considering it in particular for $X=R^{k}$, we get

$$\sum_{1=1}^{\infty} R^{1bk} \int_{0}^{t} f(\mu,\theta,1-1,h)dh$$

$$= \frac{\mu\ \theta R^{bk}}{(\mu(1-R^{bk})-\theta)} \left(\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-\mu(1-Rbk)t}}{\mu(1-R^{bk})}\right. \qquad \text{(XII)}$$

Using (VII) and (XII) in (XI), we get

$$\nu_4(t) = \frac{R^{(b-a+1)k}\mu\theta}{(\mu(1-R^{bk})-\theta} P_{a-1\ k} \sum_{m=0}^{a-2} R^{mk}\ Q_m(t)\ (e^{-\theta t}-e^{-\mu(1-Rbk)t})$$

$$+ \left(\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-\mu(1-Rbk)t}}{\mu(1-R^{bk})}\right) \sum_{m=0}^{a-2} R^{mk}\ S_m(t)$$

Substituting for $P_{a-1\ k}$ in terms of E and using the expressions for $w_i(t)$, we can write

$$\nu_4(t) = E\mu\ w_4(t)-w_2(t)-\left(\frac{1}{\theta} - \frac{1}{\mu(1-R^{bk})}\sum_{m=0}^{a-2} R^{mk}\ S_m(t)\right) \qquad \text{(XIII)}$$

Adding equations (VIII) to (XIII), we find after simplification that

$$\nu(t) = \frac{E}{C'} \frac{z_0^{1-a} - z_0^{-b}}{1-z_0^{-1}} \mu\ e^{-\mu(1-z-b0)t} + \frac{z_0^{1-a}}{1-z_0^{-b}} W_3(t) + \mu\ w_1(t)$$

$$+ E \frac{\theta-\mu}{1-Rk} (R(a-b-1)k - 1)\ e^{-\theta t}$$

$$+ (1-\frac{\mu}{\theta})\ R^{(a-b-1)k}\ W_3(t) - \mu w_2(t)$$

$$\int_0^\infty \nu(t)\ dt = \frac{E}{C'} \frac{Z_0^{1-a} - z_0^{-b}}{(1-z_0^{-1})\ (1-z_0^{-b})} + \frac{(a-1)\ z_0^{1-a}}{1-z_0^{-b}}$$

$$+ E \frac{R^{(a-b-1)k}-1}{(1-R^k)})\ (1- \frac{\mu}{\theta}) + (1- \frac{\mu}{\theta}) R^{(a-b-1)k} (a-1)$$

$$(\text{Since} \int_0^\infty w_3(t)\ dt = (a-1) \text{ and } \int_0^\infty e^{-xt}\ (A_m(t) - \frac{S_m(t)}{x})\ dt = 0)$$

$$\therefore \int_0^\infty \nu(t)\ dt = \frac{1}{k}\ (\text{from } 2.2.32))$$

2.5 Partiular Cases

Case (i)

As $\theta \to \infty$, R, E and $\frac{E}{R^{mk}}$ $(m < (b-a+1))$ will tend to 0, whereas

$$\frac{E}{R^{(b-a+1)k}} = P_{a-1\ k} \text{ and}$$

$$\frac{E}{C'} = Q_{0\ k}$$

Thus, the equations (2.2.19), (2.2.20) and (2.2.25) respectively become

$$Q_{n\ r} = \frac{g_{oz}^{-r}}{z0n-r} Q_{0\ k}$$

$$(n \geq 1)\ ;\ (1 \leq r \leq k)$$

$$Q_{0\ r} = \frac{(1+\frac{\mu}{k\lambda})^{k-r}\ (z_0^{a} - z_o^{b}) + (z_0^{\ b+1} - z_o^{a})\, g_{zo}^{-r}}{z_0^{\ a-1}\ (z_0-1)} Q_{0k}$$

and

$$P_{n\ r} = P_{a-1\ k} + Q_{0\ k} \frac{(\ z_0^{\ 1-a} - z_0^{\ 1-n}\ g_{\ z0}^{\ -r})}{(1-z_0^{-b})}$$

$$(0 \leq n \leq a-1)\ ;\ (1 \leq r \leq k)$$

These results exactly coincide with corresponding results of Chaudhry and Easton.[9]

Case (ii)

When r=k=1, the equations (2.2.19) and (2.2.26) become

$$Q_{n\ 1} = (\frac{A}{z0} + B\ R^{n})\ P_{\ a-1\ 1} \qquad (n \geq 0)$$

$$P_{\ n\ 1} = \frac{\mu}{\lambda} \frac{A(1-z_0^{-(n+1)})}{1-z_0^{-1}} + B \frac{(1-R^{n+1})}{1-R} \qquad P_{a-1\ 1}$$

$$(0 \leq n \leq a-1)$$

$$\text{Where } B = \frac{\theta R^{(b-a+1)}}{\mu(1-R^{b})-\theta} \;;\; A = \frac{\dfrac{\lambda}{\mu} - B\,\dfrac{(1-R)^{a}}{1-R}}{\left(\dfrac{1-z_0^{-a}}{1-Z_0^{-1}}\right)}$$

and Z_0 is the real root of the equation

$z_0^{b+1} - (\lambda+\mu)\, z_0^{b} + \mu = 0$ that lies outside (0,1). And these results coincide with the corresponding results of $M/M_{(a,b)}/1$ queueing model with servers repeated vacation obtained by Afthab Begum[1].

Numerical Results

To evaluate the performance measures of the developed model, a computer program is executed for sample values of parameters.

The expressions for various measures have shown that, $E_k|M_{(a,b)}|1$ vacation model can be completely determined in terms of z_0, the root that lies outside of $|z|=1$ of the characteristic equation $(z-g_z{}^{k}=0)$. The root has been determined using Muller's method for various parameter

values of k, b and ρ $\left(= \dfrac{\lambda}{b\mu}\right)$.

The column $L_v(L_q)$ denotes the mean queue length for vacation (non-vacation) model. L_q is obtained from L_v by letting $\theta \to \infty$. It is verified that the values of L_q are exactly same as that of Chaudhry and Easton's results. The tables show that for each set of (a, b, ρ) the values of the mean queue length (L_v) converge to that of L_q, as $1/\theta$ approaches 0.

The probability that the server is on vacation, (a-1) units are waiting in the queue and the arrival is on the kth phase ($P_{a-1\ k}$) is also presented in the table.

$a = 4$; $b = 4$; $k = 3$; $\mu = 1$

ρ	$1/\theta$	L_q	L_v	$P_{a-1\ k}$
0.4	2.5	1.755547	5.755548	3.37776 E-02
0.6		2.848864	8.848862	1.717604 E-02
0.8		7.528811	15.52881	6.895514 E-03
0.4	2.0	1.755547	4.95578	3.877703 E-02
0.6		2.848864	7.648859	2.024947 E-02
0.8		7.528812	13.92881	8.248216 E-03
0.4	1.0	1.755547	3.35547	5.385505 E-02
0.6		2.848864	5.248865	3.113763 E-02
0.8		7.528812	10.72881	1.346857 E-02

TABLE 2.1

$a = 3 \; ; \; b = 4 \; ; \; k = 3 \; ; \; \mu = 1$

ρ	$1/\theta$	L_q	L_v	$P_{a-1\ k}$
0.4	2.5	1.446077	4.964855	3.449568 E-02
0.6		2.873471	7.963113	1.706761 E-02
0.8		8.022776	14.54438	6.740116 E-03
0.4	$\frac{10}{7}$	1.446077	3.317421	4.79761 E-02
0.6		2.873471	5.489035	2.51208 E-02
0.8		8.022776	11.2517	1.022372 E-02

TABLE 2.2

$a = 20; \; b = 30; \; k = 3 \; ; \; \mu = 1$

ρ	$1/\theta$	L_q	L_v	$P_{a-1\ k}$
0.4	2.5	12.7011	37.61624	4.762586 E-03
0.6		23.06628	59.19106	2.335453 E-03
0.8		59.55203	105.8244	9.165566 E-04
0.4	2.0	12.7011	31.79733	5.521177 E-03
0.6		23.06628	50.4449	2.762464 E-04
0.8		59.55203	94.17566	1.095477 E-03
0.4	1.0	12.7011	25.27757	6.712376 E-03
0.6		23.06628	40.63609	3.479477 E-03
0.8		59.55203	81.12597	1.40560 E-03

TABLE 2.3

CHAPTER - III

In Chapter-II, we have discussed single server Erlangian bulk service queueing model [$E_k/M_{(a,b)}/1$] with servers repeated vacation. We have derived the expressions for the mean queue length and the mean waiting time by solving the differential - difference equations using the Rouche's theorem and generating functions.

Mohana Dhas[3] has discussed the multiserver queueing model $M/M_{(a,b)}/c$ with servers repeated vacation and has given the numerical values of the steady state probabilities using the Matrix Geometric Approach.

In this Chapter, we consider the multiserver Erlangian queueing model [$E_k/M_{(a,b)}/c$] with servers' multiple vacation. By using the Matrix Geometric Technique, we calculate the numerical values for the steady state probabilities for this general model. Apart from this, we also present the numerical values for the mean queue length for this model.

The following results are deduced

1. When c=1, the mean queue length values for $E_k/M_{(a,b)}/1$ queueing model is obtained. It is verified numerically that the values calculated when c=1 coincide with the corresponding values obtained in Chapter-II.

2. When K=1, the values of the steady state queue size probabilities coincide with that of $M/M_{(a,b)}/c$ queueing model [3] numerically.

3. When K=1, c=1, the numerical values of the queue length coincide with $M/M_{(a,b)}/1$ queueing model [1].

PROBLEM DESCRIPTION

In $E_k/M_{(a,b)}/C$ queueing model the inter-arrival time follows an Erlangian distribution with a mean rate of $1/K\lambda$ for each phase such that the overall mean inter-arrival rate is $1/\lambda$. Service rate is assumed to follow an exponential distribution with rate $1/\mu$. There are `c' servers in the system and the service is governed by the General Bulk Service Rule [2.1.1].

When any server completes service and finds only β $(0 \leq \beta \leq a-1)$ waiting customers, he leaves for a random period of time called `vacation', whose duration is an exponentially distributed random variable with finite mean $1/\theta$. On returning from vacation, if any server finds more than a customer he starts his service according to the General Bulk Service Rule. On the other hand, if he finds less than a customer he takes another vacation. He continues his vacation until he finds atleast `a' waiting customers upon returning from a vacation. That is repeated vacation is considered.

STEADY STATE PROBABILITY VECTOR

The process can be formulated as a continuous Markov Chain with state space

$$[(i,j,r),\ i \geq 0,\ 0 \leq j \leq c,\ 1 \leq r \leq k]$$

The chain is said to be in the state (i,j,k) when there are

i units waiting in the queue
j servers are busily available in the system and
the arrival is in the kth phase.

Thus, when the system is in the state (i,j,k) (c-j) servers are on vacation.

The infinitesimal generator **Q** of the infinite state Markov model under consideration has the block partitioned structure as presented in page no. 8 .

Let $\underline{\mathbf{i}}$ = [(i,0,1),(i,1,1)...(i,c,1);(i,0,2),... (i,c,2);

...(i,0,k),(i,1,k)...(i,c,k)]

Let $\mathbf{P}_{ijr}$ denote the steady-state probabilities that the system is at state (i,j,r), $(i \geq 0)$; $(0 \leq j \leq k)$;$(1 \leq r \leq k)$

We recall here that the arrival rate, service rate and the mean vacation time are respectively 1/λ, 1/μ and 1/θ.

To facilitate the representation of the infinitesimal generator **Q** of the continuous time Markov Chain with the above state space, we define the submatrices C_0, C_1, C_2,C_3, A_0, A_1, A_2, D.

The matrices C_0, C_1 , C_2 , ${}_3C$ are square matrices of order (c+1)defined by

$$C_0 = \begin{bmatrix} -K\lambda & 0 & 0 & \cdots & 0 & 0 \\ \mu & -(K\lambda+\mu) & 0 & \cdots & 0 & 0 \\ 0 & 2\mu & -(K\lambda+2\mu) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & & c\mu & -K\lambda+C\mu \end{bmatrix}$$

$$C_1 = \begin{bmatrix} K\lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & K\lambda & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & K\lambda \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & c\theta & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mu & (c-1)\theta & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & (c-1)\mu & \theta \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & c\mu \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -(K\lambda+c\theta) & 0 & . & . & . & 0 \\ 0 & -(K\lambda+\mu+(c-1)\theta) & . & . & . & 0 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & 0 & . & . & . & -(K\lambda+c\mu) \end{bmatrix}$$

The matrices A_0, A_1, A_2, D are square matrices of order (c+1)k defined in terms of submatrices C_0, C_1, C_2, C_3 as follows :

$$A_0 = \begin{bmatrix} C_2 & & & & \\ & C_2 & & & \\ & & \ddots & & \\ & & & & C_2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} C_3 & C_1 & & & \\ & C_3 & C_1 & & \\ & & \ddots & \ddots & \\ & & & & C_1 \\ & & & & C_3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ C_1 & \cdots & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} C_0 & C_1 & & & \\ & C_0 & C_1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & C_1 \\ & & & & C_0 \end{bmatrix}$$

Now, the infinitesimal generator **Q** of the continuous time Markov Chain, with state space as defined has the block partitioned structure as shown below :

$$\mathbf{Q} = \begin{array}{c|cccccccccccc}
 & 0 & 1 & 2 & \cdots & a-1 & a & a+1 & \cdots & b-1 & b & b+1 & \cdots \\
\hline
0 & D & A_2 & & & & & & & & & & \\
1 & & D & A_2 & & & & & & & & & \\
\vdots & & & \ddots & \ddots & & & & & & & & \\
a-1 & & & & & D & A_2 & & & & & & \\
a & A_0 & & & & & A_1 & A_2 & & & & & \\
a+1 & & A_0 & & & & & A_1 & A_2 & & & & \\
\vdots & \vdots & & & & & & & \ddots & \ddots & & & \\
b-1 & A_0 & & & & & & & & & & & \\
b & A_0 & & & & & & & & & & & \\
b+1 & & A_0 & & & & & & & & & & \\
b+2 & & & A_0 & & & & & & & & & \\
\vdots & & & & \ddots & & & & & & & & \\
b+a & & & & & & A_0 & & & & & & \\
\vdots & & & & & & & \ddots & & & & &
\end{array}$$

Illustration :

Here we illustrate how the transition occurs from $\underline{i}$ to $\underline{j}$ for $\underline{i}=\underline{0}$ $\underline{j}=\underline{0}$; $\underline{i}=\underline{a-1}$ $\underline{j}=\underline{a}$.

$$
\begin{array}{c|cccccccccccc}
 & 001 & 011 & 021 & \dots & 0c1 & 002 & \dots & 0c2 & \dots & 00K & \dots & 0cK \\
\hline
001 & -K\lambda & 0 & & \dots & & 0 & K\lambda & \dots & & & & \\
011 & & \mu & -(K\lambda+\mu) & \dots & & & 0 & 0 & K\lambda & & \dots & \\
021 & & & & & & & & & & & & \\
\vdots & & & & & & & & & & & & \\
0c1 & & & & & c\mu & -(K\lambda+c\mu) & & \dots & & K\lambda & & \\
002 & & & & & & -K\lambda & & 0 & \dots & & 0 & \\
012 & & & & & & & \mu & -(K\lambda+\mu) & & \dots & & 0 \\
\vdots & & & & & & & & & & & & \\
0c2 & & & & & & & \dots. & & c\mu & & -(K\lambda+c\mu) & \\
\vdots & & & & & & & & & & & & \\
00K & & & & & & & & & & -K\lambda 0 & \dots & 0 \\
01K & & & & & & & & & & & \mu & -(K\lambda+\mu)..0 \\
\vdots & & & & & & & & & & & & \\
0cK & & & & & & & & & & & & c\mu \quad -(K\lambda+c\mu) \\
 & \mathbf{\underline{0}} & & & & & & & & & & &
\end{array}
$$

$$
\mathbf{\underline{0}} \quad D = \begin{bmatrix} C_0 & C_1 & & \\ & C_0 & C_1 & \\ & & \ddots & \ddots \\ & & & C_1 \\ & & & C_0 \end{bmatrix}
$$

$$\begin{array}{c|cccccccccccc}
 & a\text{-}101 & a\text{-}111 & \cdots & a\text{-}1c1 & a\text{-}102 & \cdots & a\text{-}1c2 & \cdots & a\text{-}10K & \cdots & a\text{-}1cK \\
\hline
a\text{-}101 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & 0 \\
a\text{-}101 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & 0 \\
\vdots & & & & & & & & & & & \\
a\text{-}1c1 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & 0 \\
a\text{-}102 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & 0 \\
\vdots & & & & & & & & & & & \\
a\text{-}1c2 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & 0 \\
\vdots & & & & & & & & & & & \\
a\text{-}10K & K\lambda & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
a\text{-}1cK & & & & K\lambda & & & & & & &
\end{array}$$

With the suitable identification of D, A_0, A_1, A_2 we find that the infinitesimal generator $\mathbf{Q}$ presented here is similar with that of $M/M_{(a,b)}/c$ queueing model discussed by Mohana Dhas [3]. When K=1, the matrices D, A_0, A_1 , ${}_2A$ reduce the exact form of the corresponding matrices discussed in Chapter IV of Mohana Dhas[3].

THE STEADY STATE PROBABILITY VECTORS AND THE STABILITY CONDITION

Let $\underline{\mathbf{X}}$ be the vector of steady state probabilities associated with $\underline{\mathbf{Q}}$, such that

$$\underline{X}\ \underline{Q} = \underline{O} \text{ and}$$

$$\underline{X}\ \underline{e} = 1 \qquad \_\_\_\_\_\_\_ (3.1)$$

where $\underline{\mathbf{e}}$ is a column vector of suitable dimension with all elements 1.

Let us partition $\underline{X}$ as

$$\underline{X} = (\underline{X}_0,\ \underline{X}_1,\ \underline{X}_2,\ \ldots\ \underline{X}_a,\ \underline{X}_{a+1},\ \ldots)$$

where $\underline{\mathbf{X}}_i$, for $i \geq 0$ are vectors of order **1x(c+1)k** with

$$X_i = (P_{io1}, P_{i11}, P_{i21}, \ldots P_{ic1}, P_{i02}, P_{i12}, \ldots P_{ic2}, \ldots$$
$$P_{iok},\ P_{i1k}, \ldots\ P_{ick})$$

Following Neuts [21a], we examine the existence of solution of the form

$$\underline{\mathbf{X}}_i = X_{a-1}\ R^{i-(a-1)},\ i \geq a -1 \qquad \_\_\_\_\_\_\_\_ (3.2)$$

where the matrix R is the unique non-negative solution , with spectral radius of the matrix equation

$A_2 + RA_1 + R^{b+1}A_0 = 0$

This equation is deduced on expanding the equation $\underline{X}\ \underline{Q} = \underline{0}$ and substituting for X_i, $i \geq a-1$ from (3.2).

Following theorem 1 of Latouche and Neuts , the matrix R is computed by successive substitution in the recurrence relation

$R(0) = 0$

$R(n+1) = -A_2A_1^{-1} - R^{b+1}\ A_0A_1-1$ for $n \geq 0$ _____(3.3)

and is the limit of the monotonically increasing sequence of matrices $\{Rn,\ n \geq 0\}$.

Consider the infinitesimal generator $A = A_0 + A_1 + A_2$, which is a square matrix of order $(c+1)k$.

A is irreducible and there is a unique row vector

$\pi = [\pi_1, \pi_2, \ldots \pi_{(c+1)k}] \geq 0$ such that

$\pi A = \underline{0}$ and

$\pi\underline{e} = 1$

Solving the equation $\pi A = \underline{0}$ we get

$$\pi_{c(k+1)} = \pi_{(c-1)(k+1)} = \ldots = \pi_{(k+1)} ; \pi_i = 0 \text{ otherwise} .$$

Substituting this in $\sum \pi i=1$ we get,

$\pi_{r(c+1)} = 1/K$ for $1 \leq r \leq K$

Following Neuts [21a] the system is stable if and only if

$$\pi A_1 \underline{e} + (b+1)\pi A_0 \underline{e} > 0$$

i.e., $b\pi A_0\underline{e} + \pi(A_1+A_0)\underline{e} > 0$

$$b\pi A_0\underline{e} + \pi(-A_2)\underline{e} > 0$$

i.e., $\pi A_2\underline{e} < b\pi A_0\underline{e}$

$$\pi A_2\underline{e} = K\lambda \, [\pi_{(k-1)(c+1)+1} + \ldots \pi_{k(c+1)}]$$

$$\pi A_0\underline{e} = c\theta[\pi_1+\pi_{(c+2)}+\ldots] + [\mu + (c-1)\theta][\pi_2 + \pi_{(c+3)}+\ldots]$$
$$+\ldots+c\mu \, [\pi_{(c+1)} +\pi_{2(c+1)} + \ldots \pi_{k(c+1)}]$$

Substituting for the π_i's we get

$\pi A_2\underline{e} < \pi A_0\underline{e}$ implies

$$\lambda < cb\mu$$

Thus, the matrices R decides the vectors X_i, $i \geq a-1$.

Finally, we have to determine the vectors

$(\underline{X}_0, \underline{X}_1, \ldots \underline{X}_{a-1})$

We define $\mathbf{Q}^*$ by

$$\mathbf{Q}^* = \left[\begin{array}{ccccccc}
D & & A_2 & & 0 & \ldots & \\
 & 0 & & D & & A_2 & \ldots \\
 & & & & D & & \\
\sum_{i=1}^{b-a+1} R^i A_0 & & R^{b-a+2} & A_0 & \ldots & & R^b A_0 + D
\end{array}\right]$$

Lemma :

$\mathbf{Q}^*$ is an infinitesimal generator.

Proof :

To prove $\mathbf{Q}^*\underline{e} = \underline{0}$, it is enough to consider the last row of $\mathbf{Q}^*$, since the other rows are identical to that of Q and Q is an infinitesimal generator. Thus,

$$(\text{Last row sum of } \mathbf{Q}^*)\ \underline{e} = \sum_{i=1}^{b-a+1} R^i A_0\underline{e} + R^{b-a+2} A_0\underline{e} + (R^b A_0 + D)\underline{e}$$

$$= (I-R)^{-1} (I-R^{b+1}) A_0\underline{e} - A_2\underline{e} - A_0\underline{e}$$

$$= (I-R)^{-1} [-R^{b+1}A_0\underline{e} + R(A_2+A_0)\underline{e} - A_2\underline{e}]$$

$$= (I-R)^{-1} [-R^{b+1}A0\underline{e} - RA_1\underline{e} - A_2\underline{e}]$$

$$= \underline{0} \text{ by equation (3.3)}$$

Hence, the Lemma.

Therefore, $\mathbf{Q}^*$ is an infinitesimal generator and is also irreducible.

Let $\underline{\mathbf{X}}^* = (\underline{X}_0, \underline{X}_1, \ldots \underline{X}_{a-1})$ be a solution of the equation

$$\underline{X}^*\underline{Q}^* = \underline{0}$$

Expanding the equation we get,

$$\underline{X}_0 D + \underline{X}_{a-1} \sum_{R=1}^{b-a+1} R^i A_0 = 0$$

i.e., $\underline{X}_0 D + \underline{X}_{a-1} R(I-R)^{-1} (I-R^{(b-a+1)})A_0 = 0$

$$\underline{X}A_2 + X_1 D + \underline{X}_{a-1} (I-R^{(b-a+1)})A_0 = 0$$

$$\underline{X}_{a-3}A_2 + \underline{X}_{a-2}D + X_{a-1} A_0 = 0$$

$$\underline{X}_{a-2}A_2 + \underline{X}_{a-1} (R^b A_0 + D) = 0$$

The vector $\underline{X}_i$ for $0 \leq i \leq a-2$ may be obtained in terms of ${}_a\underline{X}$ from the above set of equations and $\underline{X}_{a-1}$ may be obtained from the normalising condition

$$\sum_{i=0}^{a-2} \underline{X}_i\underline{e} + \underline{X}_{a-1} (I-R)^{-1} \underline{e} = 1$$

Algorithm 1 : To calculate the probability vector $\underline{X}_i$ satisfying

$\underline{X}\ \underline{Q} = 0$ and $\underline{X}\ \underline{e} = 1$

Step 1

Calculate the square matrix R of order (c+1)K recursively from the equation

$$R = -A_2A_1^{-1} - R^{b+1} A_0A_1^{-1}$$

Then the solution for X_i ,$i \geq a-1$ are given by

$$\underline{X}_i = \underline{X}_{a-1} R^{i-(a-1)},\ i \geq a-1$$

Step 2

To calculate $\{ X_i,\ 0 \leq i \leq a-1 \}$ satisfying $\underline{X}^*\ \underline{Q}^* =\underline{0}$

$$\sum_{i=0}^{a-2} \underline{X}_i\underline{e} + \underline{X}_{a-1} (I-R)^{-1}\ \underline{e} = 1 \qquad \ldots.\ I$$

The equations given in (I) are **a(c+1)k+1** in number, in **a(c+1)k** variables.

Let $\underline{Q}_1{}^*$ be the matrix obtained from $\underline{Q}^*$ replacing the last column of it by the column vector $\underline{e}$.

Calculate the inverse of the matrix $\underline{Q}_1{}^*$, using Gauss-Jordan Method.

The entries of the last row of the inverse matrix $Q_1{}^*$ give the required probability vectors.

Algorithm 2 : To calculate the mean queue length

Let L_q denote the mean queue length for the model $E_k/M_{(a,b)}/c$ with servers repeated vacation. Recalling that P_{ijr} are the steady state probabilities that there are i members in the queue, j servers are busily available in the system and the arriving customer is in the rth phase, we have

$$L_q = \sum_{n=1}^{\infty} n \sum_{j=0}^{c} \sum_{r=1}^{k} P_{njr}$$

$$= \sum_{n=1}^{\infty} n \, \underline{X}_n \underline{e}$$

Since $[\underline{X}_n = (P_{n01}, P_{n11}, \ldots P_{nc1}; P_{n02}, P_{n12}, \ldots P_{nc2}; \ldots; P_{nok}, P_{n1k} \ldots P_{nck})]$

Thus,

$$L_q = \sum_{n=1}^{a-1} n \, \underline{X}_n \underline{e} + \sum_{n=a}^{\infty} n \, \underline{X}_n \underline{e}$$

Substituting for X_n , $n \geq a$ and simplifying

$$= \sum_{n=1}^{a-1} n \, \underline{X}_n \underline{e} + X_{a-1}\{aR(I-R)^{-1} + R^2(I-R)^{-1})^2\}\underline{e}$$

Numerical results

By theorem 1 of Latouche and Neuts R is the limit of the sequence of matrices R(n), $n \geq 0$ defined by

$$-A_2 A_1^{-1} - R^{b+1} A_0 A_1^{-1}$$

a=4, b=10, c=2, k=2, θ=0.1, λ=2.1, μ=1.0

R =

```
0.0000000   0.0000000   0.0000000    0.0000000   0.0000000   0.0000000
0.0000000   0.0000000   0.0000000    0.0000000   0.0000000   0.0000000
0.0000000   0.0000000   0.0000000    0.0000000   0.0000000   0.0000000
0.9545455   0.0250566   0.0016655    0.9111571   0.0438111   0.0027214
0.0000000   0.7939361   0.00129515   0.0000000   0.6303344   0.0019057
0.0000000   0.0000000   0.6775101    0.0000000   0.0000000   0.4590200
```

The steady state probability vectors X_0 , X_1 , X_2 , X_3

```
0.0030567   0.0128382   0.0009091    0.0084089   0.0224790   0.0014397
0.0130709   0.0195806   0.0010430    0.0171382   0.0170827   0.0007707
0.0206964   0.0149445   0.0005830    0.0238207   0.0131219   0.0004526
0.0265760   0.0115721   0.00036156   0.0290177   0.0102554   0.0000297
```

$$\sum_{I=0}^{\infty} x_i = 0.9999999$$

In tables 3.1 , 3.2 , 3.3 , 3.4 the numerical values for average queue length for the model $E_k/M_{(a,b)}/C$ with servers vacation are presented for different values of $a,b,k,\mu,\theta,\lambda,c$. L_q represents the mean queue length for the model.

Table 3.1

$a=3$, $b=8$, $k=3$, $\mu=1.0$, $\theta=0.1$

c	λ	L_q	λ	L_q
1	1.6	15.8269500	3.2	31.2104700
2		7.7671150		14.6790300
3		5.1366800		9.4350630
4		3.8526350		6.9202930
5		3.1010500		5.4671320

From this table 3.1, we find that as we appoint more number of servers the mean queue length decreases considerably.This we have illustrated for two different arrival rates λ.

Table 3.2

$a=3$, $b=8$, $k=3$, $c=3$, $\mu=1.0$.

Table 3.2 shows that as the mean vacation time $1/\theta$,decreases , the mean queue length also decreases. That is earlier the server returns from vacation lesser is the mean queue length.

θ	λ	L_q	λ	L_q
.1	3.2	9.4350630	4.8	13.5036200
.2		4.6396020		6.5836060
.3		3.1764510		4.4879490
.4		2.4886290		3.5046420
.5		2.0911940		2.9377660

Table 3.3

$a=3,\ b=8,\ k=3,\ \mu=1.0,\ \theta=.1$

λ	c	L_q
3.4	2	15.5278300
3.6		16.3751000
3.8		17.2212900
3.4	3	9.9522910
3.6		10.4664300
3.8		10.9778500
3.4	4	7.2848780
3.6		7.6465310
3.8		8.0055120

From Table 3.3 we find that as the inter arrival rate increases there is a propotional increase in the mean queue length.

Table 3.4

$C=2,\ k=2,\ \mu=1.0,\ \lambda=2.5,\ \theta=.1$

a	b	L_q	a	L_q
5	15	12.3998500	10	14.4015400
	25	12.1368500		13.7266300
	35	12.0490500		13.5133100

From Table 3.4 we observe that there are only marginal changes due to the increase in the minimum and maximum capacity taken for service.

We can also obtain the mean queue length for the $M/M_{(a,b)}/c$, $M/M_{(a,b)}/1$, $E_k/M_{(a,b)}/1$.

CHAPTER IV

Conclusions and suggestions for further research work

In this dissertation Erlangian Queueing Model under repeated vacation is considered . The General Bulk Service Rule is followed.

In modelling real life transportation systems we often find that bulk service queueing structure occupy an important place. Taxi stand, unscheduled cars ferry, a single ground floor station of an elevator are some of the important practical situations wherein the model and the investigation has applications. Since bulk service queueing systems are difficult to analyse, some of the recent researchers have made many simplifying assumptions in order to obtain limited results for such systems. These assumptions are Poisson Process for the input and Exponential service. But these assumptions may not hold in many practical situations. For example, in many industrial process units may go through several exponential phases where they are out. For a concrete application of an Erlangian process one may refer to Giffin [10]. However, it may be stated that phases need not have physical meaning,they are introduced to generalise the input process.

Easton and Chaudhry [9], have considered **$E_k/M_{(a,b)}/1$ queueing model** and have given the steady state results including the expected number in queue and the waiting time distributions. Later Afthab Begum [1] has introduced vacation in $E_k/M_{(a,b)}/1$ model and derived the expected queue length and waiting time distribution for both single and repeated vacations.

The author has discussed the $E_k/M_{(a,b)}/1$ queueing model with servers multiple vacations in Chapter 2 and analysed the more general multiserver $E_k/M_{(a,b)}/c$ queueing model with servers repeated vacations. The numerical values for the expected queue length for this general model has been presented and the corresponding results for $E_k/M_{(a,b)}/1$, $M/M_{(a,b)}/1$, $M/M_{(a,b)}/c$ are deduced.

The author feels that the waiting time distribution can be discussed for the $E_k/M_{(a,b)}/c$ queueing model in future.

In this model every server who becomes idle, leaves the system for a random period of time, whether or not there are servers available in a system. So the customers who arrive during the absence of all servers, may have to wait for servers return from the vacation. But in places like hospitals, telegraph offices, telephone booths.... we expect always some server to be present in the system.

So we may introduce the following concept in $E_k/M_{(a,b)}/c$ queueing model that is if a server, soon after his service, finds no customer waiting for service and there are atleast "r" servers busily available in the system, leaves for a vacation. If a server becomes idle while there are (r-1) servers (excluding him) available (idle or busy), he will remain in the system expecting the arrivals for service. We find that Ek/M(a,b)/c queueing model without vacation (with vacation) are the particular cases of the above queueing model when r=c (r=0) respectively.

BIBLIOGRAPHY

1. **AFTHAB BEGUM. M.I** (1996), " Queueing Models with bulk service and vacation " Ph.D Dissertation Bharathiar University.
2. **ARORA .K.L** (1964), "Two server bulk service queueing processes" .Opns.Res. 286-294.
3. **AUDSIN MOHANA DHAS** (1989), "Markovian general bulk service queueing models "Ph.D Dissertation Bharathiar University.
4. **BAILEY . N.T.J** (1954) " On queueing processes with bulk services ",J.R.S.S.B,16,80-87.
5. **BORTHAKUR. A**(1971) "A Poisson queue with a general bulk service rule",J.Assam.Sci.Soc.,14,162-167.
6. **CHAMPERNOWNE.D.G** (1956) "An elementary method of solution of the queueing problem with a single server and a constant parameter ",J.Roy.Statist.Soc.B21,168-175.
7. **CHAUDHRY.M.L, MEDHI.J, SIM .S.H,AND TEMPLETON.J.G.C**(1987) "On a two heterogeneous server Markovian queue with general bulk service rules",Sankhya: The Indian Journal of Statistics, 49,B.Pt.1,36-50.
8. **CONOLLY.B.W** (1958) "A difference equation technique applied to the simple queue with arbitrary arrival interval distribution", J.Roy.Statist.Soc.B21,168-175.
9. **EASTON.G.D AND CHAUDHRY.M.L**(1982) The queueing system $E_k/M_{(a,b)}/1$ and its numerical anaalysis ", Compt.Opns.Res,197-205.
10. **GIFFIN.W.C**(1978) "Queueing theory and applications".192 Grid,Columbus,OHIO.
11. **GROSS.D AND HARRIS.C.M**(1985) "Fundamentals of queueing theory ,(the second edition),John Wiley and Sons,New York.
12. **JAISWAL.N.K** (1960) " The time dependent solution of the bulk service queueing problems".Opns.Res.8,773-781.
13. **JAYARAMAN.D** (1990) "Queueing models with servers vacation and general bulk service".Ph.D Dissertation Bharathiar University.

14. **KANDASAMY.P.R** (1990) "Matrix-geometric algorithmic approach to some Markovian queueing and inventory models ",Ph.D. Dissertation , Bharathiar University.

15. **KENDALL** 1951) " Some problems in the theory of queues".J.J.S.S.B.,13,151-185.

16. **KLIENROCK.L** (1975) " Queueing systems ",Vol.1,Theory , John Wiley And Sons,New York.

17. **LEVY.Y AND YECHIALI.U** (1976) " An M/M/S queue with servers vacation" INFOR Vol 4 .No.2. 153-163.

18. **MADILL .B.R AND CHAUDHRY .M.L**(1986) "Steady state analytic and algorithmic research on the queueing system GI/M(a,b)/c, precedings of the U.G.C Sponsored seminar on Stochastic Modelling and decision making."University of Delhi.

19. **MEDHI.J** (1981) " Stochastic processes ", Wiley Eastern Limited.

20. **NADARAJAN.R AND SUBRAMANIAN.A** (1984) " A general bulk service queue with servers vacation," Opns.Res.in Mgmt.system,Academic Pub. 127-135.

21. **NEUTS.M.F**(1967) " A general class of bulk queues with Poisson input".Ann.Math.Statist.38, 759-770.

a. **NEUTS.M.F**(1981) " Matrix - Geometric Solutions in Stochastic models and algorithmic approach ."The Johns Hopkins University Press , Baltimore and London.

22. **SHANKARANARAYANAN.G AND NADARAJAN.R** (1979) " General bulk service rule with erlang input ", Research supported by U.G.C Project No. F-23,1171-1179.

23. **SIM.S.H AND TEMPLETON.J.G.C**(1988) "Steady state results for the M/M(a,b)/c batch service". Europian Journal of Opns.Res.21,260-267.

24. **SITRARASU.M.R**(1993) " Some Markovian queueing model with general bulk service". Ph.D Dissertation Bharathiar University.

25. **TIAN,ZHANGAND CAO**(1989) "The GI/M/1 queue with exponential vacation queueing system,5,331-344.