A STUDY OF S-CLOSED SPACES

By

Geetha Veni. P

A DISSERTATION SUBMITTED TO
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INTRODUCTION

INTRODUCTION

The concept of S-closed spaces was first introduced by Thompson [13] in 1976.

"A topological space is said to be S-closed iff every semi-open cover of X has a finite subcollection whose closures cover X".

Various aspects of S-closed spaces have been studied by many authors. In this thesis the following papers are taken for discussion :

(1) Thompson, T., S-closed spaces [13]

(2) Thompson, T., Semi-continuous and Irresolute images of S-closed spaces [14].

(3) Cameron, D., Properties of S-closed spaces [2].

(4) Noiri, T., Properties of S-closed spaces [8].

(5) Sivaraj, D., A note on S-closed spaces [12].

(6) Dlaska, Ergun and Ganster, On the topology generated by semi-regular sets [4].

In Chapter I, we discuss the properties of S-closed spaces and in Chapter II a study of S-closed spaces under different types of mappings is carried out. A few interesting properties of S-closed spaces are as follows :

(1) An S-closed, first countable, regular space is finite. (Hence an S-closed metrizable space is finite) [13].

(2) An S-closed regular space is extremally disconnected [13].

(3) An S-closed Hausdorff space is extremally disconnected [13].

(4) An S-closed regular space is compact [13].

(5) The property of being S-closed is semi-regular and contagious [2].

(6) The property of being S-closed is not productive [2].

(7) If the product of a collection of spaces is S-closed, then each component space is S-closed [2].

(8) Finite union of S-closed α-sets is S-closed [12].

(9) An extremally disconnected compact space is S-closed [13].

(10) An extremally disconnected QHC space is S-closed [2].

(11) A semi-open set A is S-closed in (X, τ) iff it is S-closed in (X, τ^{α}) [12].

(12) $(X, \tau SR)$ is countably compact ==> (X, τ) is countably S-closed [4].

Apart from the above mentioned results, some interesting characterization of S-closed spaces are also discussed.

For the study of S-closed spaces under mappings we have discussed the contributions of Thompson [14], Noiri [8] and Sivaraj [12]. Thompson [14] has proved that the semi-continuous image of an S-closed space onto any Hausdorff space is H-closed and the irresolute image of an S-closed space is S-closed. The main result in this paper is the following characterization :

"A Hausdorff space X is S-closed iff the irresolute image of X in any Hausdorff space is closed".

Noiri [8] has shown that S-closedness is preserved under weakly-continuous, almost-open surjections and the semi-continuous image of any S-closed space in any Hausdorff space is closed. The characterization obtained by Noiri [8] is as follows :

"A Hausdorff space X is S-closed iff every semi-continuous function of X into any Hausdorff space is almost-closed".

Sivaraj [12] has shown that the inverse image of an S-closed space under an almost-open, pre-semi-open bijection is S-closed. Regarding subsets, he [12] has proved that an irresolute image of a relative I-compact subset is relatively S-closed.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

In 1976, Thompson [13] introduced the notion of S-closed spaces. Since then characterizations of S-closed spaces, their properties, their behaviour under different types of mappings are investigated by eminent topologists - Thompson, Cameron, Noiri, Abd El.Monsef, Sivaraj and Ziqiu. We now give a brief survey of some of the articles published on S-closed spaces.

ON LOCALLY S-CLOSED SPACES [Noiri, T., 1983] [9]

In this paper, the author has investigated some conditions on functions for the images (inverse images) of locally S-closed spaces to locally S-closed. He has proved that locally S-closed spaces are preserved under weakly-open and θ-continuous surjections and that a topological space X is locally S-closed iff the semi-regularization X^* is locally S-closed. In the last section it is shown that locally S-closed spaces are inverse-preserved under s-perfect almost continuous mappings.

A NOTE ON S-CLOSED SPACES [Noiri, T., 1984] [10]

The main result is that an S-closed space in which every open set is the union of regular closed sets is extremally disconnected. This theorem improves results by T.Thompson [13].

ON EXTREMALLY DISCONNECTED, LOCALLY S-CLOSED SPACES [Ziqiu, Y., 1984] [15]

A necessary and sufficient condition for extremally disconnectedness for a locally S-closed space is given. This result generalizes two theorems of T.Noiri.

REMARKS ON S-CLOSED SPACES [Abd El. Monsef, M.E., Kozae, A.M., 1990] [1]

In this paper, several characterizations of S-closed spaces are obtained using regular semi-closed sets and some sets having two properties of near openness and near closedness at the same time. Image of S-closed spaces under some non continuous mappings are investigated. The relations between S-closedness and near compactness, co-compactness, almost co-compactness, light compactness, mild compactness are obtained. S-closed subsets relative to semi-T_2 spaces are also discussed.

CHARACTERIZATIONS OF S-CLOSED HAUSDORFF SPACES [Noiri, T., 1991] [11]

The family of all open subsets of (X, τ) whose complements are S-closed relative to (X, τ) is a base for a topology τ^* on X. The author shows that (X, τ) is S-closed Hausdorff iff (X, τ^*)

is Hausdorff. Other characterization of S-closed Hausdorff spaces are also obtained.

PRELIMINARY DEFINITIONS AND RESULTS

PRELIMINARY DEFINITIONS AND RESULTS

By a space X, it is meant the topological space (X, τ). The closure and interior of a subset A of X are denoted by Cl(A) [or $Cl_X(A)$] and Int(A) [or $Int_X(A)$] respectively. The collection of all clopen sets of X is denoted by $CO(X, \tau)$ or CO(X).

Definition : 1

A subset A of X is **semi-open** if there exists an open set G such that $G \subset A \subset Cl(G)$.

Equivalently A is semi-open if $A \subset Cl(Int(A))$.

Notation

The collection of all semi-open sets is denoted by **SO(X)**.

Properties : 2

(1) If A is open, then Cl(A) is semi-open.

(2) Every open set is a semi-open set.

Definition : 3

The complement of a semi-open set is called a **semi-closed** set.

Equivalently, A is semi-closed iff $Int(Cl(A)) \subset A$.

Remark : 4

Every closed set is semi-closed.

Definition : 5

The **semi-interior** of A is the union of all semi-open sets contained in A and is denoted by **sInt(A)** [or s $Int_X(A)$].

Definition : 6

The **semi-closure** of A is the intersection of all semi-closed sets containing A and is denoted by **sCl(A)** [or s $Cl_X(A)$].

Definition : 7

A subset A of a topological space is said to be **regular open** (respectively, **regular closed**) if Int(Cl(A)) = A (respectively. Cl(Int(A)) = A).

Notation

The collection of all regular open (respectively regular closed) sets of X is denoted by **RO(X)** (respectively **RC(X)**). The topology generated by regular open sets is denoted by τ_s.

Properties : 8

(1) Every regular open set is open.

(2) Every regular closed set is closed.

(3) Every regular closed set is semi-open.

(4) If A is regular open then X-A is regular closed.

(5) Closure of a semi-open set is regular closed.

Definition : 9

A subset A of a space X is said to be **semi-regular** if it is both semi-open and semi-closed.

Notation

The collection of all semi-regular sets of X is denoted by SR(X). The topology generated by semi-regular sets is denoted by τSR.

Definition : 10

The topology τ_s on X whose base is the regular open sets of τ is the **semi-regularization** of τ.

Properties : 11

(1) Every regular closed set is semi-regular.

(2) Every regular open set is semi-regular.

(3) $R \subset (X, \tau) \subset SR(X, \tau) \subset \tau SR$.

Definition : 12

A topological property R is **semi-regular** provided that a topological space (X, τ) has property R iff (X, τ_s) has property R.

Definition : 13

A subset A of X is an α-set if $A \subset A^+$ where $A^+ = Int(Cl(Int(A)))$.

Notation

The collection of all α-sets in X is denoted by τ^α. τ^α is a topology on X and the space (X, τ^α) is denoted by X^*. If $A \subset X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and interior of A in X^*.

Properties : 14

(1) $\tau \subset \tau^\alpha \subset SO(X)$

(2) $SO(X) = SO(X^*)$

Definition : 15

A space X is said to be extremally disconnected if $Cl(U) \in \tau$ for every $U \in \tau$.

Properties : 16

(1) If X is extremally disconnected, then

(i) the closure of every semi-open set in X is open.

(ii) regular open sets are clopen.

(iii) $\tau^{\alpha} = SO(X)$.

(2) If a regular closed set in X is open then the space X is extremally disconnected.

(3) A space X is extremally disconnected iff $Cl(U) = sCl(U)$ for every $U \in SO(X)$.

(4) Extremally disconnected semi-regular spaces are regular.

Lemma : 17

A topological space X is extremally disconnected iff every two disjoint open sets in X have disjoint closures.

Definition : 18

A Hausdorff space X is **H-closed** iff every open cover $\{U_a / a \in I\}$ there exists a finite subfamily $\{U_{a_i} / i = 1,2,\ldots, n\}$ such that the union of their closures cover X.

Remark : 19

Every Hausdorff S-closed space is H-closed.

Definition : 20

A filter base $F = \{A_a\}$ **s-converges** to a point $x_0 \in X$ if for each semi-open set V containing x_0 there exists an $A_a \in F$ such that $A_a \subset Cl(V)$.

Definition : 21

A filter base $F = \{A_a\}$ **s-accumulates** to a point $x_0 \in X$ if for each semi-open set V containing x_0 and $A_a \in F$, $A_a \cap Cl(V) \neq \emptyset$

Result : 22

Let F be a maximal filterbase in X. Then F s-accumulates to a point $x_0 \in X$ iff F s-converges to x_0.

Definition : 23

A topological space X is called R_0 if $x \in G \in \tau$ gives $Cl\{x\} \subseteq G$.

Definition : 24

A function $f : X \to Y$ is said to be **irresolute (semi-continuous)** if $f^{-1}(V)$ is semi-open in X for every semi-open (open) subset V in Y.

Definition : 25

A function $f : X \to Y$ is said to be **pre-semi-open (semi-open)** if f(V) is semi-open in Y for every semi-open (open) subset V in X.

Definition : 26

A bijection $f : X \to Y$ is called a **semi-homeomorphism** if f is both irresolute and pre-semi-open.

Definition : 27

A function $f : X \to Y$ is **almost-open**, if $f^{-1}(Cl_Y(V)) \subset Cl_X(f^{-1}(V))$ for each open set V in Y.

Definition : 28

A function $f : X \to Y$ is said to be **semi-closed (almost-closed)** if $f(F)$ is semi-closed (closed) for every closed (regular closed) subset F of X.

Definition : 29

A space X is **I-compact,** if every cover of X by regular closed sets has a finite subfamily whose interiors cover X.

Definition : 30

A subset A of a space X is **I-compact relative to X** if every cover of A by regular closed subsets of X has a finite subfamily whose interiors cover A.

PROPERTIES OF S-CLOSED SPACES

CHAPTER I

PROPERTIES OF S-CLOSED SPACES

In this chapter we discuss some interesting properties of S-closed spaces. The results are collected from the papers published by Thompson [13], Cameron [2], Noiri [8], Sivaraj [12] and Dlaska, Ergun and Ganster [4]. The results from each paper are discussed sectionwise.

SECTION : 1.1

Thompson [13] has first introduced the concept of S-closed spaces. He [13] has obtained a characterization of S-closed spaces and has proved that every S-closed metrizable space is finite. Using the other results Thompson [13] has concluded that a regular compact space is S-closed iff it is extremally disconnected.

Definition : 1.1.1

A topological space X is **S-closed** iff for every semi-open cover $\{ U_a \ / \ a \in I \}$ of X there exists a finite subfamily $\{ U_{a_i} \ / \ i = 1, 2, .., n \}$ such that the union of their closures cover X.

Characterization

Theorem : 1.1.2

For a topological space the following are equivalent :

(i) X is S-closed.

(ii) For each family of semi-closed sets $\{ F_a \}$ such that $\cap (F_a) = \emptyset$, there exists a finite subfamily $\{ F_{a_i} \}_{i=1}^{n}$ such that $\bigcap_{i=1}^{n} \mathrm{Int}(F_{a_i}) = \emptyset$.

(iii) Each filter base $F = \{ A_a \}$ s-accumulates to some point $x_0 \in X$.

(iv) Each maximum filterbase F s-converges.

Proof

(i) ⟹ (iv)

Let $F = \{A_a\}$ be a maximum filter base. Suppose (iv) is false. Hence by Result 22, F does not s-accumulates to any point.

∴ For every $x \in X$, there exists a semi-open set $V(x)$ containing x and $A_{a(x)} \in F$ such that $A_{a(x)} \cap Cl(V(x)) = \emptyset$.

The collection $\{ V(x) / x \in X \}$ is a semi-open cover of X.

Since X is S-closed, there exists a finite subfamily $\{ V(x_i) / i = 1, 2, \ldots, n \}$ such that

$$X = \bigcup_{i=1}^{n} Cl\ V(x_i).$$

Consider the corresponding $A_{a(x_i)} \in F$.

Since F is a filtre base on X, there exists $A_0 \in F$ such that $A_0 \subset \bigcup_{i=1}^{n} A_{a(x_i)}$.

Claim : $\bigcap_{i=1}^{n} A_{a(x_i)} = \emptyset$

Suppose not, there exists an $x \in \bigcap_{i=1}^{n} A_{a(x_i)}$

$\Longrightarrow x \in A_{a(x_i)}$, for $i = 1, 2, \ldots, n$

$\Longrightarrow x \notin Cl(V(x_i))$ for $i = 1, 2, \ldots, n$

$\Longrightarrow x \notin X$

which is a contradiction.

Hence we get $\bigcap_{i=1}^{n} A_{a(x_i)} = \emptyset$

Hence the claim

Since $A_0 \subset \bigcap_{i=1}^{n} A_{a(x_i)}$, we get $A_0 = \emptyset$

which is a contradiction to the fact that $A_0 \neq \emptyset$. Hence each maximum filter base F s-converges.

(iv) ==> (iii)

Follows from the fact that each filterbase is contained in a maximal filterbase.

(iii) ==> (ii)

Let $\{ F_a \}$ be a collection of semi-closed sets such that $\cap F_a = \emptyset$.

Suppose for every finite subfamily, $\bigcap_{i=1}^{n} \text{Int}(F_{a_i}) \neq \emptyset$.

$\therefore$ $F = \{ \bigcap_{i=1}^{n} \text{Int}(F_{a_i}) \,/\, n \in z^+,\ F_{a_i} \in \{ F_a \}\}$ forms a filterbase.

By assumption, F s-accumulates to some point $x_0 \in X$.

Hence for every semi-open set $V(x_0)$ containing x_0 $\text{Int}(F_a) \cap \text{Cl}\, V(x_0) \neq \emptyset$ for every $a \in$ ... (1)

Since $x_0 \notin \cap F_a$, there exists an $a_0 \in I$ such that $x_0 \notin F_{a_0}$.

Hence, x_0 belongs to the semi-open set $X - F_{a_0}$.

From (1), we get $\text{Int}(F_{a_0}) \cap \text{Cl}(X - F_{a_0}) \neq \emptyset$

$$\text{Int}(F_{a_0}) \cap (X - \text{Int}(F_{a_0})) \neq \emptyset$$

which is a contradiction.

(ii) ==> (i)

Let $\{ V_a \}$ be a semi-open covering of X. Then $X - V_a$ is semi-closed and $\cap (X - V_a) = \emptyset$. By hypothesis, there exists a finite subfamily such that $\bigcap_{i=1}^{n} \text{Int}(X-V_{a_i}) = \emptyset$.

i.e., $\bigcap_{i=1}^{n} (X - \text{Cl}(V_{a_i})) = \emptyset$

i.e., $\bigcup_{i=1}^{n} \text{Cl}(V_{a_i}) = X$

i.e., X is S-closed.

Theorem : 1.1.3

If X is regular and S-closed space then X is compact.

Proof

Let $X = \bigcup_{\alpha \in I} V_\alpha$, V_α open in X

Take $x \in X$ ==> there exists α such that $x \in V_\alpha$. Since X is regular, there exists U_α such that $x \in U_\alpha \subset \text{Cl}\ U_\alpha \subset V_\alpha$.

$\therefore$ $\{U_\alpha / \alpha \in I\}$ is an open cover of X.

==> $\{U_\alpha / \alpha \in I\}$ is a semi-open cover of X.

Since X is S-closed, there exists a finite subset I_0 of I such that

$$X = \bigcup_{\alpha \in I_0} \text{Cl}\ U_\alpha \subset \bigcup_{\alpha \in I_0} V_\alpha$$

Hence X is compact.

Theorem : 1.1. 4

Each S-closed, first countable, regular space is finite.

Proof

Let X be an S-closed, first countable, regular space.

Suppose X is infinite.

Since X is compact (Theorem 1.1.3), it is not discrete.

$\therefore$ There exists an $x \in X$ such that $\{x\}$ is not open.

Hence every open set containing x must intersect X in some point other than x.

$\therefore$ x is an accumulation point of X.

Since X is first countable and regular there is a local base at x, say, $\{U_n / n \in N\}$

Such that $U_1 = X$, U_n is open in X and $Cl(U_{n+1}) \subset U_n$ for each $n \in N$.

Let $\{N_k / k \in N\}$ be a family of pairwise disjoint infinite subsets of N such that

$$\cup \{N_k / k \in N\} = N$$

For each $k \in N$, we set

$$V_k = \{x\} \cup (\cup \{Cl(U_n) - Cl(U_{n+1})) / n \in N_k\}$$

Then the collection $\{V_k / k \in N\}$ is a semi-open cover of X.

If $n \in N$, then $\cup \{Cl(V_k) / k \leq n\} \neq X$

Hence X is not S-closed which is a contradiction.

Hence X is finite.

Corollary : 1.1.5

Each S-closed metrizable space is finite.

Proof

The proof follows from the above theorem 1.1.4 by using the fact that every metrizable is first countable and regular [6].

Theorem : 1.1.6

If Y is a regularly closed subset in an S-closed space X, then Y is S-closed.

Theorem : 1.1.7

Each extremally disconnected, compact space is S-closed.

Proof

Let X be an extremally disconnected, compact space.

Let $\{V_\alpha / \alpha \in I\}$ be a semi-open cover of X.

Since X is extremally disconnected, $Cl(V_\alpha)$ is open in X.

$\therefore$ $\{Cl(V_\alpha) / \alpha \in I\}$ is an open cover of X.

Since X is compact, there exists a finite subfamily I_0 of I such that

$$X = \bigcup_{\alpha \in I_0} Cl(V_\alpha)$$

Hence X is S-closed.

Corollary : 1.1.8

βN is S-closed [βN is the stone-cech compactification of N].

Proof

Since βN is extremally disconnected and compact, by the above theorem 1.1.7 we get, βN is S-closed.

Theorem : 1.1.9

If X is a S-closed regular space, then X is extremally disconnected.

Proof

Suppose X is not extremally disconnected. Then there exists a regular open set $0 \subset X$ such that $Cl(0) - 0$ and $X-Cl(0)$ are non empty. Let $X \in Cl(0) - 0$

Then for every neighbourhood V of x, $V \cap 0 \neq \emptyset$

$\therefore$ $F = \{(V \cap 0)\}$ forms a filterbase in $Cl(0)$. Since $Cl(0)$ is S-closed, by Theorem 1.1.2 F s-accumulates to some point x_0 in $Cl(0)$. Obviously the filterbase F also converges to x.

Claim : $x_0 \notin Cl(0) - 0$

Suppose not, ie., if $x_0 \in Cl(0) - 0$ then $x_0 \in X-0$. Since $X-0$ is semi-open (since every regular closed set is semi-open) we get by definition, every member of F must intersect $Cl(X-0)$.

Since 0 is also open, (X-0) is closed. Hence every member of F must intersect X-0 itself which is impossible.

Thus, $x_0 \in 0$. Since X is regular, there exists an open set U such that $x_0 \in U \subset Cl(U) \subset 0$ and $x \in X-Cl(U)$.

But since F converges to x, there exists a neighbourhood V of x such that $(V \cap 0) \subset X-Cl(U)$.

$\Longrightarrow$ $(V \cap 0) \cap Cl(U) = \emptyset$

which is a contradiction to the fact that F s-accumulates to x_0.

Hence X is extremally disconnected.

Theorem : 1.1.10

If X is a Hausdorff S-closed space, then X is extremally disconnected.

Proof

Quite similar to the proof of the previous theorem.

Corollary : 1.1.11

Let X be a regular compact space. Then X is S-closed iff X is extremally disonnected.

Proof

Follows from theorem 1.1.7 and 1.1.9.

SECTION : 1.2

Cameron [2] has characterized S-closed spaces as those spaces whose covers by regular closed sets have finite subcovers. He [2] has also proved that S-closed is semi-regular and contagious but is not productive. Moreover an extremally disconnected QHC space is S-closed. We start with the definition of QHC space.

Definition : 1.2.1

A topological space X is **quasi-H-closed** (denoted QHC) if every open cover has a finite proximate subcover (every open cover has a finite subfamily whose closures cover the space).

Characterization using regular semi-open cover

Theorem : 1.2.2

A topological space X is S-closed iff every cover of regular semi-open sets has a finite proximate subcover.

Proof

Assume X is S-closed.

Let $\{V_\alpha \,/\, \alpha \in I\}$ be a regular semi-open cover of X. Since regular semi-open sets are semi-open $\{V_\alpha \,/\, \alpha \in I\}$ will be a semi-open cover of X.

Since X is S-closed, the results follows.

Conversely, assume the given condition.

If the space X is not S-closed, then there exists a semi-open cover $\{ A_\beta / \beta \in I \}$ which has no finite proximate subcover. Then $\{ Int(Cl(A_\beta)) \cup A_\beta / \beta \in I\}$ is a regular semi-open cover X.

Suppose $X = \cup \{ Cl(Int(Cl(A_\beta))) \cup A_\beta / \beta \in I_0 \}$, where I_0 is a finite subset of I.

Since $A_\beta \subset Int(Cl(A_\beta)) \cup A_\beta \subset Cl\ A_\beta$,

$Cl(A_\beta) \subset Cl(Int(Cl(A_\beta)) \cup A_\beta) \subset Cl(A_\beta)$

$\Longrightarrow Cl(Int(Cl(A_\beta)) \cup A_\beta) = Cl(A_\beta)$

$\therefore X = \cup \{ Cl(A_\beta) / \beta \in I_0 \}$

$\Longrightarrow \{ A_\beta / \beta \in I \}$ has a finite proximate subcover which is contradiction.

Hence every regular semi-open cover has a finite proximate subcover.

$\therefore$ X is S-closed.

Corollary : 1.2.3

An extremally disconnected QHC space is S-closed.

Proof

In an extremally disconnected space regular open sets are clopen.

Hence regular semi-open sets are clopen. Thus every regular semi-open cover of X becomes a clopen cover of X.

Since X is QHC, this cover has a finite proximate subcover.

Hence X is S-closed.

Characterization using regular closed cover

Theorem : 1.2.4

A topological space X is S-closed iff every cover by regular closed sets has a finite subcover.

Proof

Assume X is S-closed.

Let $\{V_\alpha / \alpha \in I\}$ be a regular closed cover of X. Since every regular closed set is semi-open, we get $\{V_\alpha / \alpha \in I\}$ is a semi-open cover of X. Since X is S-closed, there exists a finite subfamily I_0 of I such that

$$X = \bigcup_{\alpha \in I_0} Cl(V_\alpha)$$

$$= \bigcup_{\alpha \in I_0} V_\alpha$$

Conversely, assume the given condition.

Let $\{V_\beta / \beta \in I\}$ be a semi-open cover of X. Since V_β is semi-open, $Cl(V_\beta)$ is regular closed. $\therefore$ $\{Cl(V_\beta) / \beta \in I\}$ is a regular closed cover of X. By assumption, there exists a finite subfamily I_0 of I such that

$$X = \bigcup_{\alpha \in I_0} Cl(V_\beta).$$

Hence X is S-closed.

Theorem : 1.2.5

S-closed is semi-regular.

Proof

This follows from theorem 1.2.4 and the fact that τ and τ_s have the same regular closed sets.

Theorem : 1.2.6

S-closed is contagious. [A property R is contagious if a space has the property whenever a dense suset has the property].

Proof

Let A be an S-closed dense subset of X. Let $\{ V_\beta / \beta \in I \}$ be a semi-open cover of X. Then $\{ V_\beta \cap A / \beta \in I \}$ will be a semi-open cover of A.

Since A is S-closed, there is a finite proximate subcover $\{ V_{\beta_i} \cap A / i = 1, 2, \ldots, m \}$ of A.

$$\text{i.e.,} \quad A = \bigcup_{i=1}^{m} Cl(V_{\beta_i} \cap A)$$

Since A is dense in X, $X = ClA = \bigcup_{i=1}^{m} Cl(V_{\beta_i} \cap A)$

$$X = ClA = \bigcup_{i=1}^{m} Cl(V_{\beta_i} \cap A)$$

$$\subset \bigcup_{i=1}^{m} (Cl(V_{\beta_i}) \cap Cl(A))$$

$$X \subset \bigcup_{i=1}^{m} Cl(V_{\beta_i})$$

Hence X is S-closed.

The following theorem shows that if the product space is S-closed, then each component space is also S-closed.

Theorem : 1.2.7

If $(\pi_I X_\alpha, \pi_I \tau_\alpha)$ is S-closed, then (X_α, τ_α) is S-closed for each $\alpha \varepsilon I$.

Proof

Let $\{ Cl_{\tau_{\alpha_0}} U_\beta \,/\, \beta \varepsilon I \}$ be a regular closed cover of $(X_{\alpha_0}, \tau_{\alpha_0})$.

Then $\{ p^{-1} Cl_{\tau_{\alpha_0}} U_\beta \,/\, \beta \varepsilon I \}$ is a regular closed cover of $\pi_I X_\alpha$ (where p denotes the projection mapping). Since πX_α is S-closed, there exists a finite subcover $\{ p^{-1} Cl_{\tau_{\alpha_0}} (U_{\beta_i})/i=1,2,..n \}$

$\Longrightarrow$ $\{ Cl_{\tau_{\alpha_0}} U_{\beta_i} \,/\, i = 1, 2, \ldots, n \}$ is a finite subcover of X_{α_0}.

Hence $(X_{\alpha_0}, \tau_{\alpha_0})$ is S-closed.

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.8

Products of S-closed spaces need not be S-closed.

By corollary 1.1.8 βN is S-closed.
Since $\beta N \times \beta N$ is not extremally disconnected but Hausdorff, by theorem 1.1.10, we conclude that $\beta N \times \beta N$ is not S-closed.

SECTION : 1.3

In this section we discuss the properties of S-closed spaces as given by Noiri [8]. He [8] has given a characterization of S-closed spaces and two sufficient conditions for a space to be S-closed. The following definitions and theorems are need for discussion.

Definition : 1.3.1

A subset A of X is **S-closed in X**, if A is S-closed as a subspace of X.

Definition : 1.3.2

A subset A of a space X is said to be **S-closed relative to X**, if every cover of A by semi-open sets of X has a finite subfamily whose closures cover A.

Theorem : 1.3.3

A subset G of a space X is S-closed relative to X iff every cover of G by regular-closed sets of X has a finite subcover.

Proof

Similar to the proof of Theorem 1.2.4.

Theorem : 1.3.4 [7]

An open set G of a space X is S-closed iff G is S-closed relative to X.

Theorem : 1.3.5 [7]

A space X is S-closed iff every proper regular open set of X is S-closed.

The following theorem to proved by Noiri in his paper entitled "On S-closed subspaces".

Theorem : 1.3.6

Let A be a subset of a space X. If A is S-closed relative to X, then Cl(A) and Int(Cl(A)) are S-closed relative to X.

Characterization

Theorem : 1.3.7

A space X is S-closed iff every proper regular closed set of X is S-closed relative to X.

Proof

Assume X is S-closed.

Let F be any proper-regular closed set of X. Then Int(F) is proper regular open.

By theorem 1.3.5, we get Int(F) is S-closed. Since Int(F) is open, by theorem 1.3.4 we get, Int(F) is S-closed relative to X.

Hence F = Cl(Int(F)) is S-closed relative to X, by theorem 1.3.6.

Conversely, assume the given condition

Let F be a proper - regular closed set of X. Then, by assumption, F is S-closed relative to X. By theorem 1.3.6, Int(F) is S-closed relative to X. Since Int(F) is proper regular-open, X-Int(F) is proper regular-closed.

By assumption, X - Int(F) is S-closed relative to X.

$\therefore$ X = Int(F) U (X - Int(F))

Hence X is S-closed being a finite union of sets which are S-closed relative to X.

Theorem : 1.3.8

If a space X is the union of a finite number of S-closed open subsets, then X is S-closed.

Proof

Let $X = \bigcup_{k=1}^{n} A_k$, A_k is S-closed and open. By theorem 1.3.5 we get A_k is S-closed relative to X.

$\therefore$ X is S-closed being the union of a finite number of sets S-closed relative to X.

Theorem 1.3.9

If there exists a dense subset G of X which is S-closed relative to X, then X is S-closed.

Proof

Let $\{ F_\alpha \,/\, \alpha \in I \}$ be a regular-closed cover of X. Since G is S-closed relative to X, by theorem 1.3.3 there exists a finite subfamily I_0 of I such that

$G \subset \bigcup \{ F_\alpha \,/\, \alpha \in I_0 \}$

Since $X = Cl(G)$, $X \subset \bigcup \{ F_\alpha \,/\, \alpha \in I_0 \}$

Hence X is S-closed.

SECTION : 1.4

Sivaraj [12] has obtained properties of S-closed spaces using α-sets. First we discuss the results on semi-open sets and α-sets which are used to prove the main results on S-closed spaces.

Theorem : 1.4.1

In a space X,

(i) if A is semi-open, then $Cl(A) = Cl^{*}(A)$, and

(ii) if A is semi-closed, then $Int(A) = Int^{*}(A)$.

Proof

To prove $Cl^{*}(A) \subset Cl(A)$

$Cl^{*}(A) = \cap \{ B / X-B \in \tau^{\alpha} \}$

$Cl(A) = \cap \{ K / X-K \in \tau \}$

Since $\tau \subset \tau^{\alpha}$, $Cl^{*}(A) \subset Cl(A)$ ... (1)

Let $x \notin Cl^{*}(A)$

Then there exists an open set V in X^{*} such that $x \in V$ and $V \cap A = \emptyset$.

$\Longrightarrow$ $Int(A) \cap Int(V) = \emptyset$

$\Longrightarrow$ $Int(A) \cap Cl(Int(V)) = \emptyset$

$\Longrightarrow$ $Int(A) \cap Int(Cl(Int(V))) = \emptyset$

$\Longrightarrow$ $Int(A) \cap V^{+} = \emptyset$

$\Longrightarrow$ $Cl(Int(A)) \cap V^{+} = \emptyset$

Since A is semi-open, $A \subset Cl(Int(A))$.

$\Longrightarrow \quad A \cap V^+ = \emptyset$

Moreover $x \in V^+$ as $V \subset V^+$

Hence $x \notin Cl(A)$

$\therefore \quad Cl(A) \subset Cl^*(A)$ ... (2)

From (1) and (2) we get

$Cl(A) = Cl^*(A)$

(ii) A is semi-closed

$\therefore$ X - A is semi-open.

By (i), $Cl(X-A) = Cl^*(X-A)$

Since $Cl(X-A) = X - Int(A)$, we get

$X - Int(A) = X - Int^*(A)$

$\Longrightarrow \quad Int(A) = Int^*(A)$

Hence the theorem.

Corollary : 1.4.2

In a space X, if A is semi-open then $Int(Cl(A)) = Int^*(Cl(A)) = Int^*(Cl^*(A))$.

Proof

Follows immediately from the above theorem.

Corollary : 1.4.3

If X is extremally disconnected and A is semi-open, then $sCl(A) = Cl(A)$.

Proof

In an extremally disconnected space, $SO(X) = \tau^{\alpha}$

Hence $s\,Cl(A) = Cl^*(A)$ for any subset A of X. By theorem 1.4.1(i), since A is semi-open, $sCl(A) = Cl^*(A) = Cl(A)$.

Corollary : 1.4.4

Let A be a semi-open subset of a space X.

(i) If B is semi-open in A, then $Cl_A(B) = Cl_A^*(B)$, and

(ii) if B is semi-closed in A, then $Int_A(B) = Int_A^*(B)$.

Proof

(i) If B is semi-open in A, $B = C \cap A$, C is semi-open in X, then $B \in SO(X)$.

By theorem 1.4.1,

$$Cl(B) = Cl^*(B)$$

Hence $Cl_A(B) = Cl_A^*(B)$

(ii) Since A-B is semi-open in A, proof follows from (i).

Theorem : 1.4.5

If A is a semi-open subset of a space (X, τ), then $SO(A, \tau/A) = SO(A, \tau^{\alpha}/A)$.

Proof

Let $B \in SO(A, \tau/A)$

$\Longrightarrow$ $B \in SO(X) = SO(X^*)$

By theorem "Let $A \subset Y \subset X$ where X is a topological space and Y is a subspace. Let $A \in SO(X)$. Then $A \in SO(Y)$" in [5], we get

$B \in SO(A, \tau^{\alpha}/A)$

Conversely,

B is semi-open in $(A, \tau^{\alpha}/A)$

$\Longrightarrow$ $B \in SO(X^*)$ since A is semi-open.

$\Longrightarrow$ $B \in SO(X)$

$\Longrightarrow$ $B \in SO(A, \tau/A)$

Hence $SO(A, \tau/A) = SO(A, \tau^{\alpha}/A)$.

Theorem : 1.4.6

Let X be a topological space,

(i) X is S-closed iff X^* is S-closed.

(ii) $A \subset X$ is S-closed relative to X iff A is S-closed relative to X^*.

Proof

Follows from theorem 1.4.1 and the fact that $SO(X) = SO(X^*)$.

Theorem : 1.4.7

A semi-open set A is S-closed in X iff A is S-closed in X^*.

Proof

Since $A \in SO(X)$,

By Theorem 1.4.5, we get $SO(A, \tau/A) = SO(A, \tau^{\alpha}/A)$. Since A is S-closed as a subspace of X.

==> Any cover $\{A_{\alpha}\}$ of X, where $A_{\alpha} \in SO(A, \tau/A) = SO(A, \tau^{\alpha}/A)$ must have a finite subcover $\{A_i\}_{i=1}^{n}$ such that

$$A \subset \bigcup_{i=1}^{n} Cl_A(A_i)$$

By corollary 1.4.4, we get $A \subset \bigcup_{i=1}^{n} Cl_A^{*}(A_i)$

Thus A is S-closed in X^*.

Corollary : 1.4.8

If a space X is the union of a finite number of S-closed α-sets, then X is S-closed.

Proof

Let $X = \bigcup_{k=1}^{n} A_k$, A_k is S-closed and an α-set.

By the above theorem, each A_k is S-closed in X^*. Since each α-set is open in X^*.

==> each A_k is open in X^*.

==> X is union of finite number S-closed open sets (in X^*).

==> X^* is S-closed (by theorem 1.3.8)

==> X is S-closed (by theorem 1.4.7)

SECTION : 1.5

Regarding the properties of S-closed spaces the main result proved by Dlaska, Ergun and Ganster [4], is the following :

A space (X, τ_s) is S-closed and extremally disconnected.
iff it is compact and extremally disconnected
iff $(X, \tau SR)$ is compact.

Moreover $(X, \tau SR)$ is countably compact $\Longrightarrow$ (X, τ) is countably S-closed.

Definition : 1.5.1

A topological space X is called **countably S-closed** if every countable cover of X by the regular closed sets has a finite subcover.

Definition : 1.5.2

A topological space X is called **almost extremally disconnected** if $\partial U = Cl(U) - U$ is finite for every $U \in RO(X, \tau)$.

Theorem : 1.5.3

$$(X, \tau) \text{ is } R_0 \Longrightarrow (X, \tau_s) \text{ is } R_0.$$

Proof

Take $G \in \tau_s$ and $x \in G$
Then $G \in \tau$ and $x \in G$

Since (X, τ) is R_0, we get $Cl_\tau\{x\} \subset G$

Since $\tau_s \subset \tau$

$\Longrightarrow$ $Cl_{\tau_s}\{x\} \subset Cl_\tau\{x\} \subset G$

$\Longrightarrow$ $Cl_{\tau_s}\{x\} \subset G$

Hence (X, τ_s) is R_0.

Theorem : 1.5.4

(X, τ) is extremally disconnected iff $SR(X, \tau) = CO(X, \tau)$.

Proof

Assume (X, τ) is extremally disconnected.

Take $A \in CO(X, \tau)$

$\Longrightarrow$ A is open and closed.

$\Longrightarrow$ A is semi-open and semi-closed.

$\Longrightarrow$ $A \in SR(X, \tau)$

Hence $CO(X, \tau) \subset SR(X, \tau)$ ... (1)

Take $A \in SR(X, \tau)$

$\Longrightarrow$ A is semi-open and A is semi-closed.

By theorem 1.4.3, $Cl(A) = sCl(A)$

$= A$ (Since A is semi-closed)

$\Longrightarrow$ A is closed.

Since closure of every semi-open set in an extremally disconnected space is open, we get

$Cl(A)$ is open.

$\Longrightarrow$ A is open.

$\therefore$ $A \in CO(X, \tau)$

Hence $SR(X, \tau) \subset CO(X, \tau)$ ... (2)

From (1) and (2), we get

$$SR(X, \tau) = CO(X, \tau)$$

Corollary : 1.5.5

The following are equivalent for any topological space X.

(1) (X, τ) is extremally disconnected.

(2) $\tau SR = \tau_s$

(3) $\tau SR \subseteq \tau$

Proof

(1) $\Longrightarrow$ (2)

If (X, τ) is extremally disconnected, then

$SR(X, \tau) = RO(X, \tau) \subset \tau_s$

$\Longrightarrow$ $\tau SR \subset \tau_s$

Always $\tau_s \subset \tau SR$

$\Longrightarrow$ $\tau SR = \tau_s$

(2) $\Longrightarrow$ (3)

Since $\tau_s \subset \tau$

We get $\tau SR \subset \tau$

(3) ==> (1)

If $k \in RC(X, \tau)$ then $k \in SR(X, \tau)$ [since every regular closed set is semi-regular].

==> $k \in SR(X, \tau) \subset \tau SR \subset \tau$

Thus every regular closed set is open.

Hence (X, τ) is extremally disconnected.

Corollary : 1.5.6

Let (X, τ) be a semi-regular R_0 space. Then the following are equivalent.

(1) $(X, \tau SR)$ is compact.

(2) (X, τ) is S-closed and extremally disconnected

(3) (X, τ) is compact and extremally disconnected.

Proof

(1) ==> (2)

Let $\{ V_\alpha / \alpha \in I \}$ be a regular closed cover of X. Since every regular closed set is semi-regular, we get $\{ V_\alpha / \alpha \in I\}$ is a semi-regular cover of X. By (1) there exists a finite subset I_0 of I such that

$$X = \bigcup_{\alpha \in I_0} V_\alpha$$

X is S-closed.

Since (X, τ) is R_0, by theorem 1.5.3 we get (X, τ_s) is R_0. Hence (X, τ) is extremally disconnected (by the result "An S-closed space is extremallydisconnected iff (X, τ_s) is R_0").

(2) $\Longrightarrow$ **(3)**

Since extremally disconnected semi-regular spaces are regular and regular S-closed spaces are compact (by theorem 1.1.3).

Hence (X, τ) is compact.

(3) $\Longrightarrow$ **(1)**

Since $\tau SR \subset \tau$, by corollary 1.5.5 we get, $(X, \tau SR)$ is compact.

Theorem : 1.5.7

If $(X, \tau SR)$ is countably compact, then (X, τ) is countably S-closed and almost extremally disconnected.

Proof

Since every regular closed set is semi-regular. We get by assumption (X, τ) is countably S-closed.

Since $RC(X, \tau) \subseteq SR(X, \tau) \subseteq \tau SR$.

Suppose there exists an $U \in RO(X, \tau)$ such that ∂U is infinite.

Then ∂U can be written as a countable partition

$$\text{ie.,} \quad \partial U = \bigcup_{n=1}^{\infty} A_n$$

Now the countable cover $\{X - Cl(U)\} \cup \{U \cup A_n\}$ by semi-regular sets does not have finite subcover which is a contradiction to $(X, \tau SR)$ is countably compact. Hence ∂U is finite and (X, τ) is almost extremally disconnected.

STUDY OF S-CLOSED SPACES UNDER DIFFERENT TYPES OF MAPPINGS

CHAPTER II

STUDY OF S-CLOSED SPACES UNDER DIFFERENT TYPES OF MAPPINGS

This chapter is devoted to the study of S-closed spaces under irresolute, semi-continuous, almost-open, weakly continuous, semi-closed, pre-semi-open and semi-open mappings. Thompson [14] has concentrated on the semi-continuous and irresolute images of S-closed spaces. The main result proved in his paper is : "A Hausdorff space X is S-closed iff the irresolute image of X in any Hausdorff space is closed". Noiri [8] has improved some of the results of Thompson [14] and has obtained a characterization of S-closed spaces in terms of semi-continuous mappings. Sivaraj [12] has improved results of Noiri [8] using α-sets and I-compact sets.

SECTION : 2.1

Thompson [14] has investigated the images of S-closed spaces under semi-continuous and irresolute mappings. In this section we discuss these results.

Theorem : 2.1.1

The semi-continuous image of an S-closed space onto any Hausdorff space is H-closed.

Proof

Let $f : X \to Y$ be a semi-continuous surjection. Let $\{ V_\alpha \,/\, \alpha \in I \}$ be an arbitrary open cover of Y. Since f is semi-continuous, $\{ f^{-1}(V_\alpha) \,/\, \alpha \in I \}$ is a semi-open cover of X. By hypothesis, there exists a finite subfamily $\{ f^{-1}(V_{\alpha_i}),\ i = 1, 2, \ldots, n \}$ such that

$$X = \bigcup_{i=1}^{n} Cl(f^{-1}(V_{\alpha_i}))$$

Since $\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$ is dense in X,

$$X = sCl \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$$

$$Y = f(X) = f(sCl(\bigcup_{i=1}^{n} (f^{-1}(V_{\alpha_i}))))$$

$$\subset f(Cl \bigcup_{i=1}^{n} (f^{-1}(V_{\alpha_i})))$$

$$\subset Cl (\bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))) \quad \text{[Since f is surjection]}$$

$$\subset Cl \bigcup_{i=1}^{n} (V_{\alpha_i})$$

$$\subset \bigcup_{i=1}^{n} Cl(V_{\alpha_i})$$

Hence Y is H-closed.

Theorem : 2.1.2

If $f : X \to Y$ is an irresolute surjection of an S-closed space X, then Y is S-closed.

Proof

Let $\{ V_\alpha \ / \ \alpha \in I \}$ be a semi-open cover of Y. Since f is irresolute, $\{ f^{-1}(V_\alpha) \ / \ \alpha \in I \}$ is a semi-open cover of X. The remaining part of the proof is same as the proof of the previous theorem.

Theorem : 2.1.3

The irresolute image of any S-closed Hausdorff space in any Hausdorff space is closed.

Proof

Let $f : X \to Y$ be an irresolute function from S-closed space X to a Hausdorff space Y. Let $y \in Cl(f(X))$ and N(y) be the open neighbourhood filterbase about y.

Since X is S-closed, by theorem 1.1.2 the filterbase $F = f^{-1}(N(y))$ has an s-accumulation point x.

Claim : The filterbase $f(F)$ accumulates to $f(x)$.

Let V be any open set containing $f(x)$. Then V is semi-open. Hence $f^{-1}(V)$ is a semi-open set containing x and therefore for every $W \in N(y)$, $f^{-1}(W) \in F$, and $f^{-1}(W) \cap Cl(f^{-1}(V)) \neq \emptyset$.

By lemma 17, we have $Int(f^{-1}(W)) \cap Int(f^{-1}(V)) \neq \emptyset$.

$$\begin{aligned} \text{Therefore } \emptyset &\neq f[f^{-1}(Int\ f^{-1}(W) \cap Int\ f^{-1}(V))] \\ &\subset f[f^{-1}(W) \cap f^{-1}(V)] \\ &\subset W \cap V \end{aligned}$$

Since W and V are arbitrary chosen, we have $f(F)$ accumulates to $f(x)$.

But $f(F)$ is a finer filterbase than $N(y)$, hence $N(y)$ accumulates to $f(x)$.

Since $N(y)$ converges to y, and since Y is Hausdorff we get that $f(x) = y$.

Hence $y \in f(X)$ and $f(X)$ is closed in Y.

The converse of the above theorem is given in the following theorem.

Theorem : 2.1.4

If every irresolute image of a Hausdorff space X in any Hausdorff space Y is closed, then X is S-closed.

Proof

Suppose that X is not S-closed. Then by theorem 1.1.2, there exists a filterbase F with no s-accumulation point. Thus for every $x \in X$, there exists an open set $V(x)$ containing x and an element $F_{a(x)}$ of F such that $F_{a(x)} \cap Cl(V(x)) = \emptyset$.

Let S denote the collection of all finite intersections of sets of the form $X - Cl(V(x))$.

This collection forms an open filterbase. Select an object ∞ not in X and consider the space $\hat{X} = X \cup \{\infty\}$ with following topology :

Neighbourhoods of points in X are unchanged, and a basic neighbourhood system of ∞ is

$$N(\infty) = \{ S_b \cup \{ \infty \} / S_b \in S \}$$

It is easily seen that with this topology $\hat{X}$ is Hausdorff.

Consider the inclusion map $i : X \to \hat{X}$

This i is irresolute and that i(X) is not closed in $\hat{X}$ which is a contradiction.

Hence X is S-closed.

Combining theorem 2.1.3 and 2.1.4 we have the following main result.

"A Hausdorff space X is S-closed iff the irresolute image of X in any Hausdorff space is closed".

SECTION : 2.2

In this section we discuss the results of Noiri [8]. He has studied the behaviour of sets S-closed relative to a space under (1) irresolute, (2) weakly continuous, (3) almost open and (4) semi-closed mappings. Special attention is given to semi-continuous images of S-closed spaces.

Sets S-closed relative to X

Theorem : 2.2.1

Let X be an extremally disconnected space and $f : X \to Y$ an irresolute function. If G is S-closed relative to X, then f(G) is S-closed relative to Y.

Proof

Let $\{ V_\alpha / \alpha \in I \}$ be a cover of f(G) by semi-open sets in Y.

Since f is irresolute, $\{ f^{-1}(V_\alpha) / \alpha \in I \}$ is a cover of G by semi-open sets of X.

Since G is S-closed relative to X, there exists a finite subfamily I_0 of I such that $G \subset \cup \{ Cl_X f^{-1}(V_\alpha) / \alpha \in I_0 \}$. Since X is extremally disconnected, we get

$$G \subset \cup \{ sCl_X f^{-1}(V_\alpha) / \alpha \in I_0 \}$$

ie. $G \subset sCl_X(\cup \{ f^{-1}(V_\alpha) / \alpha \in I_0 \})$

$$\therefore \quad f(G) \subset f(sCl_X(\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)))$$

$$\subset sCl_Y(f(\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)))$$

$$\subset Cl_Y(f(\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)))$$

$$\subset Cl_Y(\bigcup_{\alpha \in I_0} V_\alpha)$$

$$\subset \bigcup_{\alpha \in I_0} Cl_Y(V_\alpha)$$

Hence f(G) is S-closed relative to Y.

Theorem : 2.2.2

Let $f : X \to Y$ be an irresolute function. If G is an open S-closed set of X, then f(G) is S-closed in Y.

Proof

Let $f_G : G \to f(G)$ be a function defined by $f_G(x) = f(x)$, for every $x \in G$.

To show that f_G is irresolute.

Let $V_0 \in SO(f(G))$

Then there exists a $V \in SO(Y)$ such that $V_0 = V \cap f(G)$.

Since f is irresolute, $f^{-1}(V) \in SO(X)$

Therefore $f^{-1}(V) \cap G \in SO(X)$

Therefore $f_G^{-1}(V_0) = (f^{-1}(V) \cap G) \in SO(G)$

Hence f_G is irresolute.

Since G is S-closed and f_G is irresolute by theorem 2.1.2 we get, $f_G(G) = f(G)$ is S-closed in Y.

The following lemma is useful in proving the theorem 2.2.4.

Lemma : 2.2.3

If a function $f : X \to Y$ is weakly continuous and almost-open then $f^{-1}(F)$ is regular closed in X for every regular closed set F of Y.

Theorem : 2.2.4

Let $f : X \to Y$ be a weakly continuous and almost-open function. If G is S-closed relative to X, then f(G) is S-closed relative to Y.

Proof

Let $\{F_\alpha / \alpha \in I\}$ be any cover of f(G) by regular closed sets of Y.

Then by lemma 2.2.3, $\{f^{-1}(F_\alpha) / \alpha \in I\}$ is a cover of G by regular closed sets of X.

By theorem 1.3.3, $G \subset \cup \{ f^{-1}(F_\alpha) / \alpha \in I \}$ for some finite subfamily I_0 of I

Hence $f(G) \subset \cup \{ F_\alpha / \alpha \in I_0 \}$

Hence f(G) is S-closed relative to Y (by theorem 1.3.3)

Corollary : 2.2.5

If X is an S-closed space and $f : X \to Y$ is a weakly-continuous almost-open surjection, then Y is S-closed.

Proof

Follows immediately from the above theorem.

Theorem : 2.2.6

Let X be an extremally disconnected space, $f : X \to Y$ a semi-closed, almost-open surjection and $f^{-1}(y)$ S-closed relative to X for each point $y \in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.

Proof

Let $\{ F_\alpha / \alpha \in I \}$ be a cover of $f^{-1}(G)$ by regular-closed sets of X.

Take $y \in G$.

Since $f^{-1}(y)$ is S-closed relative to X, by theorem 1.3.3 there exists a finite subfamily $I(y)$ of I such that

$$f^{-1}(y) \subset \cup \{ F_\alpha / \alpha \in I(y) \}$$

Since X is extremally disconnected, for each $\alpha \in I$

$$F_\alpha = Cl_X(Int_X(F_\alpha)) \text{ is open in X.}$$

Hence we get, $f^{-1}(y) \subset \cup \{ Cl_X(Int_X(F_\alpha)) \,/\, \alpha \in I(y) \}$

Put $U(y) = \cup \{ F_\alpha / \alpha \in I(y) \}$, since f is semi-closed and surjection, there exists a $V(y) \in SO(Y)$ such that $y \in V(y)$ and $f^{-1}(V(y)) \subset U(y)$. Then $\{ V(y) \,/\, y \in G \}$ is a cover of G by semi-open sets of Y. Since G is S-closed relative to Y, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in G such that

$$G \subset \cup \{(Cl_Y(V(y_j))) \,/\, j = 1, 2, \ldots, n\}$$

Since f is almost open, we get

$$f^{-1}(G) \subset \bigcup_{j=1}^{n} f^{-1}(Cl_Y(V(y_j)))$$

$$\subset \bigcup_{j=1}^{n} Cl_X(f^{-1}(V(y_j))) \qquad \text{(by theorem 2.3.5)}$$

$$\subset \bigcup_{j=1}^{n} Cl_X(U(y_j))$$

$$f^{-1}(G) \subset \bigcup_{j=1}^{n} \; \bigcup_{\alpha \in I(y_i)} F_\alpha$$

By theorem 1.3.3, we get $f^{-1}(G)$ is S-closed relative to X.

Corollary : 2.2.7

Let X be an extremally disconnected space and $f : X \to Y$ a closed open surjection with compact point inverses. If Y is an S-closed space, then X is S-closed.

Proof

This follows immediately from theorem 1.3.3 and 2.2.6.

Corollary : 2.2.8

If X is compact, Y is S-closed and X x Y is extremally disconnected, then X x Y is S-closed.

Proof

Since X is compact, the natural projection π_Y : X x Y → Y is a closed open surjection with compact point inverses. Therefore, it follows from corollary 2.2.7 that X x Y is S-closed.

Remark : 2.2.9

In corollary 2.2.8, the condition extremally disconnected on X x Y cannot be dropped because β N is S-closed and compact, but βN x βN is not S-closed [Example 1.2.8].

Semi-Continuous images of S-closed spaces

Lemma : 2.2.10

If f : X → Y is a semi-continuous function and G is an open set of X, then the function f_G : G → f(G) defined by $f_G(x) = f(x)$ for every x ε G, is semi-continuous.

Proof

Let V be an open set in $f(G)$.

Then $V = f(G) \cap A$, where A is open in Y.

Since f is semi-continuous, $f^{-1}(A)$ is semi-open in X.

Therefore $f_G^{-1}(V) = G \cap f^{-1}(A)$ is semi-open in X

$\Longrightarrow f_G^{-1}(V) = (G \cap f^{-1}(A)) \in SO(G)$

$\Longrightarrow f_G$ is semi-continuous.

Theorem : 2.2.11

The semi-continuous image of any S-closed space in any Hausdorff space is closed.

Proof

Let $f : X \rightarrow Y$ be a semi-continuous function from an S-closed space X to a Hausdorff space Y.

Let $f_X : X \rightarrow f(X)$

Then by lemma 2.2.10, we get f_X is semi-continuous. Since f_X is semi-continuous and surjection, by theorem 2.1.1, we get $f(X)$ is H-closed.

$\Longrightarrow$ $f(X)$ finite union of closed sets.

Hence $f(X)$ is closed.

Characterization of S-closed spaces in terms of semi-continuous function

Theorem : 2.2.12

A Hausdorff space X is S-closed iff every semi-continuous function of X into any Hausdorff space Y is almost-closed.

Proof

Assume X is an S-closed Hausdorff space. Let $f : X \rightarrow Y$ be a semi-continuous function. Since X is S-closed Hausdorff, it is extremally disconnected (by theorem 1.1.10).

Let F be any regular-closed set of X, then $F = Cl_X(Int_X(F))$ is open in X.

Therefore by Lemma 2.2.10, $f_F : F \rightarrow f(F)$ is semi-continuous. Since X is S-closed, F is S-closed (by theorem 1.1.6). Hence $f_F(F) = f(F)$ is H-closed (by theorem 2.2.1). Therefore f(F) is closed in Y. Hence f is almost-closed.

Conversely, assume the given condition

Let Y be any Hausdorff space and $f : X \rightarrow Y$ any irresolute function.

Then, f is semi-continuous.

Hence, by hypothesis, f is almost-closed.

Thus, f(X) is closed in Y.

Hence X is S-closed (by theorem 2.1.4)

Corollary : 2.2.13

If X is an S-closed Hausdorff space, Y is a Hausdorff space and $f : X \to Y$ is a semi-continuous bijection, then f is irresolute and Y is S-closed.

Proof

Let V be any semi-open set of Y. Since f is semi-continuous, $f^{-1}(Int_Y(V)) \in SO(X)$. Now, put $F = Cl_X(Int_X(f^{-1}(Int_Y(V))))$, then we have $f^{-1}(Int_Y(V)) \subset F$ and hence $Int_Y(V) \subset f(F)$. By theorem 2.2.12, f is almost-closed and hence f(F) is closed in Y.
Therefore, we have $Cl_Y(Int_Y(V)) \subset f(F)$.
Therefore we obtain $f^{-1}(V) \subset f^{-1}(Cl_Y(Int_Y(V))) \subset F \subset Cl_X(Int_X f^{-1}(V))$
This shows that $f^{-1}(V) \in SO(X)$.
Therefore, f is irresolute.
Hence Y is S-closed (by theorem 2.1.2).

SECTION : 2.3

This section is devoted to the study of S-closed spaces using α-sets and I-compact spaces. Sivaraj [12] has proved that the inverse image of an S-closed space under an almost open, pre semi-open (semi-open) bijection is S-closed (QHC). He [12] has also established that an irresolute image of a relative I-compact subset is relatively S-closed.

Lemma : 2.3.1

If a function $f : X \to Y$ is semi-continuous and G is open in X^*, then f_G is semi-continuous.

Proof

Let V be open in $f(G)$.

$V = A \cap f(G)$, where A is open in Y.

Since f is semi-continuous, $f^{-1}(A)$ is semi-open in X.

$$f_G^{-1}(V) = f_G^{-1}(A \cap f(G))$$
$$= G \cap f^{-1}(A)$$

$\Longrightarrow$ $f_G^{-1}(V)$ is semi-open in X.

Therefore, $f_G^{-1}(V)$ is semi-open in G.

Hence f_G is semi-continuous.

Lemma : 2.3.2

If $f : X \to Y$ is an irresolute function and G is open in X^*, then f_G is an irresolute function.

Proof

Let V be semi-open in $f(G)$.

Then $V = f(G) \cap A$, where A is semi-open in Y.

Therefore, $f_G^{-1}(V) = G \cap f^{-1}(A)$

Since f is irresolute, $f^{-1}(A)$ is semi-open in X.

$\Longrightarrow$ $f_G^{-1}(V)$ is semi-open in X.

$\Longrightarrow$ $f_G^{-1}(V)$ is semi-open in G.

$\Longrightarrow$ f_G is an irresolute function.

Theorem : 2.3.3

Let $f : X \to Y$ be an irresolute function. If G is open in X^* which is also an S-closed subspace of X, then f(G) is an S-closed subspace of Y.

Proof

Since f is an irresolute function, by the above lemma we get, f_G is an irresolute function. Hence by theorem 2.1.2, we get $f_G(G) = f(G)$ is an S-closed subspace of Y.

Lemma : 2.3.4

If $f : X \to Y$ is a semi-homeomorphism, and G is open in X^*, then f_G is a semi-homeomorphism.

Proof

Since f is a semi-homemorphism, f is both irresolute and pre-semi-open. By lemma 2.3.2, we get f_G is irresolute. Let B be a semi-open set in G. Then B is semi-open in X, as G is semi-open in X. Since f is pre-semi-open, f(B) is semi-open in Y. Since $f_G(B) = f(B) \subset f(G)$, $f_G(B)$ is semi-open in f(G).

$\Longrightarrow$ f_G is pre-semi-open. Hence f_G is a semi-homeomorphism.

Lemma : 2.3.5

A function $f : X \to Y$ is almost open iff $f^{-1}(Cl_Y(V)) \subset Cl_X(f^{-1}(V))$ for each semi-open set V in Y.

Proof

Assume f is almost open. Let V be semi-open in Y.

Then there exists an open set G in Y such that

$$G \subset V \subset Cl_Y(G)$$

Since f is almost open and G is open in Y, we get

$$f^{-1}(Cl_Y(G)) \subset Cl_X(f^{-1}(G))$$

$$V \subset Cl_Y(G) \implies f^{-1}(Cl_Y(V)) \subset f^{-1}(Cl_Y(G)) \subset Cl_X(f^{-1}(G)) \subset Cl_X(f^{-1}(V))$$

Hence $f^{-1}(Cl_Y(V)) \subset Cl_X(f^{-1}(V))$

Conversely, assume the given condition

Let V be open in Y.

Then V is semi-open in Y.

$\implies f^{-1}(Cl_Y(V)) \subset Cl_X(f^{-1}(V))$

$\implies$ f is almost open.

Theorem : 2.3.6

If a function $f : X \to Y$ is an almost open, pre-semi-open bijection, and Y is S-closed, then X is S-closed.

Proof

Let $\{ V_\beta \ / \ \beta \in I \}$ be a semi-open cover of X. Since f is pre-semi-open, $f(V_\beta)$ is semi-open in Y.

$\implies \{ f(V_\beta) \ / \ \beta \in I \}$ is a semi-open cover of Y. Since Y is S-closed, there exists a finite subfamily I_0 of I such that $Y = \cup \{ Cl_Y(f(V_\beta)) \ / \ \beta \in I_0 \}$. Since f is bijection,

$$X = \cup \{ f^{-1}(Cl_Y(f(V_\beta))) \ / \ \beta \in I_0 \}$$

$$= \cup \{ Cl_X(V_\beta) \ / \ \beta \in I_0 \}$$

Hence X is S-closed.

Following a similar proof as in the above theorem one gets immediately the following results.

Corollary : 2.3.7

If a function $f : X \to Y$ is an almost-open, pre-semi-open bijection and G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.

Theorem : 2.3.8

If a function $f : X \to Y$ is an almost-open, semi-open bijection and Y is S-closed, then X is QHC.

Proof

Let $\{ V_\beta \,/\, \beta \in I \}$ be an open cover of X. Since f is semi-open, we get each $f(V_\beta)$ is semi-open in Y. Hence $\{ f(V_\beta) \,/\, \beta \in I \}$ is a semi-open cover of Y. Since Y is S-closed, there exists a finite subfamily I_0 of I such that,

$$Y = \cup \{ Cl_Y(f(V_\beta)) \,/\, \beta \in I_0 \}$$

$$X = \cup \{ f^{-1}(Cl_Y(f(V_\beta))) \,/\, \beta \in I_0 \}$$

Since f is almost open, and f is bijection.

$$X = \cup \{ Cl_X(V_\beta) \,/\, \beta \in I_0 \}$$

$\Longrightarrow$ X is QHC.

Theorem : 2.3.9

A space X is I-compact iff every semi-open cover of X has a finite subfamily, the interior of whose closures cover X.

Proof

The result follows from the fact that closure of a semi-open set is regular closed and a regular closed set is semi-open.

Corollary : 2.3.10

A subset A of a space X is I-compact relative to X iff every cover of A by semi-open sets of X has a finite subfamily, the interiors of whose closures cover A.

Remark : 2.3.11

From the above corollary, it is easy to see that if A is I-compact relative to X, then A is S-closed relative to X.

Theorem : 2.3.12

If a function $f : X \to Y$ is irresolute and $A \subset X$ is I-compact relative to X, then f(A) is S-closed relative to Y.

Proof

Let $\{ V_\beta \, / \, \beta \in I \}$ be a cover of f(A) by semi-open sets of Y.

Since f is irresolute, $\{ f^{-1}(V_\beta) \, / \, \beta \in I \}$ is a cover of A by semi-open sets of X.

Since A is I-compact relative to X,

$A \subset \cup \{ Int_X(Cl_X(V_\beta)) \, / \, \beta \in I_0 \}$ for some finite subfamily I_0 of I .

$\Longrightarrow$ $A \subset \cup \{ sCl_X(f^{-1}(V_\beta)) \,/\, \beta \in I_0 \}$

$f^{-1}(V_\beta) \subset f^{-1}(sCl_Y(V_\beta))$

$sCl_X(f^{-1}(V_\beta)) \subset sCl_X(f^{-1}(sCl_Y(V_\beta)))$

$= f^{-1}(sCl_Y(V_\beta))$ [Since $f^{-1}(sCl_Y(V_\beta))$ is semi-closed in X, since f is irresolute]

Therefore $A \subset \cup \{ f^{-1}(sCl_Y(V_\beta)) \,/\, \beta \in I_0 \}$

$f(A) \subset \cup \{ sCl_Y(V_\beta) \,/\, \beta \in I_0 \}$

$\subset \cup \{ Cl_Y(V_\beta) \,/\, \beta \in I_0 \}$

$\Longrightarrow$ f(A) is S-closed relative to X.

Theorem : 2.3.13

Let $f : X \to Y$ be a semi-closed almost open surjection and $f^{-1}(y)$ be I-compact relative to X for each $y \in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.

Proof

Let $\{ V_\beta \,/\, \beta \in I \}$ be a cover of $f^{-1}(G)$ by semi-open sets of X.

Since $f^{-1}(y)$ is I-compact relative to X, for each $y \in G$, there exists a finite subfamily $I(y)$ of such that

$$f^{-1}(y) = \cup \{ Int_X(Cl_X(V_\beta)) \,/\, \beta \in I(y) \}$$

Since f is a semi-closed surjection, there exists a semi-open set G_y in Y such that $y \in G_y$ and

$$f^{-1}(G_y) \subset \cup \{ \text{Int}_X(\text{Cl}_X(V_\beta)) \,/\, \beta \in I(y) \} \qquad \ldots \quad (1)$$

The collection $\{ G_y \,/\, y \in G \}$ is a semi-open cover of G. Since G is S-closed relative to Y, there exists $y_1, y_2, \ldots, y_n$ in G such that

$$G \subset \bigcup_{i=1}^{n} (\text{Cl}_Y(G_{y_i}))$$

$$f^{-1}(G) \subset \bigcup_{i=1}^{n} f^{-1}(\text{Cl}_Y(G_{y_i}))$$

Since f is almost open, by lemma 2.3.5

$$f^{-1}(G) \subset \bigcup_{i=1}^{n} \text{Cl}_X(f^{-1}(G_{y_i}))$$

Using (1), we get $f(G) \subset \bigcup_{i=1}^{n} \{ \text{Cl}_X(V_\beta) \,/\, \beta \in I(y_i),\ 1 \leq i \leq n \}$

$\Longrightarrow$ $f^{-1}(G)$ is S-closed relative to X.

Corollary : 2.3.14

If a function $f : X \to Y$ is a semi-closed, almost open surjection, $f^{-1}(y)$ is I-compact relative to X for each $y \in Y$ and Y is S-closed then X is S-closed.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

The concept of S-closed spaces was first introduced by Thompson in 1976. Since then a number of papers have been published on this topic discussing various aspects of S-closed spaces. In this thesis, we have made an attempt to study the contributions of Thompson [13], Cameron [2], Noiri [8], Dlaska, Ergun and Ganster [4] in detail. The following characterization of S-closed spaces are obtained :

(1) If X is a regular compact space, then X is S-closed iff X is extremally disconnected [13].

(2) A space X is S-closed iff every regular closed cover has a finite subcover [2].

(3) A space X is S-closed iff every proper regular - closed set of X S-closed relative to X [8].

(4) If X is a semi-regular R_0 extremally disconnected space, then X is S-closed iff it is compact [4].

Apart from the above characterizations, many interesting properties have been discussed. Moreover, a study of S-closed spaces under irresolute, semi-continuous, almost-open, weakly continuous, semi-closed, pre-semi-open and semi-open mappings has also been made.

When we go through the literature on S-closed spaces, we find that a special case of S-closed spaces, namely s-closed spaces has been introduced Di Maio and Noiri [3]. Though the details of this paper are not given in this thesis, we find that an interesting development of s-closed spaces has been carried out in this paper.

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