

Some Bulk - Service Queueing Models

BY

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Chapter I

INTRODUCTION

Queueing theory was founded by the work of A.K.Erlang (6), who began in 1908 to study problems of congestion in telephone service for the Copenhagen telephone company. The theory provides predictions about waiting times, the average number of waiting customers, the length of a busy period and so forth. These predictions help us to anticipate situations and to take appropriate measures to shorten the queues.

A further attractive feature of the theory is the quite astonishing range of its applications. Some of the more prominent of these are telephone conversation, machine repair, toll booths, taxi stands, inventory control, the loading and unloading of ships, scheduling patients in hospital clinics, production flow and applications in the computer field with respect to program-scheduling etc.

A queueing system may be described as one having a service facility, at which units of some kind (called 'customers') arrive for service and whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line develops. The waiting units take their turn for service according to a pre-assigned

rule, and after service they leave the system. Thus the input to system consists of the customers demanding service and the output is the serviced customers.

I.CHARACTERISTICS OF QUEUEING PROCESSES

The basic characteristic features are

- 1.Input process
2. Service mechanism
3. Queue discipline
4. Number of servers
5. System capacity

1.INPUT PROCESS

The input process describes the manner in which units arrive (either singly or in a group) and join the system. The interval between two consecutive arrivals is called the interarrival time.

2.SERVICE MECHANISM

The service mechanism describes the manner in which service is rendered. A unit may be served either singly or in

a batch. The time required for servicing a unit (or a group, in case of batch service) is called the service time.

3.QUEUE DISCIPLINE

Queue discipline refers to the manner by which customers are selected for service when a queue has formed. The most common discipline is

First In, First Out (FIFO)

the other disciplines are

Last In, First Out (LIFO)

Service in Random Order

Serving in Priority

4.NUMBER OF SERVERS

A queueing system may contain a number of service channels (either parallel or series or a combination of both) to provide service. When the system has a number of parallel servers it is known as multi server model.

5.SYSTEM CAPACITY

The source or the population from which the input comes is finite or infinite.

PRIORITY DISCIPLINE

The occurrence of interruptions in customer's service due to the arrival of some customers is called the priority. There are two types of priorities.

1. Non-preemptive rule.
2. Preemptive rule.

NOTATION

Kendall proposed a convenient notation to denote a queueing systems. A queueing process is described by the notation $A|B|C|X|Y$, where

- A:** represents the interarrival time distribution of the customers from the source population,
- B:** the service time distribution of the given service facility
- C:** the number of parallel service channels
- X:** the capacity of the system and
- Y:** the type of queue discipline.

A and B usually take one of the following symbols,

- M** - for exponential (Markovian) distribution
- E_k** - for Erlang - k distribution
- G** - for arbitrary (General) distribution
- D** - for fixed (deterministic) distribution

STEADY STATE

A system is said to be in steady state when its operating characteristics are independent of time. That is the number of arrivals during a certain interval becomes independent of time.

TRANSIENT STATE

A system is said to be in transient state if its operating characteristics are dependent on time.

MARKOVIAN QUEUEING MODELS

Queueing models with inter-arrival time of customers, and service time exponentially distributed are called Markovian queueing models.

NON-MARKOVIAN QUEUEING MODELS

Queues in which inter arrival and/or service time distributions are other than exponential are known as non-Markovian queues. Since these systems do not have memory less property, the study of non-Markovian systems is much more complicated.

REVIEW OF LITERATURE

BULK SERVICE MODEL

There are situations particularly in transportation system where the service provided is such that a group (or batch) of customers can be served simultaneously. Examples include shuttle - bus service, freight trains, express elevators, tour guides & batch servicing in manufacturing processes.

There could be a number of policies or rules according to which batches for bulk service may be formed. The following are the types of bulk service policies or rules frequently discussed in the literature.

1. Bailey (1) considers that units are served in batches of not more than 'b' (say). If immediately after the completion of a service the server finds more than 'b' units waiting, he takes a batch of 'b' for service while others wait; if he finds 'r' units ($0 \leq r \leq b$) he takes all the 'b' units in a batch for service.

Blomena (3), Jaiswal (9), Neuts (15) consider the same rule with the restriction that $r \neq 0$ ($1 \leq r \leq b$), i.e., the service facility stops until a unit arrives. This rule will be called the usual bulk service rule, while Bailey's rule

will be its modified type (it will be also called bulk service rule with intermittently available server). Jaiswal points out that the distribution of the queue length for the modified rule can be obtained from that of the usual rule.

2. The rule with a fixed batch 'k' has been considered by Fabens (7) Takacs (20) and others. In this case the server waits until there are 'k' units.

3. Neuts (15) considers this rule: If, immediately after the completion of a service, the server finds less than 'a' units present, he waits until there are 'a' units, whereupon he takes the batch of 'a' units for service, if he finds 'a' or more but at most 'b', he takes them all in the batch and if he finds more than 'b', he takes in the batch for service 'b' units, while others wait. The batch takes a minimum of 'a' units and a max of 'b' units. This rule will be called general bulk service rule as rules under (1) and (2) can be taken as particular cases.

4. Bhat (2) considers the rule that the number taken in a batch is a random variable 'Y'. The corresponding Markov model is denoted by $M/M(y)/1$.

Borthakur (4) by assuming the service times were exponentially distributed and independent of the batch size was able to obtain explicit results for the steady state probability of queue length. Medhi (12) considered the waiting time distribution for the $M/M(a,b)/1$ system and gave

explicit results for the first two moments of this distribution. Easton and Chaudhry (5) obtained steady state results for the system $E_k/M(a,b)/1$ including the waiting time distribution.

Multiserver bulk service models are discussed by several authors. Notable among them are Medhi (13) who obtained an analytic expression for a passenger waiting time distribution in terms of the steady state probability $P_{0,0}$ that there are no idle servers and no waiting customers in the queue at a random epoch. Neuts and Nadarajan (16) expressed the joint distribution of customer waiting time and service group size and also the distribution of the number of the waiting customers and the number of free servers in algorithmically useful form. Sim and Templeton (17) have derived an exact recursive algorithm to study the steady state probabilities that there are 'm' idle servers and 'j' waiting customers in the queue. This differs from the Gauss Seidel iteration in the matrix geometric approach of Neuts & Nadarajan (16). Analytical expression for steady state system performance measures are derived in terms of $P_{0,0}$ which can be computed using the recursive algorithm. These results generalised those given by Medhi and Borthakur (11).

Madill and Chadhury (10) have discussed the queueing system $GI/M(a,b)/1$ and derived the expression for the moments of the steady state waiting time in queue and the moments of

the conditional waiting times of customer who are m^{th} in their batches.

BULK SERVICE QUEUE WITH ACCESSIBLE AND NON-ACCESSIBLE BATCHES

Under the general bulk service rule discussed by Neuts (15), the late arrivals could not join the batch already in service till the service of the batch is completed, even when the batch size under service is less than the maximum capacity. Thus the server's full capacity is not utilized while a batch of 'q' units such that $q < b$ is in service. Kleinrock (8) and Sivasamy (18) have considered the queueing models of this category. The service rule is defined as, while a batch of 'q' units ($a \leq q < b$) is in service, if additional units arrive before the completion of service of the batch concerned, they join the batch already in service subject to a maximum of 'b-q' units i.e., if a batch is being served and does not utilize its full capacity for service, it remains accessible for customers arriving during the service time of this batch until its full capacity for service is utilized. The service time of the batch is not altered by joining customers so that these customers only obtain service for the residual service time of the batch and leave the system with all other members of the batch.

Kleinrock (8) has discussed the $M/M(a,b)/1$, system under this service rule. Sivasamy (18) has considered $M/M(a,b)/1$ model under this rule and obtained steady state probabilities and waiting time distribution for this system.

PRIORITY QUEUES

Many situations in which waiting lines develop, are characterised by the occurrence of interruptions in customers service. Such interruptions may be caused by break down of a machine that provides service or by the customer assigned priority. The influence of service interruptions upon waiting-line behaviour, due to the priority customers can be investigated under two cases,

1. Non-preemptive priority
2. Preemptive priority

Queues with two independent classes of units, namely type 1 unit (higher priority) and type-2 unit (lower priority) have been studied by many authors. If type-2 units is being serviced and a unit of type-1 the higher priority arrives, it may either wait till the unit under service completes its service or it may preempt the unit under service. The former is called the non-preemptive priority discipline and the latter the preemptive priority discipline.

Kleinrock (8), Morse (14) have analysed non-preemptive priority queue with single server and Gail, Hantler and Taylor (21) have studied the non-preemptive priority with multiple servers.

Sivasamy (18), (19) have discussed priority of both types with single server. In the non-preemptive priority case he has considered the general bulk service rule for the low priority unit and in the preemptive priority queueing system the higher priority units are served in batches.

Synopsis

SYNOPSIS

In chapter - II $M/M(a,b)/2$ model is analysed explicitly. In this model, the inter arrival times are exponentially distributed with parameter λ and there are two servers doing service in batches according to the general bulk service rule introduced by Neuts [15]. The service times of both the servers are exponentially distributed with the same parameter μ . The service times are independent of batch size. The object of the chapter is to give the steady-state queue size probabilities in a closed form in terms of the unique root of the characteristic equation. The waiting time distribution of an arriving customer is calculated and the Little's formula is verified. The expected size of the service batch in which an arriving passenger will eventually be served is also derived. Numerical solutions are presented for the average queue length, expected busy period and the expected size of the service batches.

Chapter - III is devoted for a non-preemptive priority queueing problem, consisting of two types of units, with Poisson inputs and exponential service time distribution. The higher priority units are served singly in order of their arrivals and they have non-preemptive priority over the lower priority units. The lower priority units are served in batches of varying size under the usual bulk service rule. The

queue length distribution of both the classes have been obtained. Certain known results are derived as particular cases. Waiting time distribution of both the type - 1 customers are derived and the Little's formula is verified.

In chapter - IV, we consider the Non-Markovian GI/M(a,b)/1 queueing model under the bulk service rule in which the late entries are allowed to join the batch in course of on going service as long as the number of units in that batch is less than the maximum capacity 'b'. The closed form of the various performance measures (mean queue length, waiting time distribution) for the bulk service queue with accessible batches are derived.

Chapter II

M/M(a,b)/2 QUEUEING SYSTEM

Customers arrive individually at a service facility in accordance with a Poisson process having rate λ and form a queue waiting to be served. The facility consists of two homogeneous servers serving the customers under general bulk service rule. Each server will start his service if there exists atleast 'a' units in the queue. If a free server finds more than 'a' units but less than 'b' units in the queue, he takes all the customers into a batch together, for service. If he finds more than 'b' units, he takes only 'b' members in a batch for service. The service times are independently distributed random variables with an exponential distribution having mean $1/\mu$. The service distribution is independent of the batch size.

STEADY STATE RESULTS :

let $P_{i,n}$ denote the probability that there are n units in the queue and both the servers are busy if $i=2$ ($n \geq 0$) or one server is busy if $i=1$ ($0 \leq n \leq a-1$) or both the servers are idle if $i=0$ ($0 \leq n \leq a-1$) at the steady state. When both the servers are free, they have the equal chance of starting the service as soon as the minimum capacity is reached.

The steady state equations satisfied by $P_{i,n}$ are,

$$\lambda P_{0,0} = \mu_{1,0} \quad (1)$$

$$\lambda P_{0,n} = \lambda P_{0,n-1} + \lambda P_{1,n}, \quad 1 \leq n \leq a-1 \quad (2)$$

$$(\lambda + \mu)P_{1,0} = \lambda P_{0,a-1} + 2\mu P_{2,0}, \quad (3)$$

$$(\lambda + \mu)P_{1,n} = \lambda P_{1,n-1} + 2\mu P_{2,n}, \quad 1 \leq n \leq a-1 \quad (4)$$

$$(\lambda + \mu)P_{2,0} = \lambda P_{1,a-1} + 2\mu \sum_{s=a}^b P_{2,s}, \quad (5)$$

$$(\lambda + 2\mu)P_{2,n} = \lambda P_{2,n-1} + 2\mu P_{2,n+b}, \quad n \geq 1 \quad (6)$$

Equation (6) gives

$$2\mu P_{2,n+b+1} - (\lambda+2\mu)P_{2,n+1} + \lambda P_{2,n} = 0, \quad \forall n \geq 0$$

This equation can be rewritten as,

$$(2\mu E^{b+1} - (\lambda+2\mu)E + \lambda) P_{2,n} = 0, \quad \forall n \geq 0 \quad (7)$$

Where E is the forward shifting operator on the index n defined by,

$$E(P_{2,n}) = P_{2,n+1}$$

The characteristic equation of the above difference equation is

$$h(z) = 2\mu z^{b+1} - (\lambda+2\mu)z + \lambda = 0$$

$$z - \frac{2\mu z^{b+1} + \lambda}{(\lambda + 2\mu)} = 0$$

$$\text{i.e., } z - F(z) = 0$$

$$\text{Where } F(z) = \frac{2\mu z^{b+1} + \lambda}{(\lambda + 2\mu)}$$

$0 = z - F(z)$ will have exactly one real root inside $(0,1)$,

If $F'(z) > 1$ i.e., $(\lambda/2b\mu) < 1$.

Let r_0 be the root that lies inside $(0,1)$ and r_1, r_2, \dots, r_b be the other roots that lie outside the unit circle $|z| = 1$ of the characteristic equation.

$$\text{Then } P_{2,n} = Ar_0^n + \sum_{i=1}^n A_i r_i^n, \quad \forall n \geq 0 \quad (\text{from (7)})$$

$$\text{Since } \sum_{n=0}^{\infty} P_{2,n} < 1 \text{ and } |r_i| > 1, \quad i=1,2,\dots,b,$$

$$\text{We have } A_i = 0, \quad \forall i=1,\dots,n$$

$$\text{Thus } P_{2,n} = Ar_0^n, \quad \forall n \geq 0$$

$$\text{or } P_{2,n} = P_{2,0} r_0^n, \quad \forall n \geq 0 \quad (8)$$

Equation (4) implies that

$$(\lambda + \mu)P_{1,n+1} = \lambda P_{1,n} + 2\mu P_{2,n+1}, \quad 0 \leq n \leq a-2$$

$$((\lambda + \mu)E - \lambda) P_{1,n} = 2\mu P_{2,0} r_0^{n+1} \quad (\text{From (8)})$$

Using the differential equation technique

$$P_{1,n} = A_1 \frac{\lambda^n}{(\lambda + \mu)^n} + \frac{2\mu r_0^{n+1}}{(\lambda + \mu)r_0 - \lambda} P_{2,0}$$

$$P_{1,n} = \left[A \frac{\lambda^n}{(\lambda + \mu)^n} + \frac{2\mu r_0^{n+1}}{(\lambda + \mu)r_0 - \lambda} \right] P_{2,0}, \quad \forall n \geq 0$$

Where $A_1 = AP_{2,0}$

$$\text{Hence } P_{1,n} = [AR^n + Kr_0^n] P_{2,0}, \quad \forall n \geq 0 \quad (9)$$

$$\text{Where } R = \frac{\lambda}{\lambda + \mu}, \quad k = \frac{2\mu r_0}{(\lambda + \mu)r_0 - \lambda}$$

Add the second equation over $j=1$ to n .

$$\lambda P_{0,n} = \lambda P_{0,0} + \mu \sum_{j=1}^n P_{1,j} \quad \text{from (1)}$$

$$\text{i.e., } P_{0,n} = \frac{\mu}{\lambda} \sum_{j=1}^n P_{1,j}$$

Substituting for $P_{1,j}$ from (9)

$$P_{0,n} = \left[\frac{\mu}{\lambda} \frac{A(1-R^{n+1})}{1-R} + \frac{k\mu}{\lambda} \frac{(1-r_0^{n+1})}{1-r_0} \right] P_{2,0} \quad (10)$$

Thus the steady state probabilities are given by

$$P_{2,n} = r_0^n P_{2,0}, \quad \forall n \geq 0 \quad (11)$$

$$P_{1,n} = [AR^n + kr_0^n] P_{2,0} \quad \forall n \geq 0$$

$$P_{0,n} = \left[\frac{A\mu}{\lambda} \frac{(1-R^{n+1})}{(1-R)} + \frac{k\mu}{\lambda} \frac{(1-r_0^{n+1})}{(1-r_0)} \right] P_{2,0}$$

To calculate $P_{2,0}$, we use the normalising condition,

$$\sum_{n=0}^{\infty} P_{2,n} + \sum_{n=0}^{a-1} P_{1,n} + \sum_{n=0}^{a-1} P_{n,0} = 1$$

substituting the values of $P_{2,n}$, $P_{1,n}$, $P_{0,n}$ from (11), we have

$$\left[\sum_{n=0}^{\infty} r_0^n + \sum_{n=0}^{a-1} [AR^n + kr_0^n] + \frac{A\mu}{\lambda} \frac{(1-R^{n+1})}{(1-R)} + \frac{k\mu}{\lambda} \frac{(1-r_0^{n+1})}{(1-r_0)} \right] P_{2,0} = 1$$

On simplification, we find

$$P_{2,0} = \left[\frac{1}{1-r_0} + \frac{a\mu}{\lambda} \left[\frac{A}{1-R} + \frac{k}{1-r_0} \right] - \frac{2\mu r_0(1-r_0)}{\lambda(1-r_0)^2} \right]^{-1} \quad (12)$$

A can be calculated from equation (3)

$$(\lambda + \mu)(A + k)P_{2,0} = \lambda \left[\frac{\mu A}{\lambda} \frac{(1-R^a)}{(1-R)} + \frac{k\mu}{\lambda} \frac{(1-r_0^a)}{(1-r_0)} \right] P_{2,0}$$

$$\text{i.e., } A = \frac{\mu(2 - kr_0^a)}{R^{a-1} \lambda (1 - r_0)} \quad (13)$$

Next we shall calculate the average number of customers in the queue.

MEAN NUMBER IN QUEUE

Let L_q denote the average number of members in the queue, then

$$L_q = \sum_{n=1}^{\infty} n P_{2,n} + \sum_{n=1}^{a-1} n P_{1,n} + \sum_{n=1}^{a-1} n P_{0,n}$$

Substitute for the probabilities from (11)

$$L_q = \left[\sum_{n=1}^{\infty} n r_0^n + \sum_{n=1}^{a-1} n [AR^n + kr_0^n] \right] P_{2,0}$$

$$\sum_{n=1}^{a-1} n \left(\frac{A\mu}{\lambda} \frac{(1-R)^{n+1}}{(1-R)} + \frac{k\mu(1-r_0)^{n+1}}{\lambda(1-r_0)} \right) P_{2,0}$$

we first note that,

$$\sum_{n=1}^{k-1} n x^n = \frac{x(1-x^k) - kx^k(1-x)}{(1-x)^2} \quad (14)$$

substitute this in L_q , we get

$$L_q = \left[\frac{r_0}{(1-r_0)^2} + A \sum_{n=1}^{a-1} n R^n + K \sum_{n=1}^{a-1} n r_0^n + \frac{A\mu}{\lambda(1-R)} \frac{a(a-1)}{2} \right. \\ \left. - \frac{A\mu R}{\lambda(1-R)} \sum_{n=1}^{a-1} n R^n + \frac{k\mu a(a-1)}{\lambda(1-r_0)^2} - \frac{k\mu r_0}{\lambda(1-r_0)} \sum_{n=1}^{a-1} n r_0^n \right] P_{2,0}$$

we have,

$$L_q = \left[\frac{r_0}{(1-r_0)^2} + \left[\frac{A\mu}{(1-R)\lambda} + \frac{k\mu}{\lambda(1-r_0)} \right] \frac{a(a-1)}{2} \right. \\ \left. - \frac{2\mu r_0^2}{\lambda(1-r_0)^3} [(1-r_0)^a - ar_0^{a-1}(1-r_0)] \right] P_{2,0}$$

DISTRIBUTION OF THE WAITING TIME

Let the random variable T denote the waiting time in the queue for an arriving unit and $v(t)$ be the probability density function of T .

Let $f(\alpha, k; t)$ be the p.d.f. of gamma distribution with parameters α, k

$$\text{Then } f(\alpha, k; t) = \frac{\alpha^k t^{k-1} e^{-\alpha t}}{\Gamma(k)}, \quad t > 0, k = 1, 2, \dots$$

$$\int_0^x f(\alpha, k; t) = 1 - \sum_{s=0}^{k-1} \frac{e^{-\alpha x} (\alpha x)^s}{s!} = \Gamma t(\alpha, k)$$

An arriving unit may find the system in one of the following states

1. $(0, a-1)$
2. $(1, a-1)$
3. $(0, n), 0 \leq n \leq a-1$
4. $(1, n), 0 \leq n \leq a-1$
5. $(2, kb + m), k = 0, 1, 2, \dots, a-1 \leq m \leq b-1$
6. $(2, kb + m), k = 0, 1, 2, \dots, 0 \leq m \leq a-1$

In the case of (1) and (2) the arriving unit does not wait, the probability of zero delay is $(P_{0, a-1} + P_{1, a-1})$.

In the case of (3) and (4) the arriving unit has to wait for the arrivals of $(a - 1 - n)$ units. The time needed for $(a - 1 - n)$ arrivals has a gamma distribution with parameters $(\lambda, a - 1 - n)$

In the case of (5) the arriving unit has to wait for the completion of services of $(k + 1)$ batches, the time required for this has a gamma distribution with parameters $2\mu, (k + 1)$.

In the case of (6) the arriving unit has to wait till either the services of (k+1) groups are completed or (a-1-m) units arrive, whichever occurs later. The duration which is given by the maximum of two gamma variates may be denoted by the random variable Z. i.e.,

$$Z = \max (\text{gamma variate with parameters } \lambda, a-1-m; \\ \text{gamma variate with parameters } 2\mu, k+1)$$

The distribution function $F_Z(t)$ and the p.d.f $h_Z(t)$ of Z are given by,

$$F_Z(t) = \text{pr}(Z \leq t) = \text{Max}\{\Gamma t(\lambda, a-1-m), \Gamma t(2\mu, k+1)\} \\ h_Z(t) = F_Z'(t) \\ = f(\lambda, a-1-m; t) \Gamma t(2\mu, k+1) + \Gamma t(\lambda, a-1-m) f(2\mu, k+1, t)$$

The p.d.f. $v(t)$ of T is given by

$$v(t) = \left[\sum_{n=0}^{a-2} (P_{0,n} + P_{1,n}) \frac{\lambda e^{-\lambda t} (\lambda t)^{a-m-2}}{a-m-2!} \right. \\ + \sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} P_{2, kb+m} \frac{2\mu e^{-2\mu t} (2\mu t)^k}{k!} \\ \left. + \sum_{k=0}^{\infty} \sum_{m=0}^{a-2} P_{2, kb+m} \left[\left(\frac{2\mu e^{-2\mu t} (2\mu t)^k}{k!} \int_0^t \frac{\lambda e^{-\lambda t} (\lambda t)^{a-m-2}}{(a-m-2)!} \right) \right. \right.$$

$$+ \frac{\lambda e^{-\lambda t} (\lambda t)^{a-m-2}}{a-m-2!} \int_0^t \frac{2\mu e^{-2\mu t} (2\mu t)^k}{k!} dt \Big] P_{2,0}$$

On simplifying

$$v(t) = \left[\frac{\mu}{\lambda} \left[\frac{\mu}{1-R} + \frac{A k}{1-r_0} \right] \sum_{m=0}^{a-2} s_m(t) - \frac{2\mu r_0}{\lambda(1-r_0)} \sum_{m=0}^{a-2} r_0^m s_m(t) \right.$$

$$\left. \left[\frac{r_0^{a-1} - r_0^b}{1-r_0} \right] 2\mu e^{-2\mu} (1 - r_0^b)^t + \sum_{k=0}^{\infty} \sum_{m=0}^{a-2} r_0^{kb+m} \right.$$

$$\left. \left(s_m(t) \left(1 - \sum_{i=0}^k \frac{e^{-2\mu t} (2\mu t)^i}{i!} \right) + Q_m(t) \frac{2\mu e^{-2\mu t} (2\mu t)^k}{k!} \right) \right] P_{2,0}$$

where $s_m(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{a-m-2}}{a-m-2!}$, $Q_m(t) = \int_0^t s_m(x) dx$.

$$v(t) = \left[\frac{\mu}{\lambda} \left[\frac{A}{1-R} + \frac{k}{1-r_0} \right] \sum_{m=0}^{a-2} s_m(t) \right.$$

$$\left. - \frac{2\mu r_0}{\lambda(1-r_0)} \sum_{m=0}^{a-2} r_0^m s_m(t) \right.$$

$$+ \left[\frac{r_0^{a-1} - r_0^b}{1-r_0} \right] 2\mu e^{-2\mu(1-r_0^b)t}$$

$$+ \frac{1}{1-r_0} \sum_{m=0}^{a-2} r_0^m s_m(t)$$

$$+ 2\mu \sum_{m=0}^{a-2} r_0^m e^{-2\mu(1-r_0^b)t} \left[Q_m(t) - \frac{s_m(t)}{(1-r_0^b)2\mu} \right] P_{2,0} \quad (15)$$

By calculation, we found that

$$\int_0^{\infty} s_m(t) dt = 1 \text{ and}$$

$$\int_0^{\infty} \left(Q_m(t) - \frac{s_m(t)}{(1-r_0^b)2\mu} \right) e^{-2\mu(1-r_0^b)t} dt = 0$$

$$\int_0^{\infty} v(t) dt = \left[\frac{r_0^{a-1} - r_0^b}{1 - r_0} - \frac{2\mu r_0}{\lambda(1 - r_0)} \left(\frac{1 - r_0^{a-1}}{1 - r_0} \right) \right. \\ \left. + \frac{\mu(a-1)}{\lambda} \left(\frac{A}{1-R} + \frac{k}{1-r_0} \right) + \frac{A}{1-r_0^b} \left(\frac{1 - r_0^{a-1}}{1 - r_0} \right) \right] P_{2,0}$$

$$= 1 - P_{0,a-1} + P_{1a-1}$$

The expected waiting time in the queue $E(T)$ is given by,

$$E(t) = \int_0^{\infty} t v(t) dt$$

$$= \left[\left(\frac{r_0^{a-1} - r_0^b}{1 - r_0} \right) \frac{2\mu}{(2\mu(1 - r_0^b))^2} \right]^2 \frac{2\mu r_0}{\lambda(1 - r_0)}$$

$$\sum_{m=0}^{a-2} \int_0^{\infty} t s_m(t) dt$$

$$\sum_{m=0}^{a-2} r_0^m \int_0^{\infty} t s_m(t) dt + \frac{\mu}{\lambda} \left(\frac{A}{1 - R} + \frac{k}{1 - r_0} \right) \sum_{m=0}^{a-2} \int_0^{\infty} t s_m(t) dt$$

$$+ \frac{1}{1 - r_0^b} \sum_{m=0}^{a-2} r_0^m \int_0^{\infty} t s_m(t) dt$$

$$+ 2\mu \sum_{m=0}^{a-2} r_0^m \int_0^{\infty} t e^{-2\mu(1 - r_0^b)t} w_m(t) dt \quad P_{2,0}$$

$$\text{Where } w_m(t) = \frac{s_m(t)}{2\mu(1 - r_0^b)}$$

It is also found that,

$$\int_0^{\infty} t s_m(t) dt = \frac{a-1-m}{\lambda} \quad \text{and} \quad \int_0^{\infty} t w_m e^{-2\mu(1 - r_0^b)t} dt = \frac{r_0^{a-m+1}}{\lambda^2(1 - r_0)^2}$$

Then,

$$E(T) = \left[\frac{A\mu}{\lambda(1-R)} + \frac{k\mu}{\lambda(1-r_0)} + \frac{a(a-1)}{2} + \frac{r_0}{\lambda(1-r_0)^2} - \frac{2\mu r_0^2}{\lambda^2(1-r_0)^2} \left[\frac{(1-r_0^a) - ar_0^{a-1}(1-r_0)}{(1-r_0)} \right] \right] P_{2,0}$$

$$[\text{Since } 2\mu r_0^{b+1} - (\lambda+2\mu)r_0 + \lambda = 0]$$

$$E(T) = \frac{L_q}{\lambda}$$

SERVICE BATCH SIZE DISTRIBUTION :

The size of the batch in which an arriving unit will be served will depend on the state at which the customer arrived.

Let (1) x be size of the service batch in which an arriving passenger will eventually be served.

(2) $g_x = p(X=x), a \leq x \leq b$

(3) $f_T(v, n+1, t) = \text{gamma density with parameters } v, n+1;$

$$f_T(v, n+1, t) = \frac{v(vt)^n e^{-vt}}{n!}, t \geq 0$$

$$(4) k_j = \binom{n+j}{j} \frac{\lambda^j}{(\lambda+v)^j} \frac{v^{n+1}}{(\lambda+v)^{n+1}} \quad j=0, 1, 2, \dots$$

An arriving passenger may find the system in one of the following classes of states:

- (1) $(m, j), 1 \leq m \leq 2, 0 \leq j \leq a-1$
- (2) $(0, j), a-1 \leq r \leq b-1, j = nb+r, n=0,1,2\dots$
- (3) $(0, j), 0 \leq r \leq a-2, j = nb+r, n=0,1,2\dots$

In case (1) an arriving passenger will eventually be served in a batch of exactly 'a' passengers.

In case of (2), an arriving passenger has to wait for the completion service of $(n+1)$ batches. The time period T required for this to occur has a gamma distribution with parameters $2\mu, n+1$. The size of the service batch in which this particular passenger will eventually be served depends on how many passengers arrive during the period T . We obtain, for $n=0,1,2,\dots, a-1 \leq r \leq b-1$.

$$p(X=x \mid \text{state } (0, nb+r)) = \begin{cases} 0, & X \leq r \\ K_{x-r-1}, & r \leq x \leq b-1 \\ \sum_{i=b-r-1}^{\infty} k_i, & x=b \end{cases}$$

Similarly it can be shown for case (3) that

for $n=0,1,2,\dots, 0 \leq r \leq a-2$

$$\begin{aligned}
p(X=x \mid \text{state } (0, nb+r)) &= \sum_{i=0}^{a-r-1} k_i, & x=a \\
& k_{x-r-1}, & a < x \leq b-1 \\
& \sum_{i=b-r-1}^{\infty} k_i, & x=b
\end{aligned}$$

It follows that the probability g_x for an arriving passenger to be eventually served in a batch of x passengers is given by,

$$\begin{aligned}
g_b &= \sum_{n=0}^{\infty} \sum_{r=0}^{b-1} \sum_{i=b-r-1}^{\infty} k_i P_{2, nb+r} \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{b-1} \sum_{i=b-r-1}^{\infty} \binom{n+i}{i} \left(\frac{\lambda}{\lambda + 2\mu} \right)^i \left(\frac{2\mu}{\lambda + 2\mu} \right)^{n+i} \\
&= \sum_{r=0}^{b-1} \sum_{i=b-r-1}^{\infty} r_0^{i-1} r_0^r \frac{2\mu}{\lambda} P_{20} \\
&= \sum_{r=0}^{b-1} \frac{r_0^b}{1-r_0} \frac{2\mu}{\lambda} P_{20} \\
&= \frac{b(r_0 - w_2)}{w_2 r_0(1-r_0)} P_{20} \text{ where } w_2 = \frac{\lambda}{\lambda + 2\mu}
\end{aligned}$$

$$g_a = \sum_{j=0}^{a-1} (P_{0j} + P_{ij}) + \sum_{n=0}^{\infty} \sum_{r=0}^{a-1} \sum_{i=0}^{a-r-1} k_i P_{2nb+r}$$

$$= \sum_{j=0}^{a-1} (P_{0j} + P_{ij}) + \sum_{r=0}^{a-1} \sum_{i=0}^{a-r-1} \sum_{n=0}^{\infty} \binom{n+i}{i} (w_2)^i (i-w_2)^{n+1} r_0^{nb+r} P_{20}$$

$$= \sum_{j=0}^{a-1} (P_{0j} + P_{ij}) + \sum_{r=0}^{a-1} \sum_{i=0}^{a-r-1} \left(\frac{w_2}{1-(1-w_2)r_0^b} \right)^i \left(\frac{1}{1-(1-w_2)r_0^b} \right) (1-w_2)r_0^r P_{20}$$

$$= 1 - \frac{P_{20}}{w_2(1-r_0)^2} [- (r_0-w_2) + r_0^{a+1}(1-w_2)(1+a(1-r_0))]$$

$$g_x = \sum_{n=0}^{\infty} \sum_{r=0}^{x-1} k_{x-r-1} P_{2,nb+r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{x-1} \binom{n+x-r-1}{x-r-1} \left(\frac{\lambda}{\lambda+2\mu} \right)^{x-r-1} \left(\frac{2\mu}{\lambda+2\mu} \right)^{nb+r} r_0^{nb+r} P_{20}$$

$$= \sum_{r=0}^{x-1} \left(\frac{\lambda}{\lambda+2\mu} \right)^{-(x-r)} r_0^{x-r} \left(\frac{\lambda}{\lambda+2\mu} \right)^{x-r-1} \left(\frac{2\mu}{\lambda+2\mu} \right) r_0^r P_{20}$$

$$= (1-w_2) w_2^{-1} r_0^x P_{20}$$

The average batch size is given by,

$$E(X) = ag_a + bg_b + \sum_{x=a+1}^{b-1} xg_x$$

$$= a \left[1 - \frac{P_{20}}{w_2(1-r_0)^2} (-(r_0-w_2) + (1-w_2)r_0^{a+1}(a+1-ar_0)) \right]$$

$$+ \frac{b^2 P_{20}(r_0-w_2)}{r_0 w_2(1-r_0)} + \sum_{x=a+1}^{b-1} \frac{x(1-w_2)}{w_2} x r_0^x P_{20}$$

Consider $\sum_{n=a+1}^{b-1} n^2 r_0^n = \sum_{x=a+1}^{b-1} (n(n-1) + nr_0^x)$

$$= r_0^2 \frac{d^2}{dr_0^2} \left(\frac{r_0^{a+1} - r_0^b}{1-r_0} \right) + r_0 \frac{d}{dr_0} \left(\frac{r_0^{b+1} - r_0^b}{1-r_0} \right)$$

Therefore $E(X)$ becomes,

$$= a \left(1 + \frac{(r_0-w_2)P_{20}}{w_2(1-r_0)^2} \right) + \frac{P_{20}}{w_2(1-r_0)^3} (1-w_2)r_0^{a+1} (a+1-ar_0+r_0)$$

$$- \frac{P_{20}(r_0-w_2)}{r_0 w_2(1-r_0)^3} [-b^2(1-r_0)^2$$

$$+ (b(1-r_0)^2 + r_0 + r_0^2 - 2br_0^2 + 2br_0)]$$

(since $2\mu r_0^{b+1} - r_0(\lambda + 2\mu) - \lambda = 0$)

$$\begin{aligned}
E(X) = a \left[1 + \frac{(r_0 - w_2)}{w_2(1-r_0)^2} P_{20} \right] \\
+ \frac{(1-w_2) r_0^{a+1} P_{20}}{w_2(1-r_0)^3} (1 + r_0 + a - ar_0) \\
- \frac{(r_0 - w_2) P_{20}}{r_0 w_2(1-r_0)^3} [(1+2b)r_0 + (1-2b)r_0^2]
\end{aligned}$$

BUSY PERIOD

A busy period for two servers starts with the instance at which two channels become busy and lasts up to the first subsequent instant when one of the channels becomes idle.

The difference differential equations of the busy period are

$$P_{1,n}'(t) = 2\mu P_{2,n}(t), \quad 0 \leq n \leq a-1$$

$$P_{2,0}'(t) = -(\lambda+2\mu)P_{2,0}(t) + 2\mu \sum_{s=a}^b P_{2,s}(t)$$

$$P_{2,n}'(t) = -(\lambda+2\mu)P_{2,n}(t) + \lambda P_{2,n-1}(t) + 2\mu P_{2,n+b}(t), \quad n \geq 1$$

The initial conditions are

$$P_{2,0}(0) = 1 \text{ and } P_{2,n}(0) = 0, n \geq 1$$

Taking Laplace transformation on both sides,

$$sP_{1,n}(s) - P_{1,n}(0) = 2\mu P_{2,n}(s)$$

$$sP_{2,0}(s) - P_{2,0}(0) = -(\lambda+2\mu)P_{2,0}(s) + 2\mu \sum_{s=a}^b P_{2,0}(s)$$

$$sP_{2,n}(s) - P_{2,n}(0) = -(\lambda+2\mu) P_{2,n}(s) + \lambda P_{2,n-1}(s) + 2\mu P_{2,n+b}(s)$$

$$\text{i.e., } sP_{1,n}(s) = 2\mu P_{2,n}(s) \quad (1)$$

$$sP_{2,0}(s) - 1 = -(\lambda+2\mu)P_{2,0}(s) + 2\mu \sum_{s=a}^b P_{2,0}(s) \quad (2)$$

$$sP_{2,n}(s) = -(\lambda+2\mu) P_{2,n}(s) + \lambda P_{2,n-1}(s) + 2\mu P_{2,n+b}(s) \quad (3)$$

The characteristic equation of (3) becomes,

$$2\mu z^{b+1} - (\lambda + 2\mu + s)z + \lambda = 0$$

Using Rouché's theorem,

$$P_{2,n}(s) = (r_0(s))^n P_{2,0}(s), \forall n \geq 0 \quad (4)$$

$$P_{1,n}(s) = \frac{2\mu}{s} P_{2,n}(s) , \text{ (from (1))}$$

$$= \frac{2\mu}{s} (r_0(s))^n P_{2,0}(s) \text{ (from (4))}$$

Substitute for $P_{2,n}(s)$, $P_{1,n}(s)$ in (1), we get

$$sP_{2,0}(s) - 1 = -(\lambda + 2\mu)P_{2,0}(s) + 2\mu \sum_{s=a}^b P_{2,0}(s)$$

$$(s+\lambda+2\mu) P_{2,0}(s) - 1 = 2\mu \sum_{n=a}^b (r_0(s))^n P_{2,0}$$

$$P_{2,0}(s) = \frac{1 - r_0(s)}{s + 2\mu(1-r_0^a(s))}$$

The p.d.f of T is given by,

$$f(1,s) = \sum_{n=0}^{a-1} s P_{1,n}(s)$$

$$= \sum_{n=0}^{a-1} s \frac{2\mu}{s} (r_0(s))^n P_{2,0}(s)$$

$$= 2\mu \left(\frac{1-r_0^a(s)}{1-r_0(s)} \right) P_{2,0}(s)$$

Mean time $E(T)$

$$\begin{aligned} E(T) &= - \frac{d}{ds} f(1,s) \Big|_{s=0} \\ &= - \frac{d}{ds} 2\mu \left(\frac{1-r_0^a(s)}{1-r_0(s)} \right) \left(\frac{1-r_0(s)}{s+2\mu(1-r_0^a(s))} \right) \end{aligned}$$

On simplification, at $s = 0$, we get,

$$= \frac{1}{2\mu(1-r_0^a)}$$

$$\text{Therefore } E(T) = \frac{1}{2\mu(1-r_0^a)}$$

NUMERICAL RESULTS

A Computer programme is written to calculate

1. The Mean queue length L_q
2. The Probability that one of the server is busy P_1
3. The Probability that two servers are busy P_2
4. The Average batch size $E(X)$
5. The expected busy period for two servers $B(s)$

From the table its is found that the length of the queue and the batch size in which the arriving customer served increase as ρ increase and both the servers together are found to be busy for more percentage of time as e increases. Hence when the arrival rate or ρ is less, the servers are found to be idle for more percentage of time.

$$P_I = 1 - P_1 - P_2$$

**TABLE 1 PERFORMANCE MEASURES OF THE M/M (10,30)/2 SYSTEM
GIVEN THAT $\mu=1$**

λ	P	L_q	B(S)	E(X)	P_1	P_2
12	.2	5.329029	11.45376	.6387475	.4598802	.3288056
18	.3	7.587034	14.43617	.7889988	.342645	.5604431
24	.4	11.29100	17.73631	.9854207	.2349406	.717639
30	.5	16.6888	20.71825	1.2543566	.1565608	.8190175
36	.6	24.74254	23.25436	1.6511756	.1017145	.8853993
42	.7	37.97516	25.38286	2.305220	.0630949	.9302032
48	.8	64.11391	27.17343	3.6042261	.03538047	.9613818
54	.9	141.8888	28.69666	7.4852722	.01510665	.9836798

**TABLE 2 PERFORMANCE MEASURES OF THE M/M (29,30)/2 SYSTEM
GIVEN THAT $\mu=1$**

λ	P	L_q	B(S)	E(X)	P_1	P_2
6	.1	9.500316	20.00056	.501592	.285635	0.007179283
12	.2	9.559132	20.07374	.5247601	.4556719	0.0716618
18	.3	10.04709	20.49235	.5774772	.4962169	0.1940821
24	.4	11.58348	21.37993	.660203	.4582925	.3421467
30	.5	14.91085	22.63695	.7832927	.382858	.4907596
36	.6	21.18921	24.10296	.9728904	.2953695	.6267363
42	.7	32.97771	25.64406	1.2928152	.2091396	.7451981
48	.8	57.97418	27.16983	1.9364297	.1301064	.8457391
54	.9	134.8598	28.63245	3.8719552	.06065349	.9299268

TABLE 3 PERFORMANCE MEASURES OF THE M/M (20,50)/2 SYSTEM WITH $\mu=1$

λ	P	Lq	B(S)	E(X)	P1	P2
10	.1	9.517134	20.0338	.5133963	.4148388	0.04230576
20	.2	10.24062	21.26968	.5889707	.493275	.2364046
30	.3	13.03118	24.8631	.7055046	.4066207	.4582644
40	.4	18.37097	29.54802	.8628182	.2975259	.6326768
50	.5	26.75445	34.18893	1.0811042	.2067253	.756097
60	.6	39.69923	38.35202	1.4053069	.1380986	.8418553
70	.7	61.28887	41.9576	1.9415615	0.0873395	.9020795
80	.8	104.1914	45.04755	3.0087325	0.04966503	.9451671
90	.9	232.0963	47.70191	6.2001856	0.02143645	.9766163

TABLE 4 PERFORMANCE MEASURES OF THE M/M (25,50)/2

λ	P	Lq	B(S)	E(X)	P1	P2
10	.1	12.00381	25.0074	.5052992	.3597164	0.02010007
20	.2	12.32427	25.5173	.5519838	.4965814	0.1466601
30	.3	14.07638	27.56153	.6361277	.4667294	0.3315673
40	.4	18.22412	30.86291	.7559947	.3772549	0.5084722
50	.5	25.59599	34.63622	.9263352	.2817371	.6532427
60	.6	37.78189	38.25925	1.1824231	.1983218	.7649329
70	.7	58.83176	41.7961	1.6088112	.130369	.8495509
80	.8	101.3606	44.87213	2.4603888	0.07933634	.913613
90	.9	229.0112	47.59522	5.0117245	0.03370394	.9624344

Chapter III

A NON-PREEMPTIVE PRIORITY-M,M/M,M(1,b)/1 QUEUEING MODEL

In most of the queueing models the service is rendered on first come first service basis. This is obviously not the only manner of service and there are many alternatives such as last come first served, selection in random order and selection by priority etc. In priority schemes, customers with the highest priorities are selected for service ahead of those with lower priorities, independent of their time of arrivals into the system. There are two types of priority situations namely, preemption and non-Preemption. In preemptive cases, the customer with the highest-priority is called to enter the service immediately, even if another with lower priority is already present in service as soon as he arrives. A priority discipline is said to be non-Preemptive if there is no interruption and the highest priority customer just goes to the head of the queue to wait his turn.

In this chapter a non-Preemptive priority queueing model consisting of two types of units with Poisson inputs and exponential service time distribution has been analysed. The higher priority units are served singly in order of their arrival and they have non-Preemptive priority over the lower

priority units. The later units are served in batches of varying size under the usual bulk service rule. i.e., the size of the batch in service is $1 \leq x < b$.

The performance measures for the non-preemptive priority queueing system are usually obtained by considering two dimensional generating functions. But calculating the measures from two dimensional generating function is very tedious and so we calculate the measures using differential equation technique and single variable generating function.

The measures calculated for the model are the following

1. The probability that the server is busy with type- i ($i=1,2$) customers.
2. Mean number of type-1 units in system and type-2 units in queue.
3. The waiting time distribution of type-1 customer.
4. Verification of Little's formula.

Model Description

The system $M, M/M, M(1, b)/1$.

In this model two types of units namely type-1 units (higher priority units) and type-2 units (lower priority units) arrive at a service facility in independent Poisson streams with means λ_1, λ_2 respectively. The type-1 units are served one at a time and have non-preemptive priority over type-2 units. The service is rendered to type-2 units in batches of size $\max(1, b)$. Let the service times of the type-1 units and type-2 units be independent and identically distributed variables and have the probability densities,

$$dF_1(t) = \mu_1 e^{-\mu_1 t} dt \text{ and}$$

$$dF_2(t) = \mu_2 e^{-\mu_2 t} dt \text{ respectively.}$$

the service time distribution of a group of type-2 units of size $k(1 \leq k \leq b)$ is also assumed to be independent of the batch size.

If there is no higher priority units in the system, a lower priority batch consisting of $k(1 \leq k \leq b)$ of type-2 units if present is taken into service; otherwise the server is at rest. On the otherhand at the completion of a service, if both type-1 and type-2 units are present the server will start his service with type-1 units, Since the type-2 units are given the lower priority than the type-1 units. The situation that the system consisting of type-1 units in the queue and type-2 units in service can occur, if the type-1 units would have arrived after the service of type-2 units had started.

DEFINITION:

Let us define the following steady-state probabilities.

R_0 : Probability that, the system is empty, and the server is idle.

$P_{m,n}$: Probability that, the server is busy with the type-1 unit and there are m type-1 units in the system (including the person being served) and n type-2 units in the queue, $m \geq 1, n \geq 0$.

$Q_{m,n}$: Probability that, the server is busy with the type-2 units and there are m type-1 units in the system and n type-2

units (excluding the person being included in the batch under service) in the queue, $m \geq 0, n \geq 0$.

Let us define the following partial generating functions

$$P_m(Y) = \sum_{n=0}^{\infty} P_{m,n} Y^n, \quad m \geq 1$$

$$Q_m(Y) = \sum_{n=0}^{\infty} Q_{m,n} Y^n, \quad m \geq 0$$

STEADY STATE EQUATIONS AND THE DISTRIBUTION OF THE QUEUE LENGTH

Assuming steady state solution exist and following the usual arguments, we get the balance equations:-

$$(\lambda_1 + \lambda_2) R_0 = \mu_1 P_{1,0} + \mu_2 Q_{0,0} \quad (1)$$

$$(\lambda_1 + \lambda_2 + \mu_1) P_{1,0} = \lambda_1 R_0 + \mu_1 P_{2,0} + \mu_2 Q_{1,0} \quad (2)$$

$$(\lambda_1 + \lambda_2 + \mu_1) P_{1,n} = \lambda_2 P_{1,n-1} + \mu_1 P_{2,n} + \mu_2 Q_{1,n}, \quad n \geq 1 \quad (3)$$

$$(\lambda_1 + \lambda_2 + \mu_1) P_{m,0} = \lambda_1 P_{m-1,0} + \mu_1 P_{m+1,0} + \mu_2 Q_{m,0}, \quad m \geq 2 \quad (4)$$

$$(\lambda_1 + \lambda_2 + \mu_1)P_{m,n} = \lambda_1 P_{m-1,n} + \lambda_2 P_{m,n-1} + \mu_1 P_{m+1,n} + \mu_2 Q_{m,n}, \quad m \geq 2, n \geq 1 \quad (5)$$

$$(\lambda_1 + \lambda_2 + \mu_2)Q_{0,0} = \lambda_2 R_0 + \mu_1 \sum_{k=1}^b P_{1,k} + \mu_2 \sum_{k=1}^b Q_{0,k} \quad (6)$$

$$(\lambda_1 + \lambda_2 + \mu_2)Q_{0,n} = \lambda_2 Q_{0,n-1} + \mu_1 P_{1,n+b} + \mu_2 Q_{0,n+b}, \quad n \geq 1 \quad (7)$$

$$(\lambda_1 + \lambda_2 + \mu_2)Q_{m,0} = \lambda_1 Q_{m-1,0}, \quad m \geq 1 \quad (8)$$

$$(\lambda_1 + \lambda_2 + \mu_2)Q_{m,n} = \lambda_1 Q_{m-1,n} + \lambda_2 Q_{m,n-1}, \quad m, n \geq 1 \quad (9)$$

Adding the 5th equation multiplied by y^n over $n=1$ to and 4th equation we get,

$$(\lambda_2 y - (\lambda_1 + \lambda_2 + \mu_1))P_m(y) + \lambda_1 P_{m-1}(y) + \mu_1 P_{m+1}(y) + \mu_2 Q_m(y) = 0, \quad m \geq 2 \quad (10)$$

Similarly from equation (2) and (3)

$$(\lambda_2 y - (\lambda_1 + \lambda_2 + \mu_1))P_1(y) + \mu_1 P_2(y) + \mu_2 Q_1(y) = -\lambda_1 R_0 \quad (11)$$

Equation (8) and (9) imply,

$$(\lambda_2 y - (\lambda_1 + \lambda_2 + \mu_1))Q_m(y) = -\lambda_1 Q_{m-1}(y), \quad m \geq 1 \quad (12)$$

By similar arguments, from equations (6) and (7) we get,

$$\begin{aligned}
 (\lambda_1 + \lambda_2 + \mu_2)Q_0(y) &= \lambda_2 y Q_0(y) + \frac{\mu_1}{y^b} \sum_{n=1}^{\infty} P_{1,n+b} y^{n+b} \\
 &+ \frac{\mu_2}{y^b} \sum_{n=1}^{\infty} Q_{0,n+b} y^{n+b} + \lambda_2 R_0 + \mu_1 \sum_{k=1}^b P_{1,k} + \mu_2 \sum_{k=1}^b Q_{0,k}
 \end{aligned}$$

Simplifying further,

$$\begin{aligned}
 (\lambda_2 y^{b+1} - (\lambda_1 + \lambda_2 + \mu_2) y^b + \mu_2) Q_0(y) + \mu_1 P_1(y) & \quad (13) \\
 &= \sum_{n=1}^{b-1} (\mu_1 P_{1,n} + \mu_2 Q_{0,n}) (y^n - y^b) + \lambda_1 R_0
 \end{aligned}$$

Recursively from equation (12) we get,

$$Q_m(y) = \left(\frac{\lambda_1^m}{(\lambda_1 + \lambda_2 + \mu_2 - \lambda_2 y)^m} \right) Q_0(y), \quad \forall m \geq 0$$

Equation (10) implies that,

$$(\mu_1 E^2 - [(\lambda_1 + \lambda_2 + \mu_1) - \lambda_2 y] E + \lambda_1) P_m(y) = -\mu_2 \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_2 - \lambda_2 y} \right)^{m+1} Q_0(y)$$

where $E(P_m(y)) = P_{m+1}(y)$

Using the differential equation technique,

$$P_m(y) = A(y)(w(y))^m - \left(\frac{\lambda_1 \mu_2}{\lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2 - \lambda_2 y)} \right) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_2 - \lambda_2 y} \right)^{m-1} Q_0(y) \quad (15)$$

$$\text{where } w(y) = \frac{(\lambda_1 + \lambda_2 + \mu_1 - \lambda_2 y) - [(\lambda_1 + \lambda_2 + \mu_1 - \lambda_2 y)^2 - 4\lambda_1 \mu_1]^{\frac{1}{2}}}{2\mu_1}$$

$$|w(y)| < 1 \text{ for } |y| < 1$$

Substituting for $P_m(y)$, $Q_m(y)$ in (11), we get

$$\frac{\mu_2(\lambda_1 + \lambda_2 + \mu_2 - \lambda_2 y)}{\lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2 - \lambda_2 y)} Q_0(y) - A(y) = -R_0 \quad (16)$$

Also from (13) we obtain

$$\frac{(\lambda_1 + \lambda_2 + \mu_2 - \lambda_2 y) [\mu_2(\mu_2 - \mu_1) - (\lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2))] y^b + \lambda_2(\mu_2 - \mu_1) y^{b+1}}{\lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2 - \lambda_2 y)} Q_0(y) \quad (17)$$

$$+ \mu_1 A(y) w(y) = \sum_{k=0}^{b-1} (\mu_2 P_{1,k} + \mu_2 Q_{0,k}) (y^k - y^b) + \lambda_1 y^b R_0$$

Let us now calculate the following performance measures for the system

- (1) The probability that the system contains m units of type-1 when the server is busy with the type-1 unit is given by,

$$\sum_{n=0}^{\infty} P_{m,n} = P_m(1)$$

$$= A(1)(w(1))^m - \left(\frac{\lambda_1 \mu_2}{\lambda_1 \mu_2 + (\mu_2 - \mu_1) \mu_2} \right) \left(\frac{(\lambda_1)^{m-1}}{(\lambda_1 + \mu_2)^{m-1}} \right) Q_0(1) \quad (18)$$

Thus the probability that server is busy with type-1 unit is

$$P_1 = \sum_{m=1}^{\infty} P_m(1)$$

$$P_1 = A(1) \left(\frac{p_1}{1-p_1} - \frac{\lambda_1}{\mu_2} \right) + \frac{\lambda_2}{\mu_2} R_0 \quad (19)$$

where $p_1 = \frac{\lambda_1}{\mu_1}$

(2) The probability that there are m units of type-1 in the system when the server is busy with a batch of type-2 units is given by,

$$\sum_{n=0}^{\infty} Q_{m,n} = Q_m(1) = \left(\frac{(\lambda_1)^m}{(\lambda_1 + \mu_2)^m} \right) Q_0(1) \quad (20)$$

Therefore the probability that a batch of type-2 is in service is

$$\begin{aligned}
 P_2 &= \sum_{m=0}^{\infty} Q_m(1) \\
 &= \left(\frac{\lambda_1 + \mu_2 - \mu_1}{\mu_2} \right) (A(1) - R_0) \quad (\text{using (16) at } y=1) \quad (21)
 \end{aligned}$$

(3) The normalising condition implies that

$$P_1 + P_2 + R_0 = 1$$

$$A(1) = \frac{(\mu_2 - R_0 \mu_1)(\mu_1 - \lambda_1)}{\mu_1(\lambda_1 + \mu_2 - \mu_1)} \quad (22)$$

(4) Mean number of type-1 units in the system is

$$\begin{aligned}
 L_1 &= \sum_{m=1}^{\infty} m \sum_{n=1}^{\infty} P_m(y) + \sum_{m=1}^{\infty} m \sum_{n=1}^{\infty} Q_m(y) \\
 &= \sum_{m=1}^{\infty} m P_m(1) + \sum_{m=1}^{\infty} m Q_m(1)
 \end{aligned}$$

substitute for $P_m(1)$ and $Q_m(1)$ from (18) & (20)

$$L_1 = A(1) \left[\frac{p_1}{(1-p_1)^2} - \frac{\mu_1 \lambda_1}{\mu_2^2} \right] + \frac{\lambda_1 \mu_1 R_0}{\mu_2^2} \quad (23)$$

(5) Mean number of type 2 units in the queue is

$$L_{q2} = \sum_{n=1}^{\infty} n \sum_{m=1}^{\infty} P_m(y) + \sum_{n=1}^{\infty} n \sum_{m=1}^{\infty} Q_m(y)$$

$$= \left[\sum_{m=1}^{\infty} P_m(y) \right]'_{y=1} + \left[\sum_{m=0}^{\infty} Q_m(y) \right]'_{y=1}$$

where the dash denotes the derivative with respect to y .

$$L_{q2} = A'(1) \frac{p_1}{1-p_1} + \frac{\mu_2 - \mu_1}{\mu_2} + \frac{\lambda_1 \lambda_2 \mu_1}{(\mu_1 - \lambda_1)^3} A(1) \quad (25)$$

(6) Waiting time distribution of type-1 units in queue

Let $w_{q1}(t)$ denote the waiting time of the type-1 arriving customer in the queue. Let $w_{q1}(t)$ denote the probability density function, then the arriving customer has to wait in the following cases:-

(1) If he finds the system in state $(m, n, 1)$ (where m and n denote the system has m type-1 units, n type-2 units and the server is busy with type-1 units), the arriving unit has to wait for $m+1$ service completion whose density function is given by

$$\frac{\mu_1 e^{-\mu_1 t} (\mu_1 t)^m}{m!}$$

(2) Similarly if the customer finds the system in $(m, n, 2)$, (the server is busy with type-2 units), then he has to wait for the completion of the service of the batch of type-2 units and for the service completion of m type-1 units, which has density function.

$$\int_0^t \mu_2 e^{-\mu_2(t-h)} \frac{\mu_1 e^{-\mu_1 h} (\mu_1 h)^m}{m!} dh$$

If the type-1 unit arrives when the system is empty then his waiting time in the system is his service time.

Then,

$$w_{q1}(t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} P_{m,n}(y) \frac{\mu_1 e^{-\mu_1 t} (\mu_1 t)^m}{m!} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_m(y) \int_0^t \mu_2 e^{-\mu_2(t-h)} \frac{\mu_1 e^{-kh} (\mu_1)^m}{m!} dh + R_0 \mu_1 e^{-\mu_1 t}$$

$$w_{q1} = \sum_{m=1}^{\infty} P_m(1) \frac{\mu_1 e^{-\mu_1 t} (\mu_1 t)^m}{m!} + \sum_{m=0}^{\infty} Q_m(1) \int_0^t \mu_2 e^{-\mu_2(t-h)} \frac{e^{(\mu_1 h)^m} e^{-\mu_1 h}}{m!} dh$$

substituting for $P_m(1)$, $Q_m(1)$ and simplifying

$$w_{q1}(t) = A(1) \mu_1 [e^{-(\mu_1 - \lambda_1)t} - e^{-\mu_1 t}] + Q_0(1) \mu_1 \left(\frac{\lambda_1 + \mu_2}{\lambda_1 + \mu_2 - \mu_1} \right) (e^{-\mu_1 t} - e^{-\mu_2 t}) + R_0 \mu_1 e^{-\mu_1 t}$$

The expected waiting time is

$$W_{q1} = \int_0^{\infty} t W_{q1}(t) dt$$

$$= \frac{A(1)\mu_1}{(\mu_1 - \lambda_1)^2} - \frac{Q_0(1)\mu_1}{\mu_2^2} \left(\frac{\lambda_1 + \mu_2}{\lambda_1 + \mu_2 - \mu_1} \right)$$

$$W_{q1} = \frac{L_1}{\lambda_1} \quad (\text{from 23})$$

We have derived some known results of the non-preemptive priority queueing model as particular cases.

Particular cases

We calculate $A(1)$ and $A'(1)$ for the case $b=1$.

Equation (17) becomes

$$\frac{[(\lambda_1 + \lambda_2 - \lambda_2 Y)(\mu_2(\mu_2 - \mu_1) - \lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2))Y + \lambda_2(\mu_2 - \mu_1)Y^2]Q_0 Y}{\lambda_1 \mu_2 + (\mu_2 - \mu_1)(\lambda_2 + \mu_2 - \lambda_2 Y)}$$

$$+ \mu_1 A(Y)W(Y) = (\mu_2 P_{1,0} + \mu_2 Q_{0,n})(1-Y) + \lambda_1 Y R_0$$

after simplification, we get

$$A(Y) = \frac{R_0 Y \mu_2 \lambda_1 + (\mu_2 - \mu_1) R_0 (1-Y)(\lambda_2 Y - \mu_2) + R_0 \mu_2 (\lambda_2 Y - \lambda)}{\mu_2 \lambda_1 Y + (\mu_2 - \mu_1)(1-Y)(\lambda_2 Y - \mu_2) - \mu_1 \mu_2 W(Y)}$$

Since $A(1)$ and $A'(1)$ are of $0/0$ form, they can be eliminated using L' hospital rule. Thus after simplification

$$A(1) = \frac{R_0(\lambda - G_2)}{G_1 - G_2}$$

where,

$$\lambda = \lambda_1 + \lambda_2$$

$$G_1 = \frac{(\mu_1 - \lambda)\lambda_1}{\mu_1 - \lambda_1}$$

$$G_2 = \frac{(\mu_2 - \mu_1)(\lambda_2 - \mu_2)}{\mu_2}, \quad \text{and}$$

$$A'(1) = \left(\frac{\lambda_1 \lambda_2 (\lambda - G_2)}{(\mu_1 - \lambda_1)^3 (G_1 - G_2)} + \frac{G_2 \lambda_2 \mu_1}{(G_1 - G_2)(\mu_1 - \lambda_1)(\lambda_2 - \mu_2)} \right)$$

Substituting for $A(1)$ and $A'(1)$ in L_1 and Lq_2 , we get

$$L_1 = \frac{p_1}{1-p_1} \left(1 + \frac{p_2 \mu_1}{\mu_2} \right) \quad (26)$$

$$Lq_2 = \frac{p_2}{1-p_1-p_2} \left(\frac{p_1 \mu_1 + p_1 \mu_2}{\mu_1 (1-p_1)} \right) \quad (27)$$

Chapter IV

GI/M(a,b)/1 QUEUEING MODEL

Model Description

Consider the GI/M(a,b)/1 Queuing model, in which the inter arrival times are assumed to be independent and identically distributed random variable with distribution $A(t)$ and mean inter arrival time $1/\lambda$. The service is exponentially distributed with mean $1/\mu$. We assume that the service is done in batches following the bulk service rule, which is quite different from the general bulk service rule, introduced by Neuts (15). Under the general bulk service rule, the late arrivals cannot join the batch already in service till the service of batch is completed, even if the batch size under service is less than the maximum capacity.

Madill and Chaudhry (5) have discussed GI/M(a,b)/1 model with the general bulk service rule in the steady state and obtained the steady state queue - size probabilities, waiting time distribution and the moments of the conditional waiting time in the queue for the model.

In this chapter, we analyse all these results at pre-arrival epochs and random epochs for the GI/M(a,b)/1 queuing model in which the service is done in batches with

a provision that the late entries are allowed to join the batch of ongoing service.

The batch service rule discussed here is as follows :

At every departure epoch, before initiating the next service, the server may find the queue length n in one of the three categories ;

1. $0 \leq n \leq a-1$
2. $a \leq n < b$
3. $n \geq b$

In case (1) the server cannot initiate service with the number of units available in the queue. Hence he remains idle until the queue length becomes 'a', where upon he starts service on the batch with the minimum of 'a' units.

In case (2) the server takes the entire queue for batch service and admits the subsequent arrivals in the batch while the service is on, till the service of the current batch is over or till the maximum limit 'b' is reached whichever occurs first; such a batch is called an accessible batch.

In case (3) he takes 'b' units from the head of the queue for the service.

Let us first discussed the steady state results of the system at pre-arrival epochs.

PRE-ARRIVAL EPOCH :

Let t_n , $n = 0, 1, 2, \dots$, ($T_n \geq 0$) be the epoch at which the n^{th} customer arrives. We examine the system at $t_n - 0$. We define $p_j(t_n)$ to be the probability that the system has j units at the pre-arrival epoch at $t_n - 0$.

Let g_r = probability that r batches are served in an inter arrival time

$$\text{i.e., } g_r = \int_0^{\infty} \frac{e^{-\mu t} (\mu t)^r}{r!} dA(t)$$

Equilibrium equations

$$P_j = P_{j-1} + \sum_{r=1}^{\infty} P_{rb+j-1} \hat{g}_{r-1}, \quad 1 \leq j \leq a-1$$

where,

$$\hat{g}_{r-1} = 1 - \sum_{k=0}^{r-1} g_k$$

(Since, when the system has $rb + j - 1$ ($1 \leq j \leq a-1$) members at a pre-arrival epoch at most r batches can be served in an interarrival time. i.e., $(g_0 + g_1 + \dots + g_{r-1} + g_r)$)

By similar arguments,

$$P_0 = \sum_{r=0}^{\infty} \sum_{j=a-1}^{b-1} P_{rb+j} g_r \quad 2$$

$$P_j = \sum_{r=0}^{\infty} P_{rb+j-1} g_r \quad (j \geq a) \quad 3$$

From equation (3), we get

$$P_{j+1} = \sum_{r=0}^{\infty} P_{rb+j} g_r, \quad j \geq a-1$$

$$\Rightarrow (E - \sum_{r=0}^{\infty} E^{rb} g_r) P_j = 0 \quad j \geq a-1 \quad 4$$

where, E is the shifting operator, given by

$$E(P_j) = P_{j+1}$$

Denoting $\sum_{r=0}^{\infty} g_r z^{rb}$ by $G(z^b)$,

Equation (4) becomes,

$$(E - G(E^b)) P_j = 0, \quad j \geq a-1$$

The characteristic equation $Z - G(Z^b)$ has a unique root α lying inside $(0, 1)$ if $G^1(1) > 1$

$$\begin{aligned} G'(1) &= \sum_{r=0}^{\infty} g_r b_r \\ &= \sum_{r=1}^{\infty} b_r \int_0^{\infty} \frac{e^{-\mu t} (\mu t)^r}{r!} dA(t) \end{aligned}$$

$$= \int_0^{\infty} b\mu t \, dA(t)$$

$$= b\mu \times [\text{Mean of the interarrival time } A(t)]$$

$$G'(1) = \frac{b\mu}{\lambda}$$

$$G'(1) > 1 \text{ implies } \lambda/b\mu < 1.$$

Hence, equation (4) has solution,

$$P_j = C\alpha^j, \quad j \geq a-1, \text{ if } \lambda/b\mu < 1$$

$$\text{i.e., } P_j = \alpha^{j-(a-1)} P_{a-1}, \quad j \geq a-1$$

Summing the equation (1) over $j = 1$ to k

$$P_k = P_0 + \sum_{j=1}^k \sum_{r=1}^{\infty} P_{rb+j-1} \hat{g}_{r-1}$$

$$P_k = P_0 + \frac{\alpha^{1-a} (1-\alpha^k)}{1-\alpha} P_{a-1} \sum_{r=1}^{\infty} \alpha^{rb} \hat{g}_{r-1}$$

$$\text{Consider } \sum_{r=1}^{\infty} \alpha^{rb} \hat{g}_{r-1} = \sum_{r=1}^{\infty} \alpha^{rb} \left[1 - \sum_{k=1}^{r-1} g_k \right]$$

$$= \frac{\alpha^b}{1-\alpha^b} \left(1 - \sum_{i=0}^{\infty} g_i \alpha^{ib} \right)$$

$$= \frac{\alpha^b}{1-\alpha^b} (1-G(\alpha^b))$$

$$= \frac{\alpha^b}{1-\alpha^b} (1-\alpha)$$

since $G(\alpha^b) = \alpha$

Therefore,

$$P_k = P_0 + \frac{\alpha^{b+1-a} (1-\alpha^k)}{1-\alpha^b} P_{a-1}, \quad 1 \leq k \leq a-1$$

Equation (2) implies,

$$\begin{aligned} P_0 &= \sum_{r=0}^{\infty} \sum_{j=a-1}^{b-1} P_{rb+j} g_r P_{a-1} \\ &= \sum_{r=0}^{\infty} \sum_{j=a-1}^{b-1} \alpha^{rb+j-(a-1)} \left(1 - \sum_{k=0}^r g_k\right) P_{a-1} \\ &= \frac{1-\alpha^{b-a+1}}{1-\alpha} \left(\frac{1}{1-\alpha^b} - \sum_{r=0}^{\infty} \alpha^{rb} \sum_{k=0}^r g_k \right) P_{a-1} \\ &= \frac{1-\alpha^{b-a+1}}{1-\alpha^b} P_{a-1} \end{aligned}$$

$$\left(\text{since } \sum_{r=0}^{\infty} \alpha^{rb} \sum_{k=0}^r g_k = \frac{G(\alpha^b)}{1-\alpha^b} \right)$$

Substituting P_0 in P_k ($1 \leq k \leq a-1$)

$$P_k = \frac{\alpha^{a-1} - \alpha^{b+k}}{\alpha^{a-1} (1-\alpha^b)} P_{a-1}$$

P_{a-1} can be calculated from the normalising condition

$$\sum_{j=0}^{\infty} P_j = 1$$

$$\sum_{j=0}^{a-1} \frac{\alpha^{a-1} - \alpha^{b+j}}{(1-\alpha^b)\alpha^{a-1}} P_{a-1} + \sum_{j=a}^{\infty} \alpha^{j-a+1} P_{a-1} = 1$$

$$\text{i.e., } P_{a-1} \left[\frac{\alpha\alpha^{a-1}}{\alpha^{a-1}(1-\alpha^b)} - \frac{\alpha^{b-a+1}}{1-\alpha^b} \left(\frac{1-\alpha^a}{1-\alpha} \right) + \frac{\alpha}{1-\alpha} \right] = 1$$

$$P_{a-1} = \left[\frac{(1-\alpha)(1-\alpha^b)}{a(1-\alpha) + \alpha^{1-a}(\alpha^a - \alpha^b)} \right]$$

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Therefore the simplified probabilities are,

$$P_0 = \frac{1 - \alpha^{b-a+1}}{1 - \alpha^b} P_{a-1}$$

$$P_k = P_0 + \frac{\alpha^{b+1-a} (1-\alpha^k)}{1 - \alpha^b} P_{a-1}, \quad 1 \leq k \leq a-1$$

$$= \frac{\alpha^{a-1} - \alpha^{b+k}}{\alpha^{a-1}(1-\alpha)} P_{a-1}, \quad 1 \leq k \leq a$$

$$P_{a-1} = \frac{(1-\alpha)(1-\alpha^b)}{a(1-\alpha) + \alpha^{1-a}(\alpha^a - \alpha^b)}$$

$$P_k = \alpha^{k-a+1} P_{a-1}, \quad k \geq a-1$$

Random epoch

To obtain the limiting probabilities of system size at random epoch, we examine the system at some time t , preceding an arrival epoch. Then using the relations between the two sequences of random variable $j_t(t_n \leq t \leq t_{n+1})$ and j_n ($n = 0, 1, 2, \dots$) we have the following equations satisfied by

$$P_j^* = \lim_{n \rightarrow \infty} P_{jt}$$

We use the following notations to obtain the equilibrium equations.

Let g_r^* denote the probability that r batches are served in the period of time

$$R(t) = \int_0^t \lambda (1 - A(x)) dx \quad \text{and}$$

$$g_r^* = \int_0^t \frac{e^{-\mu t} (\mu t)^r}{r!} dR(t)$$

$$= \int_0^{\infty} \frac{e^{-\mu t} (\mu t)^r}{r!} \lambda (1 - A(t)) dt$$

$$g_r^* = p \times b \left(1 - \sum_{i=0}^r g_i \right) \quad (\text{using integration by parts})$$

Thus the equations are

$$P_0^* = \sum_{r=0}^{\infty} \sum_{j=a-1}^{b-1} P_{rb+j} \hat{g}_r^* \quad 8$$

where $\hat{g}_r^* = 1 - \sum_{i=0}^r g_i^*$

$$P_j^* = P_{j-1} + \sum_{r=1}^{\infty} P_{rb+j-1} \hat{g}_{r-1}^*, \quad 1 \leq j \leq a-1 \quad 9$$

$$P_j^* = \sum_{r=0}^{\infty} P_{rb+j-1} g_r^*, \quad j \geq a \quad 10$$

Substituting for P_{rb+j-1} in (10) we have

$$P_j^* = \sum_{r=0}^{\infty} \alpha^{rb+j-1} P_{a-1} \left(pb \left(1 - \sum_{i=0}^r g_i \right) \right)$$

$$= (pbP_{a-1}) \alpha^{j-1} \frac{(1 - \alpha)}{(1 - \alpha^b)}, \quad j \geq a$$

Equation (9) implies,

$$P_j^* = \frac{\alpha^{a-1} - \alpha^{b+j-1}}{(\alpha^{a-1})(1 - \alpha^b)} P_{a-1} + \sum_{r=1}^{\infty} \alpha^{rb+j-a} \left(1 - \sum_{i=0}^{r-1} g_i^* \right)$$

$$P_j^* = \frac{P_{a-1}}{1 - \alpha^b} (1 - \alpha^{b+j-a} \left(\frac{pb(1-\alpha)}{1 - \alpha^b} \right)), \quad 1 \leq j \leq a-1$$

From equation (8)

$$\begin{aligned} P_0^* &= \sum_{r=0}^{\infty} \sum_{j=a-1}^{b-1} \alpha^{rb+j-a+1} (1 - \sum_{i=0}^r g_i^*) P_{a-1} \\ &= \frac{1 - \alpha^{b-a+1}}{1 - \alpha} \left(\frac{1}{1 - \alpha^b} - \sum_{r=0}^{\infty} \alpha^{rb} \sum_{i=0}^r g_i^* \right) P_{a-1} \\ &= \frac{1 - \alpha^{b-1+1}}{(1 - \alpha)(1 - \alpha^b)} \frac{1}{(1 - \alpha^b)} (1 - \sum_{r=0}^{\infty} \alpha^{rb} g_0^*) P_{a-1} \\ &= \frac{1 - \alpha^{b-a+1}}{(1 - \alpha)(1 - \alpha^b)} \left(1 - \frac{pb(1 - \alpha)}{(1 - \alpha^b)} \right) P_{a-1} \end{aligned}$$

The normalising condition $\sum_{j=0}^{\infty} P_j^* = 1$, is verified.

using the probabilities at random epoch.

$$\begin{aligned} &\frac{1 - \alpha^{b-a+1}}{(1 - \alpha)(1 - \alpha^b)} \left(1 - \frac{pb(1 - \alpha)}{(1 - \alpha^b)} \right) \\ &+ \sum_{j=1}^{a-1} \frac{P_{a-1}}{1 - \alpha^b} (1 - \alpha^{b+j-a} \left(\frac{pb(1 - \alpha)}{1 - \alpha^b} \right)) \\ &+ \sum_{j=a}^{\infty} \frac{pb \alpha^{j-a} (1-\alpha)}{1 - \alpha^b} P_{a-1} = 1 \end{aligned}$$

$$\text{i.e., } \frac{a(1 - \alpha) + \alpha - \alpha^{b-1+1}}{(1 - \alpha)(1 - \alpha^b)} P_{a-1} = 1$$

$$P_{a-1} = \frac{(1 - \alpha)(1 - \alpha^b)}{a(1 - \alpha) + \alpha^{1-a}(\alpha^a - \alpha^b)}$$

which coincides with equation (7).

Mean queue length at random epoch is given by,

$$\begin{aligned} L_q^* &= \sum_{n=1}^{a-1} nP_n^* + \sum_{n=1}^{\infty} nP_{n+b}^* \\ &= \sum_{n=1}^{a-1} \frac{n}{1 - \alpha^b} \left(1 - \alpha^{b+j-a} \frac{pb(1 - \alpha)}{1 - \alpha^b}\right) P_{a-1} \\ &\quad + \sum_{n=1}^{\infty} \frac{nf^b \alpha^{j-a+b}(1 - \alpha)}{(1 - \alpha^b)} P_{a-1} \\ &= \frac{1}{1 - \alpha^b} \left(\frac{a(a-1)}{2} + \frac{f^b(1 - \alpha)}{1 - \alpha^b} \alpha^{b-a+1} \sum_{n=1}^{a-1} n\alpha^{n-1} \right) \\ &\quad + f^b(1 - \alpha) \alpha^{1-a+b} \sum_{n=1}^{\infty} n\alpha^{n-1} \\ &= \frac{1}{1 - \alpha^b} \left(\frac{a(a-1)}{2} + \frac{f^b \alpha^b}{1 - \alpha^b} \left(a + \frac{\alpha - \alpha^{b-a+1}}{1 - \alpha} \right) \right) \\ L_q^* &= \frac{1}{1 - \alpha^b} \left(\frac{a(a-1)}{2} + \frac{f^b \alpha^b}{1 - \alpha^b} \left(a + \frac{(\alpha^a - \alpha^b)}{1 - \alpha} \alpha^{1-a} \right) \right) P_{a-1} \quad " \end{aligned}$$

Waiting time distribution

An arriving unit may find the system in any one of the following state,

1. $0 \leq j \leq a - 2$
2. $a - 1 \leq j \leq b - 1$
3. $j = rb + K, r = 1, 2, \dots ; a - 1 \leq K \leq b - 1$
4. $j = rb + K, r = 1, 2, \dots ; 0 \leq K \leq a - 2$

In case (1), the unit has to wait for $a - j - 1$ arrivals.

In case (2), the arriving unit need not wait

In case (3), the arrival has to wait for r service completion which has density functions.

$$\frac{\mu e^{-\mu t} (\mu t)^{r-1}}{(r - 1)!}$$

In case (4), the unit has to wait for either r service completion (or) $a - j - 1$ units to arrive whichever occurs later. In this case the distribution is given by

$$\text{Max } [A_{a-j-1}(t), B_r(t)]$$

where $B_r(t) = 1 - \sum_{i=0}^{r-1} \frac{e^{-\mu t} (\mu t)^i}{i!}$

A_{a-j-1} represents the time for $(a-j-1)$ arrivals. If $w_q(t)$ denotes the density function of waiting time of an arriving customer. We have

$$\begin{aligned}
w_q(t) = & \sum_{j=0}^{a-2} \frac{\alpha^{a-1} - \alpha^{b+j}}{(1-\alpha^b)(\alpha^{a-1})} P_{a-1} d A_{a-j-1}(t) \\
& + \sum_{r=1}^{\infty} \sum_{k=a-1}^{b-1} \alpha^{rb+k-a+1} P_{a-1} \frac{\mu e^{-\mu t} (\mu t)^{r-1}}{(r-1)!} \\
& + \sum_{r=1}^{\infty} \sum_{j=0}^{a-2} \alpha^{rb+j-a+1} P_{a-1} d(\text{Max } A_{a-j-1}(t), B_r(t))
\end{aligned}$$

Where $d(\text{Max}(A_{a-j-1}(t), B_r(t))) = d A_{a-j-1}(t) B_r(t) + A_{a-j-1}(t) d B_r(t)$

Then

$$\begin{aligned}
w_q(t) = & P_{a-1} \left[\sum_{j=0}^{a-2} \frac{d A_{a-j-1}(t)}{(1-\alpha^b)} + \frac{\alpha^b \mu (1-\alpha^{b-a+1})}{1-\alpha} e^{-\mu(1-\alpha^b)t} \right. \\
& - \sum_{j=0}^{a-2} \alpha^{j-a+1} P_{a-1} \sum_{r=1}^{\infty} \alpha^{rb} \sum_{i=0}^{r-1} \frac{e^{-\mu t} (\mu t)^i}{i!} d A_{a-j-1}(t) \\
& \left. + \sum_{j=0}^{a-2} \alpha^{j-a+1} P_{a-1} \sum_{r=1}^{\infty} \alpha^{rb} \frac{\mu e^{-\mu t} (\mu t)^{r-1}}{(r-1)!} A_{a-j-1}(t) \right]
\end{aligned}$$

Mean waiting time is given by,

$$\int_0^{\infty} t w_q(t) = P_{a-1} \left[\sum_{j=0}^{a-2} \frac{t dA_{a-j-1}(t)}{1-\alpha^b} \right]$$

$$+ \frac{1-\alpha^{b-a+1}}{1-\alpha} \frac{1}{\mu(1-\alpha^b)^2} - \sum_{j=0}^{a-2} \alpha^{j-a+1} \frac{\alpha^b}{1-\alpha^b} \int_0^{\infty} \sum_{r=0}^{\infty} \frac{\alpha^{rb} \mu^r t^{r+1}}{r!}$$

$$+ \sum_{j=0}^{a-2} \alpha^{j-a+1} \int_0^{\infty} \frac{\mu^r \alpha^{rb} e^{-\mu t} t^r}{(r-1)!} A_{a-j-1}(t) dt$$

We use the following properties of Laplace transformation to obtain the mean waiting time

$$\text{Let } \tilde{a}(\mu) = \int_0^{\infty} e^{-\mu t} dA(t)$$

$$\text{then } L(t^n dA(t)) = (-D)_{\mu}^n \tilde{a}(\mu)$$

Since Laplace transform of convolution of two functions is the product of Laplace transform,

we have,

$$\int_0^{\infty} t^{r+1} dA_{a-j-1}(t) = (-D)^{r+1} q_j$$

$$\text{where } q_j = (\tilde{a}(\mu))^{a-j-1}$$

$$\begin{aligned}
\text{Therefore, } \int_0^{\infty} t w_q(t) &= P_{a-1} \left[\sum_{j=0}^{a-2} \frac{(-D)_{\mu=0} (a(\mu))^{a-j-1}}{1-\alpha^b} \right. \\
&+ \frac{1-\alpha^{b-a+1}}{\mu(1-\alpha)(1-\alpha^b)^2} - \sum_{j=0}^{a-2} \alpha^{j-a+1} \frac{\alpha^b}{1-\alpha^b} \left(\sum_{r=0}^{\infty} \frac{\alpha^{rb} \mu^r (-D)_{\mu}^r (-D)_{\mu}}{r!} q_j(t) \right) \\
&+ \sum_{j=0}^{a-2} \alpha^{j-a+1} \sum_{r=1}^{\infty} \frac{\alpha^{rb} \mu^r (-D)^r}{(r-1)!} (q_j/\mu) \left. \right]
\end{aligned}$$

I term of R.H.S. becomes

$$\begin{aligned}
&- \frac{1}{1-\alpha^b} \sum_{j=0}^{a-2} (a-j-1) (\tilde{a}(\mu))_{\mu=0}^{a-j-2} (\tilde{a}(\mu))_{\mu=0} \\
&= \frac{1}{1-\alpha^b} \sum_{j=0}^{a-2} \frac{(a-j-1)}{\lambda}
\end{aligned}$$

$$\text{where } (\tilde{a}(\mu))' = - \int_0^{\infty} t e^{-\mu t} d A(t) \Big|_{\mu=0}$$

$$= \frac{-1}{\lambda}$$

III & IV term of R.H.S. becomes

$$\begin{aligned}
 & - \sum_{j=0}^{a-2} \alpha^{j-a+1} \alpha^b / (1 - \alpha^b) e^{-\mu \alpha^b} D\mu (-D)_\mu q_j \\
 & + \sum_{j=0}^{a-2} \alpha^{j-a+1} \alpha^b \mu e^{-\mu \alpha^b} D\mu (-D)_\mu (q_j/\mu) \\
 & = \sum_{j=0}^{a-2} \alpha^{j-a+1} \alpha^b [-1/(1 - \alpha^b) e^{-\mu \alpha^b} D\mu (-D)_\mu q_j \\
 & \quad + \mu (e^{-\mu \alpha^b} D\mu 1/\mu (-D)_\mu q_j + e^{-\mu \alpha^b} D\mu q_j (-D)_\mu 1/\mu] \\
 & = \sum_{j=0}^{a-2} \alpha^{j-a+1} \alpha^b [\mu e^{-\mu \alpha^b} D\mu (q_j) 1/\mu^2] \\
 & = \sum_{j=0}^{a-2} \frac{\alpha^{j-a+1} \alpha^b}{\mu(1-\alpha^b)^2} (e^{-\mu \alpha^b} D\mu q_j)
 \end{aligned}$$

Adding all the 4 terms, we get,

$$\begin{aligned}
 \int_0^\infty t w_q(t) dt & = P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{(1-\alpha^{b-a+1}) \alpha^b}{\mu(1-\alpha)(1-\alpha^b)^2} \right. \\
 & \quad \left. + \sum_{j=0}^{a-2} \frac{\alpha^{j-a+1} \alpha^b}{\mu(1-\alpha^b)^2} e^{\mu \alpha^b} D\mu q_j \right] \\
 & = P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{1-\alpha^{b-a+1}}{\mu(1-\alpha)(1-\alpha^b)^2} + \sum_{j=0}^{a-2} \frac{\alpha^{j-a+1} \alpha^b}{\mu(1-\alpha^b)^2} \right. \\
 & \quad \left. (a (\mu(1-\alpha^b))^{a-j-1} \right]
 \end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} t w_q(t) dt &= P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{(1-\alpha^{b-a+1}) \alpha^b}{\mu(1-\alpha)(1-\alpha^b)^2} \right. \\
&\quad \left. + \sum_{j=0}^{a-2} \frac{\alpha^{b-a+1} \alpha^j \alpha^{a-j-1}}{\mu(1-\alpha^b)^2} \right] \\
&= P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{(1-\alpha^{b-a+1}) \alpha^b}{\mu(1-\alpha)(1-\alpha^b)^2} + \frac{\alpha^b}{\mu(1-\alpha^b)^2} (a-1) \right] \\
&= P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{\alpha^b}{\mu(1-\alpha^b)^2} \left[\left(\frac{1-\alpha^{b-a+1}-1+\alpha}{1-\alpha} \right) + a \right] \right] \\
&= P_{a-1} \left[\frac{a(a-1)}{2\lambda(1-\alpha^b)} + \frac{\alpha^b}{\mu(1-\alpha^b)^2} \left[a + \alpha^{-a+1} \frac{(\alpha^a - \alpha^b)}{1-\alpha} \right] \right] \\
&= \frac{P_{a-1}}{\lambda} \left[\frac{a(a-1)}{2(1-\alpha^b)} + \frac{\alpha^b p_b}{(1-\alpha^b)^2} \left[a + \alpha^{-a+1} \frac{(\alpha^a - \alpha^b)}{1-\alpha} \right] \right] \\
&= \frac{L_q^*}{\lambda}
\end{aligned}$$

Thus the Little's formula is verified.

CONCLUSION

Sim and Templeton (17) have analysed the multiserver Markovian Bulk Service Model $M/M(a,b)/c$ and derived a recursive algorithm to obtain the steady state probabilities and various performance measures. In chapter-1, we have analysed the two server model $(M/M(a,b)/2)$ explicitly and presented the expression for various measures in a closed form so that the computation can be done easily, if the root that lies in $(0,1)$ of the characteristic equation is known.

In chapter-3 we have discussed the non-preemptive priority queueing model $M, M/M, M(1,b)/1$ model and obtained the steady state results by considering the single variable generating function and applying the differential equation technique.

Madill and Chadhury (10) have discussed $GI/M(a,b)/1$ queueing model under the general bulk service rule and have given the various performance measures for the model. In chapter-4 we have considered the $GI/M(a,b)/1$ model under the bulk service rule which allows the late entries to join the batch in course of on going service and obtained the corresponding results.

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