Chapter II

CHAPTER – II

TRAPEZOIDAL FUZZY NUMBERS AND TRAPEZOIDAL FUZZY NUMBER MATRICES

Definition : 2.1

A trapezoidal fuzzy number denoted by $\tilde{A}$ = < a, b, c, d > has the membership function.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & b < x < c \\ \dfrac{d-x}{d-c}, & c \le x < d \\ 0, & x > d \end{cases}$$

Definition : 2.2

A trapezoidal fuzzy number $\tilde{A}$ = < a, b, c, d > is said to be **non-negative (non positive) trapezoidal fuzzy number** i.e., $\tilde{A} \ge 0$ ($\tilde{A} \le 0$) if and only if $a \ge 0$ ($c \le 0$). A trapezoidal fuzzy number is said to be **positive (negative) trapezoidal fuzzy number** i.e. $\tilde{A} > 0$ ($\tilde{A} < 0$) if and only if $a > 0$ ($c < 0$).

Definition : 2.3

Two trapezoidal fuzzy number $\tilde{A}_1$ = < a, b, c, d > and $\tilde{A}_2$ = < e, f, g, h > are said to be **equal** i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if a = e, b = f, c = g, d = h.

Definition : 2.4

A **trapezoidal fuzzy number matrix** of order m x n is defined as $A = (\tilde{A}_{ij})_{mxn}$ where $\tilde{A}_{ij}$ = < a_{ij}, b_{ij}, c_{ij}, d_{ij} > is the ij^{th} element of A. < a_{ij}, b_{ij}, c_{ij}, d_{ij} > are trapezoidal fuzzy numbers.

Definition : 2.5

Let A_1 = (a, b, c, d) and A_2 = (e, f, g, h) are two non-negative trapezoidal fuzzy numbers then

(i) $A_1 \oplus A_2 = < a, b, c, d > \oplus < e, f, g, h >$

$= < a + e, b + f, c + g, d + h >$

(ii) $A_1 \ominus A_2 = < a, b, c, d > \ominus < e, f, g, h >$

$= < a - h, b - g, c - f, d - e >$

(iii) $- A_1 = - < a, b, c, d >$

$= < - d, - c, - b, - a >$

(iv) $A_1 \otimes A_2 = < a, b, c, d > \otimes < e, f, g, h >$

$= < ae, bf, cg, dh >$

(v) $1/A = < 1/d, 1/c, 1/b, 1/a >$

Definition : 2.6

A trapezoidal fuzzy number

$\tilde{M} = < a, b, c, d > \equiv 0$ if and only if $a < 0 < d$.

There are three types of near zero trapezoidal fuzzy numbers :

1. $\tilde{M} = < a, b, c, d > \equiv 0$ is called **N_1-zero fuzzy number** if and only if : $a < b \leq c < 0 < d$

2. $\tilde{M} = < a, b, c, d > \equiv 0$ is called **N_2-zero fuzzy number** if and only if : $a < b < 0 < c < d$

3. $\tilde{M} = < a, b, c, d > \equiv 0$ is called **N_3-zero fuzzy number** if and only if : $a < 0 \leq b \leq c < d$

The class of N-zero fuzzy numbers shares the same arithmetic expression for extended addition and subtraction as with non-negative fuzzy numbers. However in the class of basic arithmetic functions these numbers behave differently for extended multiplication, division and inverse.

Remark : 2.7

A **zero trapezoidal fuzzy number** is denoted by $\tilde{0}$ = < 0, 0, 0, 0 >.

Remark : 2.8

A trapezoidal fuzzy number that is not positive, not negative and not zero is a N-zero trapezoidal fuzzy numbers.

Remark : 2.9

A trapezoidal fuzzy number is a triangular fuzzy number if b = c and a N-zero triangular fuzzy number if a < 0 < d ; b = c.

Proposition : 2.10

Let $\tilde{M}$ be any trapezoidal fuzzy number then the addition with its additive inverse is not zero but is a N_2-zero fuzzy number.

Proof

Since (a, b, c, d) is trapezoidal fuzzy number, $a \leq b \leq c \leq d$.

$\tilde{M} \ominus \tilde{M} = (a, b, c, d) \ominus (a, b, c, d)$

$= (a - d, b - c, c - b, d - a)$

$\therefore (a - d) < 0, b - c < 0, c - b > 0$ and $d - a > 0$.

Definition : 2.11

For two arbitrary trapezoidal fuzzy numbers $\tilde{A}_1$ = < a, b, c, d > and $\tilde{A}_2$ = < e, f, g, h > we may describe the **extended multiplication rule** as

$< a, b, c, d > \otimes < e, f, g, h > = <$ min (ae, ah, de, dh), min (bf, cf, cg, bg), max (bf, cf, cg, bg), max (ae, ah, de, dh)>

Therefore we may describe **extended arithmetic operation** between

1. two N_1-zero fuzzy numbers as < a, b, c, d > ⊗ < e, f, g, h > ≅ < min (ah, de), cg, bf, max (dh, ae) >
2. two N_2-zero fuzzy numbers as < a, b, c, d > ⊗ < e, f, g, h > ≅ < min (ah, de), min (bg, cf), max (bf, cg), max (dh, ae) >
3. two N_3-zero fuzzy numbers as < a, b, c, d > ⊗ < e, f, g, h > ≅ < min (ah, de), bf, cg, max (dh, ae) >

Similarly, if $\tilde{M}$ = < a, b, c, d > ≡ 0 and $\tilde{N}$ = < e, f, g, h > ≥ 0 then the new multiplication rule, $\tilde{M} \otimes \tilde{N}$ can be described as :

< a, b, c, d > ⊗ < e, f, g, h > ≅ < ah, min (bg, bf), max (cg, cf), dh >.

Definition : 2.12

Using the multiplication rule for two non-negative trapezoidal fuzzy numbers from

$$\tilde{A}_1 \otimes \tilde{A}_2 = \langle a, b, c, d \rangle \otimes \langle e, f, g, h \rangle$$
$$\cong \langle ae, bf, cg, dh \rangle,$$

the n[th] power of a non negative trapezoidal fuzzy number $\tilde{M}$ = (a, b, c, d) ≥ 0 can be given by the equation :

$$\tilde{M}^n = \langle a, b, c, d \rangle^n \cong \langle a^n, b^n, c^n, d^n \rangle$$

Similarly if $\tilde{M}$ = < a, b, c, d > < 0

$$\tilde{M}^n = \langle a, b, c, d \rangle^n \cong \begin{cases} \langle a^n, b^n, c^n, d^n \rangle; & n \in \text{odd} \\ \langle d^n, c^n, b^n, a^n \rangle; & n \in \text{even} \end{cases}$$

Further if $\tilde{M}$ = < a, b, c, d > ≡ 0 and is a N_1 zero fuzzy number then

$$\tilde{M}^n = \langle a, b, c, d \rangle^n \cong \begin{cases} \langle a^n, b^n, c^n, a^{n-1}d \rangle; & |a| > |d| \text{ and } n \in \text{odd} \\ \langle ad^{n-1}, c^n, b^n, a^n \rangle; & |a| > |d| \text{ and } n \in \text{even} \\ \langle ad^{n-1}, b^n, c^n, d^n \rangle; & |a| \le |d| \text{ and } n \in \text{odd} \\ \langle ad^{n-1}, c^n, b^n, d^n \rangle; & |a| \le |d| \text{ and } n \in \text{even} \end{cases}$$

If $\tilde{M}$ = < a, b, c, d > ≡ 0 and is a N_2 zero fuzzy numbers then

$$\tilde{M}^n = <a, b, c, d>^n \cong \begin{cases} <a^n, b^n, b^{n-1}c, a^{n-1}d>; & |a|>|d|, |b|>|c|, \ n \in \text{odd} \\ <a^{n-1}d, \ b^{n-1}c, b^n, a^n>; & |a|>|d|, |b|>|c|, \ n \in \text{even} \\ <a^n, bc^{n-1}, c^n, a^{n-1}d>; & |a|>|d|, |b|\le|c|, \ n \in \text{odd} \\ <a^{n-1}d, bc^{n-1}, c^n, a^n>; & |a|>|d|, |b|\le|c|, \ n \in \text{even} \\ <ad^{n-1}, bc^{n-1}, c^n, d^n>; & |a|\le|d|, |b|\le|c| \\ <ad^{n-1}, b^n, b^{n-1}c, d^n>; & |a|\le|d|, |b|>|c|, \ n \in \text{odd} \\ <ad^{n-1}, b^{n-1}c, b^n, d^n>; & |a|\le|d|, |b|>|c|, \ n \in \text{even} \end{cases}$$

If $\tilde{M}$ = < a, b, c, d > ≡ 0 and is a N_3 zero fuzzy number then

$$\tilde{M}^n = <a, b, c, d>^n \cong \begin{cases} <ad^{n-1}, b^n, c^n, d^n>; \ |a|\le|d| \\ <a^n, b^n, c^n, a^{n-1}d>; \ |a|>|d|, \ n \in \text{odd} \\ <a^{n-1}d, b^n, c^n, a^n>; \ |a|>|d|, \ n \in \text{even} \end{cases}$$

Definition : 2.13

Let $\tilde{M}$ = < a, b, c, d > and $\tilde{N}$ = < e, f, g, h > be two trapezoidal fuzzy numbers. Then the binomial expansion of fuzzy sum denoted as $(\tilde{M} \oplus \tilde{N})^n$ which may be evaluated as follows :

If both $\tilde{M}$, $\tilde{N}$ are non-negative then

$$(\tilde{M} \oplus \tilde{N})^n = (<a, b, c, d> \oplus <e, f, g, h>)$$
$$= (<a+e>^n, <b+f>^n, <c+g>^n, <d+h>^n)$$

Similarly if $\tilde{M}$, $\tilde{N}$ both are non positive then

$(<a, b, c, d> \oplus <e, f, g, h>)^n$ =

$$\begin{cases} (<a+e>^n, <b+f>^n, <c+g>^n, <d+h>^n); \ n \in \text{odd} \\ (<d+h>^n, <c+g>^n, <b+f>^n, <a+e>^n); \ n \in \text{even} \end{cases}$$

Definition : 2.14

The square root of a trapezoidal fuzzy number $\tilde{M}$ = < a, b, c, d > is obtained by solving the fuzzy equation $\tilde{X}^2$ = (a, b, c, d). In order to solve this equation we may consider the following cases :

1. If (a, b, c, d) $\geq$ 0 then there exist two roots of the equation $\tilde{X}^2$ = (a, b, c, d). On applying the multiplication rule from definition 2.6 we get $\tilde{X} = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right)$ as the non negative root and $\tilde{X} = \left(-\sqrt{d}, -\sqrt{c}, -\sqrt{b}, -\sqrt{a}\right)$ as the non positive root.

2. If $\tilde{M}$ = (a, b, c, d) $\equiv$ 0 and is a N_1 zero fuzzy number or if $\tilde{M}$ = (a, b, c, d) < 0 then there does not exist any root of the equation $\tilde{X}^2$ = (a, b, c, d).

3. If $\tilde{M}$ = (a, b, c, d) $\equiv$ 0 and is a N_2-zero fuzzy number then there exist four roots of the quadratic equation if and only if d $\geq$ | a | and c $\geq$ | b | which may be as follows :

$$\tilde{X} = \begin{cases} \left(-\sqrt{d}, -\sqrt{c}, \dfrac{b}{-\sqrt{c}}, \dfrac{a}{-\sqrt{d}}\right) \\ \left(-\sqrt{d}, \dfrac{b}{\sqrt{c}}, -\sqrt{c}, \dfrac{a}{-\sqrt{d}}\right) \\ \left(\dfrac{a}{\sqrt{d}}, \dfrac{b}{\sqrt{c}}, \sqrt{c}, \sqrt{d}\right) \\ \left(\dfrac{a}{\sqrt{d}}, -\sqrt{c}, \dfrac{b}{-\sqrt{c}}, \sqrt{d}\right) \end{cases}$$

4. If $\tilde{M}$ = < a, b, c, d > ≡ 0 and is a N_3-zero fuzzy number then there exist four roots of the quadratic equation if and only if $d \geq |a|$ which may be as follows :

$$\tilde{X} = \begin{cases} \left(\frac{a}{\sqrt{d}}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right) \\ \left(-\sqrt{d}, \sqrt{b}, \sqrt{c}, \frac{a}{-\sqrt{d}}\right) \\ \left(-\sqrt{d}, -\sqrt{c}, -\sqrt{b}, \frac{a}{-\sqrt{d}}\right) \\ \left(\frac{a}{\sqrt{d}}, -\sqrt{c}, -\sqrt{b}, \sqrt{d}\right) \end{cases}$$

Remark : 2.15

Note that all the derived roots of the equation $\tilde{X}^2$ = (a, b, c, d) may not be entirely feasible fuzzy solutions. A solution $\tilde{x} = (\alpha, \beta, \gamma, \delta)$ of the equation $\tilde{X}^2$ = (a, b, c, d) would be termed as feasible if and only if $\alpha \leq \beta \leq \gamma \leq \delta$. A solution would be termed as a feasible solution if it satisfies this constraint otherwise the solution would be termed as infeasible weak fuzzy solution.

Remark : 2.16

The n^{th} non negative root of a non negative fuzzy number is given by the equation as follows $(a, b, c, d)^{1/n} = (a^{1/n}, b^{1/n}, c^{1/n}, d^{1/n})$ on similar computational concept. However the extension to near zero fuzzy number is complex.

Definition : 2.17

The exponential of an arbitrary trapezoidal fuzzy number can be obtained directly as

$$e^{(a, b, c, d)} = (e^a, e^b, e^c, e^d).$$

Using the above equation the following set of corollaries can be easily verified.

Corollary : 2.18

$$e^{\tilde{x}} \otimes e^{\tilde{y}} = e^{\tilde{x} \oplus \tilde{y}}$$

Corollary : 2.19

$$\left(e^{\tilde{x}}\right)^{a} = e^{a.\tilde{x}}\,;\ a \in R^{+}.$$

Corollary : 2.20

$$e^{\tilde{x}} \otimes e^{-\tilde{y}} = e^{\tilde{x} \oplus (-\tilde{y})}$$

Remark : 2.21

Logarithm of a positive trapezoidal fuzzy number may be directly obtained using the result of definition 2.17 as follows

log (a, b, c, d) = (log a, log b, log c, log d).

Remark : 2.22

Logarithm of negative or near zero fuzzy number does not exist.

Definition : 2.23

1. $\tilde{M}^{\tilde{N}}\ ;\ \tilde{M},\ \tilde{N} \geq 0.$

   Let $\tilde{M}$ = (a, b, c, d), $\tilde{N}$ = (e, f, g, h) the fuzzy exponent is

   $$(a, b, c, d)^{(e, f, g, h)} = e^{(e, f, g, h) \otimes \ln (a, b, c, d)} = (a^{e}, b^{f}, c^{g}, d^{h})$$

2. $a^{\tilde{x}}\,;\ a \geq 1$ and $\tilde{x} \geq 0$

   Let $\tilde{x}$ = (e, f, g, h) the real fuzzy exponent is given by the following rule.

   $$a^{\tilde{x}} = e^{\tilde{x} \ln a} = (a^{e}, a^{f}, a^{g}, a^{h}).$$

Example : 2.24

Find all the non negative solution of the fuzzy non linear equation $e^{\tilde{x}} = \sqrt{(9,12,12,16)}$.

Solution

The non negative square root and logarithm is as follows :

$$\tilde{x} = \ln\left(\sqrt{(9,12,12,16)}\right)$$

$$= (\ln 3,\ \ln 2\sqrt{3},\ \ln 2\sqrt{3},\ \ln 4)$$

Example : 2.25

A ball is thrown from a height of (13, 16, 17, 17) meters under gravity $g = 9.8\ ms^{-2}$. Compute the approximate time when the ball will hit the ground. Use the equation $\tilde{t} = \sqrt{\dfrac{2\tilde{s}}{g}}$.

Solution :

Using the above equation we evaluate the square root of a trapezoidal fuzzy number and get the result as $\tilde{t}$ = (1.6, 1.8, 1.85, 1.85) seconds.

Definition : 2.26

A trapezoidal fuzzy number $< a_{ij}^{I}, a_{ij}^{II}, a_{ij}^{III}, a_{ij}^{iv} >$ is called a **generalized trapezoidal fuzzy number** if $a_{ij}^{I} \leq a_{ij}^{II} \leq a_{ij}^{III} \leq a_{ij}^{iv}$.

We proposed a method to make a score value by standardizing each element $\tilde{A}_{ij} = < a_{ij}^{I}, a_{ij}^{II}, a_{ij}^{III}, a_{ij}^{iv} >$ of a TRFNM as follows.

Step 1

Each generalized trapezoidal fuzzy number is standardized as follows :

$$\overline{a_{ij}} = \left\langle \frac{a_{ij}^{I}}{a_{ij}^{iv}}, \frac{a_{ij}^{II}}{a_{ij}^{iv}}, \frac{a_{ij}^{III}}{a_{ij}^{iv}}, \frac{a_{ij}^{iv}}{a_{ij}^{iv}} \right\rangle$$

$$= < a_{ij}^{I*}, a_{ij}^{II*}, a_{ij}^{III*}, a_{ij}^{iv*} >$$

Step 2

Calculate the defuzzified value $x_{a_{ij}}^{*}$ using the following formula

$$x_{a_{ij}}^{*} = \frac{a_{ij}^{I*} + a_{ij}^{II*} + a_{ij}^{III*} + a_{ij}^{iv*}}{4}$$

Step 3

Calculate the std a_{ij}^{*}

$$\text{std } a_{ij}^{*} = \sqrt{\frac{(a_{ij}^{I*} - x_{a_{ij}^{*}})^2 + (a_{ij}^{II*} - x_{a_{ij}^{*}})^2 + (a_{ij}^{III*} - x_{a_{ij}^{*}})^2 + (a_{ij}^{iv*} - x_{a_{ij}^{*}})^2}{4}}$$

Step 4

Score (a_{ij}^{*}) i.e., the score value of standardized generalized TRFNM is as the following.

$$\text{Score } (a_{ij}^{*}) = x_{a_{ij}}^{*} (1 - k. \text{ std } a_{ij}^{*})$$

where k is the scaling parameter, (i. e.) a parameter for adjusting the degree of importance of generalized TRFNM. We take k = 1.5 usually.

Step 5

The score distance between two TRFNM A and B of order m x n is

$$SD(A, B) = \sum_{i=1}^{m} \sum_{j=1}^{n} |\, score\,(a_{ij}^{*}) - score\,(b_{ij}^{*})\,|$$

It is noted that $0 \leq SD(A, B) \leq mn$. The score distance is also a metric on M. The score distance $SD : M \times M \rightarrow R$ should satisfy the following conditions.

(i) $SD(A, B) \geq 0$ for all $A, B \in M$

(ii) $SD(A, B) = SD(B, A)$ for all $A, B \in M$

(iii) $SD(A, B) = 0$ iff $A = B$ for all $A, B \in M$

(iv) $SD(A, B) \leq SD(A, C) + SD(C, B)$ for all $A, B, C \in M$

Definition : 2.27

The normalized score distance is denoted by $SD^{*}(A, B)$ and is defined as $SD^{*}(A, B) = \dfrac{SD(A,B)}{mn}$.

It is noted that $0 \leq SD^{*}(A, B) \leq 1$.

Example : 2.28

Consider two circulant generalized trapezoidal fuzzy number matrices.

A =

$$\begin{bmatrix} <20,23,25,27> & <9,11,13,15> & <14,16,18,20> & <19,20,21,22> \\ <19,20,21,22> & <20,23,25,27> & <9,11,13,15> & <14,16,18,20> \\ <14,16,18,20> & <19,20,21,22> & <20,23,25,27> & <9,11,13,15> \\ <9,11,13,15> & <14,16,18,20> & <19,20,21,22> & <20,23,25,27> \end{bmatrix}$$

B =

$$\begin{bmatrix} <1,3,5,7> & <2,4,5,7> & <10,12,16,17> & <21,23,25,27> \\ <21,23,25,27> & <1,3,5,7> & <2,4,5,7> & <10,12,16,17> \\ <10,12,16,17> & <21,23,25,27> & <1,3,5,7> & <2,4,5,7> \\ <2,4,5,7> & <10,12,16,17> & <21,23,25,27> & <1,3,5,7> \end{bmatrix}$$

Consider < 20, 23, 25, 27 > = $\left\langle \frac{20}{27}, \frac{23}{27}, \frac{25}{27}, \frac{27}{27} \right\rangle$

= < 0.7407, 0.8519, 0.9259, 1 >

$$x^*_{a_{ij}} = \frac{(0.7404 + 0.8519 + 0.9259 + 1)}{4}$$

= 0.8796.

Similarly for the other elements,

$$A = \begin{pmatrix} 0.8796 & 0.7999 & 0.85 & 0.9318 \\ 0.9318 & 0.8796 & 0.7999 & 0.85 \\ 0.85 & 0.9318 & 0.8796 & 0.7999 \\ 0.7999 & 0.85 & 0.9318 & 0.8796 \end{pmatrix}$$

| A | = 0.9318 which is the largest element in A.

$$\text{adj } A = \begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix}$$

$$= \begin{pmatrix} 0.8796 & 0.8796 & 0.8796 & 0.9318 \\ 0.9318 & 0.8796 & 0.8796 & 0.8796 \\ 0.8796 & 0.9318 & 0.8796 & 0.8796 \\ 0.8796 & 0.8796 & 0.9318 & 0.8796 \end{pmatrix}$$

$$A(\text{adj } A) = \begin{pmatrix} 0.9318 & 0.8796 & 0.8796 & 0.8796 \\ 0.8796 & 0.9318 & 0.8796 & 0.8796 \\ 0.8796 & 0.8796 & 0.9318 & 0.8796 \\ 0.8796 & 0.8796 & 0.8796 & 0.9318 \end{pmatrix}$$

$$(\text{adj } A)\, A = \begin{pmatrix} 0.9318 & 0.8796 & 0.8796 & 0.8796 \\ 0.8796 & 0.9318 & 0.8796 & 0.8796 \\ 0.8796 & 0.8796 & 0.9318 & 0.8796 \\ 0.8796 & 0.8796 & 0.8796 & 0.9318 \end{pmatrix}$$

Thus A(adj A) = (adj A) A.

a^{1*}_{11} = < 0.7407, 0.8519, 0.9259, 1 >

$$x_{11}^{*} = \frac{(0.7404 + 0.8519 + 0.9259 + 1)}{4}$$

$$= 0.8796$$

$$\text{Std } a_{11}^{1*} = \sqrt{\frac{(0.7407 - 0.8796)^2 + (0.8518 - 0.8796)^2 + (0.9259 - 0.8796)^2 + (1 - 0.8796)^2}{4}}$$

$$= 0.0959$$

Score (a_{11}^{1*}) = 0.8796 (1 – (1.5) (0.0959) = 0.07531

Similarly for the other elements we have

$$\text{Score } A^{*} = \begin{bmatrix} 0.75 & 0.62 & 0.70 & 0.83 \\ 0.83 & 0.75 & 0.62 & 0.70 \\ 0.70 & 0.83 & 0.75 & 0.62 \\ 0.62 & 0.70 & 0.83 & 0.75 \end{bmatrix}$$

$$\text{Score } B^{*} = \begin{bmatrix} 0.46 & 0.50 & 0.44 & 0.36 \\ 0.36 & 0.46 & 0.50 & 0.44 \\ 0.44 & 0.36 & 0.46 & 0.50 \\ 0.50 & 0.44 & 0.36 & 0.46 \end{bmatrix}$$

$$\text{SD (A, B)} = \sum_{i=1}^{m} \sum_{j=1}^{n} |\text{score } a_{ij}^{*} - \text{score } b_{ij}^{*}|$$

$$= 4.56$$

$$\text{SD}^{*}\text{(A, B)} = \frac{\text{SD(A, B)}}{\text{m.n}} = \frac{4.56}{16}$$ = 0.285 which lies between 0 and 1.