
Chapter 8

δP_S -Homeomorphisms

8.1 Introduction

In this chapter, the notions of δP_S -open and δP_S -closed functions are taken for study and their behaviours are characterized in locally indiscrete space. δP_S -irresoluteness and contra δP_S -irresoluteness was defined and few of their properties were investigated. Two types of homeomorphisms namely δP_S -homeomorphism and $\delta P_S\mathcal{C}$ -homeomorphism are developed and their properties are obtained. It is observed that the set of all $\delta P_S\mathcal{C}$ -homeomorphisms form a group under composition of functions.

8.2 δP_S -open and δP_S -closed functions

Definition 8.2.1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is

- (a) **δP_S -open** if $f(A)$ is δP_S -open for every open set A in X .
- (b) **δP_S -closed** if $f(A)$ is δP_S -closed for every closed set A in X .

Proposition 8.2.2. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map with $f(Int(A)) \subseteq \delta P_S Int(f(A))$, for every $A \subseteq X$. Then f is δP_S -open.

Proof. Let A be an open subset of X . By hypothesis, $f(Int(A)) \subseteq \delta P_S Int(f(A))$. Since A is open, $f(A) \subseteq \delta P_S Int(f(A))$. Thus $f(A)$ is δP_S -open and hence f is δP_S -open.

Proposition 8.2.3. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be δP_S -closed then $\delta P_S Cl(f(A)) \subseteq f(Cl(A))$, for each $A \subseteq X$.

Proof. Let $A \subseteq X$. Then $Cl(A)$ is closed in X . By hypothesis, $f(Cl(A))$ is δP_S -closed in Y . Since $f(A) \subseteq f(Cl(A))$, $\delta P_S Cl(f(A)) \subseteq \delta P_S Cl(f(Cl(A))) = f(Cl(A))$.

Theorem 8.2.4. For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta)$, $f: (X, \tau) \rightarrow (Y, \sigma), g: (Y, \sigma) \rightarrow (Z, \eta)$ and $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ the following results are true.

- (a) If $g \circ f$ is δP_S -open and f is continuous, surjective then g is δP_S -open,
- (b) If $g \circ f$ is open and g is δP_S -continuous, injective then f is δP_S -open.

Proof. (a) Let B be open in Y . Then $f^{-1}(B)$ is open in X . By hypothesis, $(g \circ f)f^{-1}(B) = g(f(f^{-1}(B))) = g(B)$ is δP_S -open in Z . Thus, g is δP_S -open.

(b) Let A be open in X . By hypothesis, $(g \circ f)(A) = g(f(A))$ is open in Z . Therefore $g^{-1}(g(f(A))) = f(A)$ is δP_S -open, as g is an injective δP_S -continuous function. Hence f is δP_S -open.

Theorem 8.2.5. For a bijective map $f: (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent.

- (a) f is δP_S -open;
- (b) f is δP_S -closed;
- (c) f^{-1} is δP_S -continuous.

Proof.

(a) \Rightarrow (b) Let $A \subseteq X$ be closed. Then $X \setminus A$ is open in X . By (a), $f(X \setminus A)$ is δP_S -open in Y . Since f is bijective, $f(X \setminus A) = Y \setminus f(A)$ which implies $f(A)$ is δP_S -closed in Y . Hence f is δP_S -closed.

(b) \Rightarrow (c) Let $A \subseteq X$ be closed. Since f is bijective, $(f^{-1})^{-1}(A) = f(A)$ is δP_S -closed in Y . Therefore f^{-1} is δP_S -continuous.

(c) \Rightarrow (a) Let $A \subseteq X$ be open. Since f^{-1} is δP_S -continuous and f is bijective, $(f^{-1})^{-1}(A) = f(A)$ is δP_S -open. Hence f is δP_S -open.

Remark 8.2.6. The above theorem gives the conditions under which δP_S -open and δP_S -closed maps coincide.

Theorem 8.2.7. A bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -closed iff for each subset B in Y and each open set U in X containing $f^{-1}(B)$, there exists a δP_S -open set V in Y such that

- (a) $B \subseteq V$ and
- (b) $f^{-1}(V) \subseteq U$.

Proof. Necessity: Let f be δP_S -closed and $B \subseteq Y$. Take an open set U in X such that $f^{-1}(B) \subseteq U$. Then $X \setminus U$ is closed in X . By hypothesis, $f(X \setminus U)$ is δP_S -closed. Define $V = Y \setminus (f(X \setminus U))$, then V is δP_S -open in Y .

Claim-(a): $B \subseteq V$.

Proof: Suppose that $v \notin V$ then $v \in f(X \setminus U) \Rightarrow f^{-1}(v) \in X \setminus U \Rightarrow f^{-1}(v) \notin U$ then $f^{-1}(v) \notin f^{-1}(B) \Rightarrow v \notin B$. Hence $B \subseteq V$.

Claim-(b): $f^{-1}(V) \subseteq U$.

Proof: Now, $v \in V \Rightarrow v \notin f(X \setminus U) \Rightarrow f^{-1}(v) \notin X \setminus U$ then $f^{-1}(v) \in U$.

Hence $f^{-1}(V) \subseteq U$.

Sufficiency: Let A be closed in X . Then $X \setminus A$ is open in X , $f(X \setminus A) \subseteq Y$ and $Y \setminus (f(X \setminus A)) \subseteq Y$. Now, $Y \setminus f(A) \subseteq f(X \setminus A) \Rightarrow f^{-1}(Y \setminus f(A)) \subseteq X \setminus A$. Then by hypothesis, there exists a δP_S -open set V in Y such that $Y \setminus f(A) \subseteq V$ and $f^{-1}(V) \subseteq (X \setminus A)$. Now, $Y \setminus f(A) \subseteq V \Rightarrow (Y \setminus V) \subseteq f(A)$ and $f^{-1}(V) \subseteq (X \setminus A) \Rightarrow V \subseteq (f(X \setminus A)) \Rightarrow Y \setminus (f(X \setminus A)) \subseteq Y \setminus V$. Thus, $Y \setminus (f(X \setminus A)) \subseteq (Y \setminus V) \subseteq f(A)$. Since f is a bijection, $Y \setminus (f(X \setminus A)) = f(A) \Rightarrow (Y \setminus V) = f(A)$. Since $Y \setminus V$ is δP_S -closed, so is $f(A)$. Hence f is δP_S -closed.

Corollary 8.2.8. A bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -open iff for each $B \subseteq Y$ and each closed set U containing $f^{-1}(B)$, there exists a δP_S -closed set $V \subseteq Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Similar to previous theorem.

Remark 8.2.9. Composition of two δP_S -closed functions is not a δP_S -closed function.

Example 8.2.10. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{b, c\}\}$, $\sigma = \{Y, \emptyset, \{a\}\}$ and $\eta = \{Z, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an identity function and $g: (Y, \sigma) \rightarrow (Z, \eta)$

be defined by $g(a) = b$, $g(b) = a$ and $g(c) = c$. Then f and g are both δP_S -closed functions but $g \circ f$ is not a δP_S -closed function as $g(f(\{c\})) = \{c\}$ is not δP_S -closed in (Z, η) .

Proposition 8.2.11. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be any function with (Y, σ) being alocally indiscrete space. Then every δP_S -open function is an open function.

Proof. Let A be open in X . Since f is δP_S -open, $f(A)$ is δP_S -open. Since (Y, σ) is a locally indiscrete space, we know that $f(A)$ is open in Y , by Theorem 2.2.31 that in a locally indiscrete space, $\delta P_S \mathcal{O}(X) = \tau$. Hence f is an open function.

Proposition 8.2.12. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be any function with (Y, σ) be a semi- T_1 -space. Then every δP_S -open function is a δ -preopen function.

Proof: Let A be a open set in X . Since f is a δP_S -open function $f(A)$ is a δP_S -open set in Y . Since Y is semi- T_1 , by Proposition 2.2.23 $f(A)$ is δ -preopen.

Corollary 8.2.13. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -open function, then f is preopen function where Y is semi- T_1 .

Proof: Follows from Proposition 2.2.23.

Remark 8.2.14. Composition of two δP_S -open functions is not a δP_S -open function.

Example 8.2.15. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \emptyset, \{a\}\}$ and $\eta = \{Z, \emptyset, \{a, b\}\}$ with $f(a) = b$, $f(b) = c$, $f(c) = a$ and $g(a) = c$, $g(b) = b$, $g(c) = a$. Then f and g are both δP_S -open functions but $g \circ f$ is not a δP_S -open function.

Theorem 8.2.16. For topological spaces (X, τ) , (Y, σ) , (Z, η) , $f: (X, \tau) \rightarrow (Y, \sigma)$, $g: (Y, \sigma) \rightarrow (Z, \eta)$ and $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ the following are true.

- (a) If f and g are both δP_S -open with (Y, σ) being a locally indiscrete space then $g \circ f$ is δP_S -open.
- (b) If f is open and g is δP_S -open then $g \circ f$ is δP_S -open.

(c) If f is open and g is δP_S -open with (Y, σ) being a locally indiscrete space then $g \circ f$ is open.

(d) If f is open and g is δP_S -open with (Y, σ) being semi- T_1 then $g \circ f$ is δ -preopen open function.

Proof. Obvious from the definitions and Theorem 2.2.31 and Proposition 2.2.23.

Theorem 8.2.17. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ such that $g \circ f: X \rightarrow Z$ is δP_S -open. Then the following are true.

(a) If f is continuous, surjective then g is δP_S -open map,

(b) If g is δP_S -irresolute, injective then f is δP_S -open map,

(c) If f is δP_S -continuous, surjective and X is a locally indiscrete space then g is δP_S -open map,

(d) If f is δ -continuous, surjective and X is a locally indiscrete space then g is δP_S -open map,

(e) If g is strongly δP_S -continuous, injective then f is open map.

(f) If f is a surjective δP_S -continuous function, then g is δP_S -open where X is a $\delta P_S T_\delta$ -space.

Proof. (a) Let B be open in Y . Since f is continuous, $f^{-1}(B)$ is open in X . Further, since $g \circ f$ is δP_S -open, $g(f(f^{-1}(B)))$ is δP_S -open in Z . As f is surjective, $g(f(f^{-1}(B))) = g(B)$ is δP_S -open in Z . Thus g is δP_S -open.

(b) Let A be open in X . Since $g \circ f$ is δP_S -open, $g(f(A))$ is δP_S -open in Z . As g is δP_S -irresolute, $g^{-1}(g(f(A)))$ is δP_S -open in Y . Further, $g^{-1}(g(f(A))) = f(A)$, as g is injective. This proves that f is δP_S -open map.

(c) Let B be open in Y . Since f is δP_S -continuous, $f^{-1}(B)$ is δP_S -open. Since X is a locally indiscrete space, $f^{-1}(B)$ is open in X . As $g \circ f$ is δP_S -open, $g(f(f^{-1}(B)))$ is δP_S -open in Z . Now, $g(f(f^{-1}(B))) = g(B)$ is δP_S -open in Z , as f is surjective. Hence g is δP_S -open map.

(d) Let B be open in Y . Since, f is δ -continuous, $f^{-1}(B)$ is δ -open. Since X is a locally indiscrete space, $f^{-1}(B)$ is open in X . As $g \circ f$ is δP_S -open, $g(f(f^{-1}(B)))$ is δP_S -open in Z . Now, $g(f(f^{-1}(B))) = g(B)$ is δP_S -open in Z , as f is surjective. Hence g is δP_S -open map.

(e) Let B be open in Y . Since $g \circ f$ is δP_S -open, $g(f(B))$ is δP_S -open in Z . As g is strongly δP_S -continuous, $g^{-1}(g(f(B)))$ is open in X . Further $g^{-1}(g(f(B))) = f(B)$ is open in X , as g is injective.

(f) Let V be any open set in (Y, σ) . Since f is g -continuous, $f(V)$ is g -open in (X, τ) . Since (X, τ) is a $T_{1/2}$ -space, $f(V)$ is open in (X, τ) . Since $(g \circ f)$ is δP_S -open and f is surjective, $(g \circ f)f^{-1}(V) = g(V)$ is δP_S -open in (Z, η) . Therefore, g is a δP_S -open map.

Proposition 8.2.18. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be δP_S -open maps. If (Y, σ) is a ${}_{\delta P_S}T_\delta$ -space, then their composition $(g \circ f): (X, \tau) \rightarrow (Z, \eta)$ is a δP_S -closed map.

Proof: Let A be a open set in (X, τ) . Since f is a δP_S -open map, $f(A)$ is δP_S -open in (Y, σ) . Since (Y, σ) is a ${}_{\delta P_S}T_\delta$ -space, $f(A)$ is δ -open in (Y, σ) . Since every δ -open set is open, $f(A)$ is open in (Y, σ) . Since g is a δP_S -open map, $g(f(A)) = (g \circ f)(A)$ is δP_S -closed in (Z, η) . Hence $(g \circ f)$ is a δP_S -open map.

Proposition 8.2.19. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -open iff $f(Int(A)) \subseteq \delta P_S(Intf(A))$, for every $A \subseteq X$.

Proof. Necessity: Let $A \subseteq X$. Now, $Int(A)$ is open in X . Since f is δP_S -open, $f(Int(A))$ is δP_S -open in Y . We know,

$$Int(A) \subseteq A \Rightarrow f(Int A) \subseteq f(A) \Rightarrow f(Int(A)) = \delta P_S Int(f(Int(A))) \subseteq \delta P_S Int(f(A)), \text{ as } f(Int A) \text{ is } \delta P_S\text{-open in } Y .$$

Sufficiency: Let A be an open subset of X . By hypothesis, $f(Int(A)) \subseteq \delta P_S Int(f(A))$. Since A is open, $f(A) = f(Int(A)) \subseteq \delta P_S Int(f(A))$. Thus $f(A)$ is δP_S -open and hence f is δP_S -open.

Corollary 8.2.20. If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -open then $f^{-1}(\delta P_S Cl(B)) \subseteq Cl(f^{-1}(B))$, for every $B \subseteq Y$.

Proof. Let f be δP_S -open and $B \subseteq Y$. Then $f^{-1}(B) \subseteq Cl(f^{-1}(B))$. By Corollary 8.2.8, there exists a δP_S -open set $A \subseteq Y$ such that $B \subseteq A$ and

$$f^{-1}(A) \subseteq Cl(f^{-1}(B)) \Rightarrow f^{-1}(\delta P_S Cl(B)) \subseteq f^{-1}(\delta P_S Cl(A)) = f^{-1}(A) \subseteq Cl(f^{-1}(B)),$$

since A is δP_S -open.

Proposition 8.2.21. If a function $f: X \rightarrow Y$ is δP_S -open then for each $x \in X$ and for each neighborhood U of $x \in X$, there exists a δP_S -neighborhood W of $f(x)$ such that $W \subseteq f(U)$.

Proof. Let $x \in X$, and for each neighborhood U of $x \in X$, there exists an open set G such that

$x \in G \subseteq U$. From Theorem 8.1.16, $f(G) = f(Int G) \subseteq \delta P_S Int(f(G)) \Rightarrow f(G)$ is δP_S -open in Y . Further, $f(x) \subseteq f(G) \subseteq f(U)$. By taking $f(G) = W$, the result follows.

8.3 δP_S -irresoluteness and contra δP_S -irresoluteness

Definition 8.3.1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

- (a) **δP_S -irresolute** if $f^{-1}(B)$ is δP_S -open in (X, τ) for every δP_S -open set B in (Y, σ) .
- (b) **contra δP_S -irresolute** if $f^{-1}(B)$ is δP_S -open in (X, τ) for every δP_S -open set B in (Y, σ) .

Proposition 8.3.2. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with (Y, σ) being a locally indiscrete space, every δP_S -continuous function is δP_S -irresolute.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a δP_S -continuous function and B be a δP_S -open subset of Y . By Theorem 2.2.31 in a locally indiscrete space $\tau = \delta P_S O(X)$ we have, B is open in Y . Since f is δP_S -continuous, then $f^{-1}(B)$ is δP_S -open in X and hence f is a δP_S -irresolute function.

Remark 8.3.3. δP_S -irresolute functions and irresolute functions are independent of each other.

Example 8.3.4. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = b, f(c) = a$. Then f is δP_S -irresolute but not irresolute since the inverse image of $\{a\}$ is $\{c\}$ and $\{c\}$ is not semi-open in (X, τ) .

Example 8.3.5. Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be such that $f(a) = a, f(b) = c, f(c) = b, f(d) = d$. Then f is irresolute but not δP_S -irresolute since the inverse image of $\{a\}$ is $\{a\}$ which is not δP_S -open in (X, τ) .

Proposition 8.3.6. Composition of two δP_S -irresolute functions is a δP_S -irresolute function.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$, and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be any two δP_S -irresolute functions. Let W be a δP_S -open subset of Z . Then $g^{-1}(W)$ is δP_S -open in Y . Therefore $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is δP_S -open in X . Hence $g \circ f$ is δP_S -irresolute.

Theorem 8.3.7. For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta), f: (X, \tau) \rightarrow (Y, \sigma), g: (Y, \sigma) \rightarrow (Z, \eta)$ and $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ the following results are true.

(a) If f is a δP_S -irresolute function and g is a δP_S -continuous (resp. totally δP_S -continuous, strongly δP_S -continuous, contra δP_S -continuous, contra δP_S -irresolute) function then $g \circ f$ is a δP_S -continuous (resp. totally δP_S -continuous, strongly δP_S -continuous, contra δP_S -continuous, contra δP_S -irresolute) function.

(b) If f is a contra δP_S -irresolute (resp. strongly continuous, quasi δP_S -continuous, perfectly δP_S -continuous) function and g is a δP_S -irresolute function then $g \circ f$ is a contra δP_S -irresolute (resp. strongly continuous, quasi δP_S -continuous, perfectly δP_S -continuous) function.

Proof. Follow from the definitions.

Remark 8.3.8. From the previous theorem, we observe that δP_S -irresoluteness acts as a mirror in reflecting the type of continuity in their compositions.

Theorem 8.3.9. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -irresolute then

(a) $A \subseteq X, f(\delta P_S Cl(A)) \subseteq Cl_\delta(f(A))$.

(b) $B \subseteq Y, \delta P_S Cl(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$

Proof. (a) Let $A \subseteq X$ then $Cl_\delta(f(A))$ is δ -open in Y and thus δP_S -open in Y . Since f is δP_S -irresolute, $f^{-1}(Cl_\delta(f(A)))$ is δP_S -open in X . $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}Cl_\delta(f(A))$.

$$\Rightarrow \delta P_S Cl(A) \subseteq \delta P_S Cl(f^{-1}(Cl_\delta(f(A)))) = f^{-1}(Cl_\delta(f(A)))$$

$$\Rightarrow f(\delta P_S Cl(A)) \subseteq Cl_\delta(f(A))$$

(b) Let $B \subseteq Y$ then $Cl_\delta(B)$ is δ -open in Y and thus δP_S -open in Y . By hypothesis

$$f^{-1}Cl_\delta(B) \text{ is } \delta P_S\text{-open in } X. \text{ Since } B \subseteq Cl_\delta(B), f^{-1}(B) \subseteq f^{-1}(Cl_\delta(B))$$

$$\Rightarrow \delta P_S Cl(f^{-1}(B)) \subseteq \delta P_S Cl(f^{-1}(Cl_\delta(B))) = f^{-1}(Cl_\delta(B))$$

Proposition 8.3.10. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is surjective, δP_S -irresolute and open with (X, τ) being a locally indiscrete space then every δP_S -open set is open in (Y, σ) .

Proof. Let B be δP_S -open in Y . By hypothesis $f^{-1}(B)$ is δP_S -open in X . Since X is locally indiscrete space, $f^{-1}(B)$ is open in X . Since f is an open function, $f(f^{-1}(B)) = B$ is open in Y as f is surjective. Therefore every δP_S -open set is open in Y .

Proposition 8.3.11. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective δP_S -irresolute and δ -preopen function with (X, τ) being a semi- T_1 space then δP_S -open set is δ -preopen in (Y, σ) .

Proof: The proof is as in the proof of Corollary 7.2.10 using Proposition 2.2.23, that in a semi T_1 space $\delta P_S O(X, \tau) = \delta P O(X, \tau)$.

8.4 δP_S - Homeomorphism and $\delta P_S \mathcal{C}$ -Homeomorphism

Definition 8.4.1. A bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called **δP_S -homeomorphism** if f is both δP_S -continuous and δP_S -open.

Theorem 8.4.2. If a bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S -continuous then the following statements are equivalent.

- (a) f is δP_S -open;
- (b) f is δP_S -homeomorphism;
- (c) f is δP_S -closed.

Proof. (a) \Leftrightarrow (c) is obvious.

(a) \Leftrightarrow (b) follows from the definition of δP_S -homeomorphism.

Remark 8.4.3. Homeomorphism is independent of δP_S -homeomorphism as seen from the following Example.

Example 8.4.4. Let $X = Y = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ be an identity function. Then f is a δP_S -homeomorphism but not a homeomorphism as the image of open set $\{a, b, c\}$ in (X, τ) , $f(\{a, b, c\}) = \{a, b, c\}$ is not open in σ .

Example 8.4.5. Let $X = Y = \{a, b, c, d\}, \tau = \sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ be an identity function. Then f is a homeomorphism but not a δP_S -homeomorphism as f is neither δP_S -closed nor δP_S -continuous.

Proposition 8.4.6. Every δP_S -homeomorphism from a locally indiscrete space into another locally indiscrete space is a homeomorphism.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a δP_S -homeomorphism. Then f is bijective, δP_S -continuous and δP_S -open. Let U be an open set in (X, τ) . Since f is δP_S -open and (Y, σ) is a locally indiscrete space $f(U)$ is δP_S -open which is open since $\delta P_S O(X) = \tau$ in a locally indiscrete space $f(U)$ is δP_S -open which is open since $\delta P_S O(X) = \tau$ in a locally indiscrete space by Proposition 2.2.23. Hence f is an open map. Let V be a closed set in (Y, σ) . Since f is δP_S -continuous, $f^{-1}(V)$ is δP_S -closed in (X, τ) and since (X, τ) by the same argument as above. Hence f is continuous. Thus f is a homeomorphism.

Proposition 8.4.7. Every δP_S -homeomorphism from a $\delta P_S T_\delta$ -space into another $\delta P_S T_\delta$ -space is a homeomorphism.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a δP_S -homeomorphism. Then f is bijective, δP_S -open and δP_S -continuous. Let U be an open set in (X, τ) . Since f is δP_S -open and (Y, σ) is a $\delta P_S T_\delta$ -space, $f(U)$ is δ -open which implies that $f(U)$ is open in (Y, σ) . Therefore, f is an open map. Let V be a closed set in (Y, σ) . Since f is δP_S -continuous and (X, τ) is a $\delta P_S T_\delta$ -space, $f(V)$ is δ -open in (X, τ) which implies that $f(V)$ is closed in (X, τ) . Therefore, f is continuous. Hence f is a homeomorphism.

Definition 8.4.8. A bijective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called **$\delta P_S \mathcal{C}$ -homeomorphism** if both f and f^{-1} are δP_S -irresolute.

Proposition 8.4.9. Composition of two $\delta P_S \mathcal{C}$ -homeomorphisms is a $\delta P_S \mathcal{C}$ -homeomorphism.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be two $\delta P_S \mathcal{C}$ -homeomorphisms and $g \circ f: (X, \tau) \rightarrow (Z, \eta)$. Let U be δP_S -open in Z . Now $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, g is δP_S -continuous which implies $g^{-1}(U)$ is δP_S -closed in Y . Once again by hypothesis, f is δP_S -continuous which implies $f^{-1}(V)$ is δP_S -open in X . Hence $g \circ f$ is δP_S -irresolute. Now, let G be δP_S -open in X . Then $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis, f is δP_S -open in which $f(G)$ is δP_S -open in Y . Again, by hypothesis, g is δP_S -open which implies $g(f(G))$ is δP_S -open in Z . Therefore $(g \circ f)^{-1}$ is δP_S -irresolute and hence $g \circ f$ is a $\delta P_S \mathcal{C}$ -homeomorphism.

Theorem 8.4.10. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a $\delta P_S \mathcal{C}$ -homeomorphism then

- a) For $A \subseteq X, f(\delta P_S Cl(A)) \subseteq Cl(f(A))$,
- b) For $B \subseteq Y, \delta P_S Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$,
- c) For $B \subseteq Y, f^{-1}(\delta P_S Cl(B)) \subseteq Cl(f^{-1}(B))$,
- d) For $A \subseteq X, \delta P_S Cl(f(A)) \subseteq f(Cl(A))$.

Proof: a) Follows from Theorem 5.2.7(d)

b) Follows from Theorem 5.2.7(e)

c) Follows from Proposition 8.2.19.

d) Follows from Proposition 8.2.3.

Definition 8.4.11. The set of all $\delta P_S \mathcal{C}$ -homeomorphisms of (X, τ) onto itself is denoted by **$\delta P_S \mathcal{C} h(X, \tau)$** .

Proposition 8.4.12. Let $f: (X, \tau) \rightarrow (X, \tau)$ be any function. If $f \in \delta P_S \mathcal{C} h(X, \tau)$ then $f^{-1} \in \delta P_S \mathcal{C} h(X, \tau)$.

Proof. Let $f \in \delta P_S \mathcal{C}h(X, \tau)$ then both f and f^{-1} are δP_S -irresolute. Now $f = (f^{-1})^{-1}$ is δP_S -irresolute. Thus both f^{-1} and $(f^{-1})^{-1}$ are δP_S -irresolute. This implies $f^{-1} \in \delta P_S \mathcal{C}h(X, \tau)$

Proposition 8.4.13. The set of all $\delta P_S \mathcal{C}$ -homeomorphisms of (X, τ) onto itself denoted by $\delta P_S \mathcal{C}h(X, \tau)$ forms a group under composition of functions.

Proof. Define a binary operation $\star: \delta P_S \mathcal{C}h(X, \tau) \times \delta P_S \mathcal{C}h(X, \tau) \rightarrow \delta P_S \mathcal{C}h(X, \tau)$ defined by $f \star g = g \circ f$, for all $f, g \in \delta P_S \mathcal{C}h(X, \tau)$ and \circ is the usual composition of maps. By Proposition 8.4.9, $g \circ f \in \delta P_S \mathcal{C}h(X, \tau)$. Hence the closure axiom holds good. Since composition of functions is associative, the associative axiom holds. Let $I \in \delta P_S \mathcal{C}h(X, \tau)$ is the identity element, where $I: (X, \tau) \rightarrow (X, \tau)$ be the identity function. By Theorem 8.4.10, if $f \in \delta P_S \mathcal{C}h(X, \tau)$ then $f^{-1} \in \delta P_S \mathcal{C}h(X, \tau)$ in such a way that $f \circ f^{-1} = f^{-1} \circ f = I$. Hence the existence of inverse is true. Therefore $\delta P_S \mathcal{C}h(X, \tau)$ forms a group under composition of functions.

Proposition 8.4.14. Let $\star: \delta P_S \mathcal{C}h(X, \tau) \rightarrow \delta P_S \mathcal{C}h(Y, \sigma)$ be a homeomorphism then $\delta P_S \ker(\star)$ is a normal subgroup of $\delta P_S \mathcal{C}h(X, \tau)$, where $\ker(\star) = \{h \in \delta P_S \mathcal{C}h(X, \tau) / \star(h) = I_Y\}$

Proof. Since $\star(I_x) = I_y, I_x \in \ker(\star)$ and hence $\ker(\star) \neq \emptyset$. Let $h_1, h_2 \in \ker(\star)$ then $\star(h_1) = \star(h_2) = I_y$. This implies $\star(h_1 h_2^{-1}) = \star(h_1) \star(h_2^{-1}) = I_y$. Then $h_1 h_2^{-1} \in \ker(\star)$. Hence $\ker(\star)$ is a subgroup of $\delta P_S \mathcal{C}h(X, \tau)$. Now let $h_1 \in \ker(\star)$ and $g \in \delta P_S \mathcal{C}h(X, \tau)$ then $\star(g h_1 g^{-1}) = I_y$ which implies $g h_1 g^{-1} \in \ker(\star)$. Hence $\ker(\star)$ is a normal subgroup of $\delta P_S \mathcal{C}h(X, \tau)$.

Proposition 8.4.15. The set $\delta P_S \mathcal{C}h(X, \tau)$ is a group under the composition of maps.

Proof: Let us define a binary operation $\star: \delta P_S \mathcal{C}h(X, \tau) \times \delta P_S \mathcal{C}h(X, \tau) \rightarrow \delta P_S \mathcal{C}h(X, \tau)$ by $(f \star g) = (g \circ f)$ for every $f, g \in \delta P_S \mathcal{C}h(X, \tau)$ and is the usual operation of composition of maps. Then by the Proposition 8.4.13, $(g \circ f) \in \delta P_S \mathcal{C}h(X, \tau)$. We know that the composition of maps is associative and the identity map $I: (X, \tau) \rightarrow (X, \tau)$ belongs to $\delta P_S \mathcal{C}h(X, \tau)$ serves as the identity element. If $f \in \delta P_S \mathcal{C}h(X, \tau)$ then $f \in \delta P_S \mathcal{C}h(X, \tau)$ such that $(f \circ f^{-1}) = f^{-1} \circ f = I$. So, the inverse exists for each element of $\delta P_S \mathcal{C}h(X, \tau)$. Therefore, $\delta P_S \mathcal{C}h(X, \tau)$ is a group under the operation of composition of maps.

Proposition 8.4.16. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a $\delta P_S \mathcal{C}$ -homeomorphism. Then f induces an isomorphism from the group $\delta P_S \mathcal{C}h(X, \tau)$ onto the $\delta P_S \mathcal{C}h(Y, \sigma)$.

Proof: Using the map f , let us define a map $\theta_f: \delta P_S \mathcal{C}h(X, \tau) \rightarrow \delta P_S \mathcal{C}h(Y, \sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in \delta P_S \mathcal{C}h(X, \tau)$. Then θ_f is a bijective map. Further for every

$h_1, h_2 \in \delta P_S \mathcal{C} h(X, \tau), \theta_f(h_1, h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a homomorphism. Hence, θ_f is an isomorphism induced by f .