CHAPTER 6

Distance Measure for Neutrosophic Variants

(6.1) Tangent Metrics in Neutrosophic Variants

(6.2) Tangent Metrics Applied to Neutrosophic Variants: A Novel Perspective

(6.3) Neutrosophic Spherical TOPSIS Methodology: An Approach to Decision- Making

Distance measures are used in a wide array of applications, from basic spatial calculations to complex data analysis and machine learning. They are fundamental in determining relationships between data points, whether physical locations or features in a dataset. This chapter deals with distance measure on neutrosophic variants that is Tangent Metric Neutrosophic Spherical Distance Measure (TMNSDM) and Tangent Metric Fermatean Neutrosophic Distance Measure (TMFNDM). This chapter compares our defined tangent distance measure with other existing distance measures such as Euclidean distance (ED), normalized Euclidean distance (N-ED), hamming distance (HD), normalized hamming distance (N-HD) and sine metric single- valued neutrosophic distance (SMSVND) measures. Further, it provides the technique for the TOPSIS method employing the defined Tangent metric.

6.1 Tangent Metrics for Neutrosophic Variants

6.1.1 Tangent Distance Measure for Fermatean Neutrosophic Set

In this sub-section, a Tangent Metric Fermatean Neutrosophic Distance Measure is introduced and its properties are studied.

Definition 6.1.1.1

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a universal set. Let

$$\tilde{A}_F = \{x, T_{\tilde{A}_F}(x_i), I_{\tilde{A}_F}(x_i), F_{\tilde{A}_F}(x_i) | x_i \in U\},$$

$$\tilde{B}_F = \{x_i, T_{\tilde{B}_F}(x_i), I_{\tilde{B}_F}(x_i), F_{\tilde{B}_F}(x_i) | x_i \in U\}$$

be two FNSs on U. Define a mapping $d: FNS(U) \times FNS(U) \to [0,1]$ to represent a distance function. Then, the distance between two FNS is denoted by

$$d_{TMFN}(\tilde{A}_F, \tilde{B}_F) = \frac{2}{\sqrt{3}n} \sum_{i=1}^{n} \left\{ \begin{array}{l} tan\left\{\frac{\pi}{6}\left|T_{\tilde{A}_F}(x_i) - T_{\tilde{B}_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{\tilde{A}_F}(x_i) - I_{\tilde{B}_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{\tilde{A}_F}(x_i) - F_{\tilde{B}_F}(x_i)\right|\right\} \end{array} \right\} \tag{6.1.1.1}$$

Then $d_{TMFN}(\tilde{A}_F, \tilde{B}_F)$ is called the tangent metric Fermatean neutrosophic distance measure (TMFNDM).

Theorem 6.1.1.2

Let X be a non-empty universel set. Let M_F,N_F , L_F be FNSs. Then, the TMFNDM should satisfy the following axioms

i. $d_{TMFN}(M_F, N_F) \geq 0$ for all $M_F, N_F \in FNS(X)$.

ii. $d_{TMFN}(M_F, N_F) = 0$ if and only if $M_F = N_F$ for all $M_F, N_F \in FNS(X)$.

iii. $d_{TMFN}(M_F, N_F) = d_{TMFN}(N_F, M_F)$ for all $M_F, N_F \in FNS(X)$.

iv. If $M_F \subseteq N_F \subseteq L_F$ for all $M_F, N_F, L_F \in FNS(X)$, then

$d_{TMFN}(M_F, L_F) \geq d_{TMFN}(M_F, N_F)$ and $d_{TMFN}(M_F, L_F) \geq d_{TMFN}(N_F, L_F)$.

Proof:

Part (i):

If $M_F, N_F \in FNS(X)$, then $0 \leq T_{M_F}(x_i) \leq 1$, $0 \leq I_{M_F}(x_i) \leq 1$, $0 \leq F_{M_F}(x_i) \leq 1, \forall x_i \in X$.

$$\Rightarrow 0 \leq \left|T_{M_F}(x_i) - T_{N_F}(x_i)\right| \leq 1,$$

$$0 \leq \left|I_{M_F}(x_i) - I_{N_F}(x_i)\right| \leq 1,$$

$$0 \leq \left|F_{M_F}(x_i) - F_{N_F}(x_i)\right| \leq 1.$$

$$\Rightarrow 0 \leq tan\left\{\frac{\pi}{6}\left(\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right)\right\} \leq \frac{1}{\sqrt{3}}$$

$$0 \leq tan\left\{\frac{\pi}{6}\left(\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right)\right\} \leq \frac{1}{\sqrt{3}}$$

$$0 \leq tan\left\{\frac{\pi}{6}\left(\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right)\right\} \leq \frac{1}{\sqrt{3}}$$

Then

$$0 \leq \begin{pmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{pmatrix} \leq \sqrt{3},, \forall x_i \in X$$

Multiple by 2

$$0 \leq \frac{2}{\sqrt{3}} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix} \leq 2, \forall x_i \in X$$

This implies

$$0 \leq \frac{2}{\sqrt{3}n} \sum_{i=1}^{n} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix} \leq 2$$

Thus, $0 \leq d_{TMNS}(M_F, N_F) \leq 2$.

Part (ii):

$$d_{TMNS}(M_F, N_F) = 0$$

$$\Leftrightarrow 0 \leq \frac{2}{\sqrt{3}n} \sum_{i=1}^{n} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix} \leq 2$$

$$\Leftrightarrow 0 \leq \frac{2}{\sqrt{3}} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix} = 0,, \forall x_i \in X$$

$$\Leftrightarrow \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix} = 0, \forall x_i \in X.$$

$$\Leftrightarrow tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} = 0 \forall x_i \in X$$

$$\Leftrightarrow \left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$\Leftrightarrow \left|T_{M_F}(x_i) - T_{N_F}(x_i)\right| = 0, \forall x_i \in X$$

$$\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right| = 0, \forall x_i \in X$$

$$\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right| = 0, \forall x_i \in X$$

$$\Leftrightarrow T_{M_F}(x_i) = T_{N_F}(x_i), I_{M_F}(x_i) = I_{N_F}(x_i), F_{M_F}(x_i) = F_{N_F}(x_i), \forall x_i \in X.$$

$\Leftrightarrow M_F = N_F$. Thus $d_{TMNS}(M_F, N_F) = 0$ iff $M_F = N_F$.

Part (iii):

$$d_{TMNS}(M_F, N_F) = \frac{2}{\sqrt{3}n} \sum_{i=1}^{n} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{N_F}(x_i) - T_{M_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{N_F}(x_i) - I_{M_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{N_F}(x_i) - F_{M_F}(x_i)\right|\right\} \end{bmatrix}$$

$d_{TMNS}(N_F, M_F)$. Thus $d_{TMNS}(M_F, N_F) = d_{TMNS}(N_F, M_F)$ and this proves part (iii).

Part (iv):

If $M_F \subseteq N_F \subseteq L_F$ then $T_{M_F}(x_i) \leq T_{N_F}(x_i) \leq T_{L_F}(x_i)$, $I_{M_F}(x_i) \leq I_{N_F}(x_i) \leq I_{L_F}(x_i)$ and $F_{M_F}(x_i) \geq F_{N_F}(x_i) \geq F_{L_F}(x_i)$, $\forall x_i \in X$. This implies to the following inequalities

$$\left|T_{M_F}(x_i) - T_{L_F}(x_i)\right| \geq \left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|$$

$$\left|T_{M_F}(x_i) - T_{L_F}(x_i)\right| \geq \left|T_{N_F}(x_i) - T_{L_F}(x_i)\right|$$

$$\left|I_{M_F}(x_i) - I_{L_F}(x_i)\right| \geq \left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|$$

$$\left|I_{M_F}(x_i) - I_{L_F}(x_i)\right| \geq \left|I_{N_F}(x_i) - I_{L_F}(x_i)\right|$$

and

$$\left|F_{M_F}(x_i) - F_{L_F}(x_i)\right| \geq \left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|$$

$$\left|F_{M_F}(x_i) - F_{L_F}(x_i)\right| \geq \left|F_{N_F}(x_i) - F_{L_F}(x_i)\right|.$$

From these inequalities, the following relation holds

$$\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{L_F}(x_i)\right|\right\} + \\ tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{L_F}(x_i)\right|\right\} + \\ tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{L_F}(x_i)\right|\right\} \end{bmatrix} \geq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{L_F}(x_i)\right|\right\} + \\ tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{L_F}(x_i)\right|\right\} + \\ tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{L_F}(x_i)\right|\right\} \end{bmatrix}$$

$$\frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{L_F}(x_i)\right|\right\} + \\ tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{L_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{L_F}(x_i)\right|\right\} \end{bmatrix} \geq \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{M_F}(x_i) - T_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{M_F}(x_i) - I_{N_F}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{M_F}(x_i) - F_{N_F}(x_i)\right|\right\} \end{bmatrix}$$

Therefore, $d_{TMNS}(M_F, L_F) \geq d_{TMNS}(M_F, N_F)$ is demonstrated. In a comparable way, it can be shown that $d_{TMNS}(M_F, L_F) \geq d_{TMNS}(N_F, L_F)$, which concludes the proof.

6.1.2 Tangent Distance Measure on Neutrosophic Spherical Set

In this sub-section, a distance measure for the NSS is introduced. It also enumerates some of the derived characteristics of this new metric.

Definition 6.1.2.1

Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set. Let
$A_S = \{x_i, T_{A_S}(x_i), I_{A_S}(x_i), F_{A_S}(x_i) : x_i \in X\}$,
$B_S = \{x_i, T_{B_S}(x_i), I_{B_S}(x_i), F_{B_S}(x_i) : x_i \in X\}$
be two NSSs on X. Define a mapping $d: NSS(X) \times NSS(X) \to [0,1]$ to represent a distance function. Then, the distance measue between two NSSs is denoted by

$$d_{TMNS}(A_S, B_S) = \frac{1}{\sqrt{3}n} \sum_{i=1}^{n} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} \qquad (6.1.2.1)$$

Then $d_{TMNS}(A_S, B_S)$ is called the tangent metric neutrosophic spherical distance measure (TMNSDM).

Theorem 6.1.2.2

Let X be a non-empty universel set. Let A_S, B_S, C_S be NSSs. Then, the TMNSDM should satisty the following axioms

i. $d_{TMNS}(A_S, B_S) \geq 0$ for all $A, B \in NSS(X)$.
ii. $d_{TMNS}(A_S, B_S) = 0$ if and only if $A_S = B_S$ for all $A_S, B_S \in NSS(X)$.
iii. $d_{TMNS}(A_S, B_S) = d_{TMNS}(B_S, A_S)$ for all $A_S, B_S \in NSS(X)$.
iv. If $A_S \subseteq B_S \subseteq C_S$ for all $A_S, B_S, C_S \in NSS(X)$, then
$d_{TMNS}(A_S, C_S) \geq d_{TMNS}(A_S, B_S)$and $d_{TMNS}(A_S, C_S) \geq d_{TMNS}(B_S, C_S)$.

Proof:

Part (i):

If $A_S, B_S \in NSS(X)$,

then $0 \leq T_{A_S}(x_i) \leq 1, 0 \leq I_{A_S}(x_i) \leq 1, 0 \leq F_{A_S}(x_i) \leq 1, \forall x_i \in X$

$\Rightarrow 0 \leq \left|T_{A_S}(x_i) - T_{B_S}(x_i)\right| \leq 1,$ $0 \leq \left|I_{A_S}(x_i) - I_{B_S}(x_i)\right| \leq 1,$ and $0 \leq \left|F_{A_S}(x_i) - F_{B_S}(x_i)\right| \leq 1.$

$\Rightarrow \ 0 \leq tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \leq \frac{1}{\sqrt{3}},$ $0 \leq tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \leq \frac{1}{\sqrt{3}},$ $0 \leq tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \leq \frac{1}{\sqrt{3}}$

Then

$$0 \leq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} \leq \sqrt{3}, \forall x_i \in X.$$

$$0 \leq \left[\frac{1}{\sqrt{3}} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix}\right] \leq 1, \forall x_i \in X.$$

This implies

$$0 \leq \frac{1}{\sqrt{3}n} \sum_{i=1}^{n} \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} \leq 1$$

Thus, $0 \leq d_{TMNS}(A_S, B_S) \leq 1$.

Part (ii):

$$d_{TMNS}(A_S, B_S) = 0$$

$$\Leftrightarrow 0 \le \frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} \tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} \le 1$$

$$\Leftrightarrow 0 \le \frac{1}{\sqrt{3}}\begin{bmatrix} \tan\left\{\frac{\pi}{6}\left|T_A(x_i) - T_B(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|I_A(x_i) - I_B(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|F_A(x_i) - F_B(x_i)\right|\right\} \end{bmatrix} = 0, \forall x_i \in X$$

$$\Leftrightarrow \begin{bmatrix} \tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + \tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} = 0, \forall x_i \in X$$

$$\Leftrightarrow \tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} = 0,$$

$$\tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} = 0,$$

$$\tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$\Leftrightarrow \left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} = 0,$$

$$\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} = 0,$$

$$\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} = 0, \forall x_i \in X$$

$$\Leftrightarrow \left|T_{A_S}(x_i) - T_{B_S}(x_i)\right| = 0,$$

$$\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right| = 0,$$

$$\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right| = 0, \forall x_i \in X$$

$\Leftrightarrow T_{A_S}(x_i) = T_{B_S}(x_i), I_{A_S}(x_i) = I_{B_S}(x_i), F_{A_S}(x_i) = F_{B_S}(x_i), \forall x_i \in X$

$\Leftrightarrow A_S = B_S$. Thus $d_{TMNS}(A_S, B_S) = 0$ iff $A_S = B_S$.

Part (iii):

$$d_{TMNS}(A_S, B_S) = \frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{B_S}(x_i) - T_{A_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{B_S}(x_i) - I_{A_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{B_S}(x_i) - F_{A_S}(x_i)\right|\right\} \end{bmatrix}$$

$d_{TMNS}(B_S, A_S)$. Thus $d_{TMNS}(A_S, B_S) = d_{TMNS}(B_S, A_S)$ and this proves part (iii).

Part (iv):

If $A_S \subseteq B_S \subseteq C_S$ then

$T_{A_S}(x_i) \leq T_{B_S}(x_i) \leq T_{C_S}(x_i)$,

$I_{A_S}(x_i) \leq I_{B_S}(x_i) \leq I_{C_S}(x_i)$ and

$F_{A_S}(x_i) \geq F_{B_S}(x_i) \geq F_{C_S}(x_i), \forall x_i \in X$.

This implies to the following inequalities

$\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right| \geq \left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|$, $\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right| \geq \left|T_{B_S}(x_i) - T_{C_S}(x_i)\right|$

$\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right| \geq \left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|$, $\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right| \geq \left|I_{B_S}(x_i) - I_{C_S}(x_i)\right|$

$\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right| \geq \left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|$, $\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right| \geq \left|F_{B_S}(x_i) - F_{C_S}(x_i)\right|$.

From these inequalities, the following relation holds

$$\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right|\right\} \end{bmatrix} \geq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix}$$

$$\frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right|\right\} \end{bmatrix} \geq \frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix}$$

Therefore, $d_{TMNS}(A_S, C_S) \geq d_{TMNS}(A_S, B_S)$ is proved. Similarly, it can be shown that $d_{TMNS}(A_S, C_S) \geq d_{TMNS}(B_S, C_S)$, which concludes the proof.

Theorem 6.1.2.3

$d_{TMNS}(A_S, C_S) \leq d_{TMNS}(A_S, B_S) + d_{TMNS}(B_S, C_S)$ is true for $A_S, B_S, C_S \in NSS(X)$.

Proof:

Let $A_S, B_S, C_S \in NSS(X)$ then for all $x_i \in X$, the following inequalities are true for the real numbers $\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right| \leq \left|T_{A_S}(x_i) - T_{B_S}(x_i)\right| + \left|T_{B_S}(x_i) - T_{C_S}(x_i)\right|$,

$$tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right|\right\} \leq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|T_{B_S}(x_i) - T_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

$$tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right|\right\} \leq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{B_S}(x_i) - I_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

$$tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right|\right\} \leq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{B_S}(x_i) - F_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

$$\leq \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i) - T_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i) - I_{B_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i) - F_{B_S}(x_i)\right|\right\} \end{bmatrix} + \begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{B_S}(x_i) - T_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{B_S}(x_i) - I_{C_S}(x_i)\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{B_S}(x_i) - F_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

$$\frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i)-T_{C_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i)-I_{C_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i)-F_{C_S}(x_i)\right|\right\} \end{bmatrix} \leq$$

$$\frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{A_S}(x_i)-T_{B_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|I_{A_S}(x_i)-I_{B_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|F_{A_S}(x_i)-F_{B_S}(x_i)\right|\right\} \end{bmatrix} + \frac{1}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{B_S}(x_i)-T_{C_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|I_{B_S}(x_i)-I_{C_S}(x_i)\right|\right\} \\ +tan\left\{\frac{\pi}{6}\left|F_{B_S}(x_i)-F_{C_S}(x_i)\right|\right\} \end{bmatrix}$$

Hence $d_{TMNS}(A_S, C_S) \leq d_{TMNS}(A_S, B_S) + d_{TMNS}(B_S, C_S)$ is true for $A_S, B_S, C_S \in NSS(X)$.

Remark 6.1.2.4

According to the above theorem, the distance measure for neutrosophic spherical sets satisfies all of the metric axioms.

6.2. Tangent Metrics Applied to Neutrosophic Variants: A Novel Perspective

6.2.1 Visualizing Tangent Metrics: An Illustrative Model for Fermatean Neutrosophic Sets

In this sub-section, an illustrative problem is constructed to elucidate the working rule of the defined distance measure in Fermatean neutrosophic environment.

Consider a scenario where students are selecting courses from different universities. The set of students, universities, and courses are represented as follows:

- $S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}\}$ which represents a collection of ten students.
- $U = \{U_1, U_2, U_3, U_4, U_5\}$ which represents a collection of five Universities.
- $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$, which represents a collection of six courses.

Table 6.2.1.1 depicts assumed relations between universities and courses, and Table 6.2.1.2 details the assumed relations between students and courses based on any chosen criteria employing the FNS.

Table 6.2.1.1. Assumed relations between Universities and courses

	C_1	C_2	C_3	C_4	C_5	C_6
U_1	0.4	0.5	0.3	0.65	0.35	0.7
	0.8	0.5	0.7	0.65	0.95	0.7
	0.7	0.9	0.4	0.7	0.6	0.65
U_2	0.3	0.55	0.7	0.6	0.9	0.65
	0.8	0.55	0.8	0.75	0.8	0.65
	0.9	0.95	0.3	0.75	0.45	0.7
U_3	0.4	0.4	0.9	0.55	0.85	0.6
	0.7	0.8	0.7	0.8	0.75	0.75
	0.7	0.7	0.3	0.8	0.5	0.75
U_4	0.4	0.3	0.7	0.5	0.8	0.55
	0.6	0.8	0.9	0.85	0.7	0.8

	0.9	0.9	0.2	0.85	0.55	0.8
U_5	0.3	0.4	0.5	0.45	0.75	0.5
	0.7	0.7	0.5	0.9	0.75	0.85
	0.4	0.7	0.9	0.9	0.6	0.85

Table 6.2.1.2 Assumed relations between students and courses

	C_1	C_2	C_3	C_4	C_5	C_6
S_1	0.8,0.7,0.55	0.7,0.9,0.65	0.6,0.75,0.75	0.7,0.8,0.6	0.95,0.85,0.4	0.4,0.95,0.95
S_2	0.75,0.75,0.6	0.65,0.9,0.7	0.55,0.8,0.8	0.85,0.9,0.7	0.9,0.8,0.45	0.35,0.95,1
S_3	0.7,0.7,0.65	0.6,0.9,0.75	0.5,0.85,0.85	0.9,0.9,0.3	0.85,0.75,0.5	0.9,0.8,0.45
S_4	0.65,0.65,0.7	0.55,0.8,0.8	0.45,0.9,0.9	0.95,0.95,0.4	0.8,0.7,0.55	0.85,0.75,0.5
S_5	0.6,0.75,0.75	0.5,0.85,0.85	0.4,0.95,0.95	0.9,0.9,0.45	0.75,0.65,0.6	0.8,0.7,0.55
S_6	0.55,0.8,0.8	0.45,0.9,0.9	0.95,0.85,0.4	0.85,0.85,0.5	0.7,0.9,0.65	0.75,0.75,0.6
S_7	0.5,0.85,0.85	0.4,0.95,0.95	0.9,0.8,0.45	0.8,0.8,0.55	0.65,0.9,0.7	0.7,0.7,0.65
S_8	0.45,0.9,.9	0.35,0.95,1	0.85,0.75,0.5	0.75,0.75,0.6	0.6,0.9,0.75	0.65,0.65,0.7
S_9	0.4,0.95,0.95	0.9,0.8,0.45	0.8,0.7,0.55	0.7,0.7,0.65	0.55,0.8,0.8	0.6,0.75,0.75
S_{10}	0.95,0.85,0.4	0.85,0.75,0.5	0.75,0.65,0.6	0.65,0.65,0.7	0.5,0.85,0.85	0.55,0.8,0.8

Distance measure for the attributes

Table 6.2.1.3 shows the shortest path, as measured by the ED measure $d_E(\tilde{A}, \tilde{B})$ using Equation (1.1.2), between each student (Table 6.2.1.2) and university (Table 6.2.1.1).

Table 6.2.1.3 ED

$d_E(\tilde{A}, \tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.7477	0.5439	0.5139	1.8147	0.5107
S_2	0.7365	0.6110	0.5392	0.6	0.5008
S_3	0.7153	0.6538	0.6519	0.7205	0.6916
S_4	0.6544	0.6075	0.6258	0.6652	0.6284
S_5	0.5780	0.5972	0.6278	0.6454	0.6062

S_6	0.5816	0.4232	0.3763	0.4462	0.6770
S_7	0.5392	0.4143	0.3752	0.4262	0.6435
S_8	0.5058	0.4222	0.4031	0.4320	0.6238
S_9	0.5605	0.5275	0.4991	0.6069	0.6396
S_{10}	0.6041	0.7076	0.6	0.7285	0.6191

The N-ED measure $d_{n-E}(\tilde{A}, \tilde{B})$ has been used in Table 6.2.1.4 to estimate the shortest distance between each student (Table 6.2.1.2) and each university (Table 6.2.1.1) using Equation (1.1.4).

Table 6.2.1.4. N-ED

$d_{n-E}(\tilde{A}, \tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.4317	0.3140	0.2967	1.0477	0.2948
S_2	0.4252	0.3527	0.3113	0.3464	0.2891
S_3	0.4129	0.3774	0.3763	0.4159	0.3993
S_4	0.3778	0.3507	0.3613	0.3840	0.3628
S_5	0.3337	0.3448	0.3624	0.3726	0.35
S_6	0.3358	0.2443	0.2173	0.2576	0.3908
S_7	0.3113	0.2392	0.2166	0.2460	0.3715
S_8	0.2920	0.2438	0.2327	0.2494	0.3601
S_9	0.3236	0.3045	0.2881	0.3503	0.3693
S_{10}	0.3488	0.4085	0.3464	0.4206	0.3574

The shortest path between each student (Table 6.2.1.2) and each university (Table 6.2.1.1) has been found using Equation (1.1.1) of the HD measure $d_H(\tilde{A}, \tilde{B})$ in Table 6.2.1.5.

Table 6.2.1.5. HD

$d_H(\tilde{A}, \tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	1.55	1.2	1.02	4.33	1.05

S_2	1.55	1.17	1.05	4.47	0.98
S_3	1.53	1.32	1.17	1.35	1.23
S_4	1.37	1.25	1.13	1.15	1.17
S_5	1.22	1.2	1.18	1.17	1.15
S_6	1	0.88	0.8	0.92	1.4
S_7	0.92	0.8	0.78	0.93	1.38
S_8	0.92	0.77	0.85	0.93	1.38
S_9	1.05	1	0.88	1.13	1.35
S_{10}	1.13	1.35	1.17	1.42	1.2

Using the N-HD measure $d_{n-H}(\tilde{A},\tilde{B})$ in Equation (1.1.3), Table 6.2.1.6 calculates the shortest distance between each student (Table 6.2.1.2) and each university (Table 6.2.1.1).

Table 6.2.1.6. N-HD

$d_{n-H}(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.51	0.4	0.34	1.44	0.35
S_2	0.51	0.39	0.35	1.49	0.32
S_3	0.51	0.44	0.39	0.45	0.41
S_4	0.45	0.42	0.38	0.38	0.39
S_5	0.40	0.4	0.39	0.38	0.38
S_6	0.33	0.29	0.27	0.31	0.47
S_7	0.31	0.27	0.26	0.31	0.46
S_8	0.31	0.26	0.28	0.31	0.46
S_9	0.35	0.33	0.29	0.38	0.45
S_{10}	0.38	0.45	0.39	0.47	0.4

The shortest path between each student (Table 6.2.1.2) and each university (Table 6.2.1.1) has been found using Equation (1.1.7) for the SMSVND measure $d_{SMSVN}\left(\tilde{A},\tilde{B}\right)$, as shown in Tables 6.2.1.7.

Table 6.2.1.7. $\mathcal{S}_{\mathcal{M}}\mathcal{S}_{\mathcal{V}}\mathcal{N}\mathcal{D}\mathcal{M}$

$d_{SMSVN}\left(\tilde{A},\tilde{B}\right)$	U_1	U_2	U_3	U_4	U_5
S_1	0.464	0.379	0.343	0.874	0.355
S_2	0.467	0.367	0.344	0.885	0.337
S_3	0.464	0.411	0.355	0.415	0.385
S_4	0.421	0.397	0.348	0.356	0.369
S_5	0.379	0.385	0.364	0.367	0.368
S_6	0.330	0.309	0.284	0.317	0.426
S_7	0.304	0.281	0.279	0.323	0.429
S_8	0.310	0.262	0.297	0.322	0.435
S_9	0.336	0.324	0.296	0.364	0.423
S_{10}	0.353	0.410	0.371	0.416	0.386

The shortest path between each student (Table 6.2.1.2.) and each university (Table 6.2.1.1) has been found using Equation (6.1.1.1) for TMFNDM $d_{TMFN}\left(\tilde{A},\tilde{B}\right)$ in Tables 6.2.1.8.

Table 6.2.1.8. $\mathcal{T}_{\mathcal{M}}\mathcal{F}\mathcal{N}\mathcal{D}\mathcal{M}$ Distance

$d_{TMFN}\left(\tilde{A},\tilde{B}\right)$	U_1	U_2	U_3	U_4	U_5
S_1	0.95	0.73	0.61	2.77	0.63
S_2	0.94	0.71	0.64	2.87	0.59
S_3	0.93	0.80	0.71	0.83	0.75
S_4	0.83	0.76	0.69	0.70	0.71
S_5	0.74	0.73	0.72	0.71	0.70
S_6	0.61	0.53	0.48	0.55	0.85

S_7	0.56	0.48	0.47	0.56	0.84
S_8	0.56	0.46	0.51	0.56	0.84
S_9	0.64	0.61	0.53	0.69	0.82
S_{10}	0.69	0.82	0.71	0.86	0.73

Evaluation

The distance between each student and each university has been calculated using six different distance measures. The shortest distance measure values for the students and each university are shown in Table 6.2.1.9.

Table 6.2.1.9. Comparison of distance measures

	U_1	U_2	U_3	U_4	U_5
S_1	0.74	0.54	0.51	1.81	0.51
	0.29	0.31	0.29	1.04	0.29
	1.01	1.2	1.01	4.33	1.05
	0.33	0.4	0.33	1.44	0.35
	0.34	0.37	0.34	0.87	0.35
	0.95	0.73	0.61	2.77	0.63
S_2	0.73	0.61	0.53	0.6	0.50
	0.28	0.35	0.31	0.34	0.28
	1.16	1.16	1.05	4.46	0.98
	0.51	0.38	0.35	1.48	0.32
	0.46	0.36	0.34	0.88	0.33
	0.94	0.71	0.64	2.87	0.59
S_3	0.71	0.65	0.65	0.72	0.69
	0.41	0.37	0.37	0.41	0.39
	1.53	1.31	1.16	1.35	1.23
	0.51	0.43	0.38	0.45	0.41
	0.46	0.41	0.35	0.41	0.38
	0.93	0.80	0.71	0.83	0.75
S_4	0.65	0.60	0.62	0.66	0.62
	0.37	0.35	0.36	0.38	0.36
	1.36	1.25	1.13	1.15	1.16
	0.45	0.41	0.37	0.38	0.38
	0.42	0.39	0.34	0.35	0.36
	0.83	0.76	0.69	0.70	0.71
S_5	0.57	0.59	0.62	0.64	0.60
	0.33	0.34	0.36	0.37	0.35

	1.21	1.2	1.18	1.16	1.15
	0.40	0.4	0.39	0.38	0.38
	0.37	0.38	0.36	0.36	0.36
	0.74	0.73	0.72	0.71	0.70
S_6	0.58	0.42	0.37	0.44	0.67
	0.33	0.24	0.21	0.25	0.39
	1	0.88	0.8	0.91	1.4
	0.33	0.29	0.26	0.30	0.46
	0.33	0.30	0.28	0.31	0.42
	0.61	0.53	0.48	0.55	0.85
S_7	0.53	0.41	0.37	0.42	0.64
	0.31	0.23	0.21	0.24	0.37
	0.91	0.8	0.78	0.93	1.38
	0.30	0.26	0.26	0.31	0.46
	0.30	0.28	0.27	0.32	0.42
	0.56	0.48	0.47	0.56	0.84
S_8	0.50	0.42	0.40	0.43	0.62
	0.29	0.24	0.23	0.24	0.36
	0.91	0.76	0.85	0.93	1.38
	0.30	0.25	0.28	0.31	0.46
	0.31	0.26	0.29	0.32	0.43
	0.56	0.46	0.51	0.56	0.84
S_9	0.56	0.52	0.49	0.60	0.63
	0.32	0.30	0.28	0.35	0.36
	1.05	1	0.88	1.13	1.35
	0.35	0.33	0.29	0.37	0.45
	0.33	0.32	0.29	0.36	0.42
	0.64	0.61	0.53	0.69	0.82
S_{10}	0.60	0.70	0.6	0.72	0.61
	0.34	0.40	0.34	0.42	0.35
	1.13	1.35	1.16	1.41	1.2
	0.37	0.45	0.38	0.47	0.4
	0.35	0.41	0.37	0.41	0.38
	0.69	0.82	0.71	0.86	0.73

In the above Table 6.2.1.9 the minimum distance measure values between each student and the corresponding universities are highlighted in yellow. Accordingly, TMFNDM $d_{TMFN}(\tilde{A},\tilde{B})$ yields the most frequent and optimal solution, which is indicated in red.

Results and Discussion

Table 6.2.1.10 Comparison

	$d_E(\tilde{A},\tilde{B})$	$d_{n-E}(\tilde{A},\tilde{B})$	$d_H(\tilde{A},\tilde{B})$	$d_{n-H}(\tilde{A},\tilde{B})$	$d_{SMSVN}(\tilde{A},\tilde{B})$	$d_{TMFN}(\tilde{A},\tilde{B})$
S_1	$U_3\&U_5$	$U_3\&U_5$	U_3	$U_1\&U_3$	$U_1\&U_3$	U_3
S_2	U_5	$U_1\&U_5$	U_5	U_5	U_5	U_5
S_3	U_3	U_3	U_3	U_3	U_3	U_3
S_4	U_2	U_2	U_3	U_3	U_3	U_3
S_5	U_1	U_1	U_5	U_5	U_3	U_5
S_6	U_3	U_3	U_3	U_3	U_3	U_3
S_7	U_3	U_3	U_3	U_3	U_3	U_3
S_8	U_3	U_3	U_2	U_2	U_2	U_2
S_9	U_3	U_3	U_3	U_3	U_3	U_3
S_{10}	U_3	U_3	U_5	U_1	U_1	U_1

From the above table, it is observe that the most frequently occurring shortest distance is found in the Tangent Metric Fermatean Neutrosophic Distance Measure (TMFNDM). Hence it is the generalized distance measure of all other taken distance measure. Similarlly check this for NSS, which is given in the below section.

6.2.2 Visualizing Tangent Metrics: An Illustrative Model for Neutrosophic Spherical Sets

In this sub-section, the same illustrative problem used in the Section 6.2.1 is considered to demonstrate the application of the defined TMNSDM in a neutrosophic spherical environment.

Table 6.2.2.1 presents data related to university and courses using the NSS, while Table 6.2.2.2 shows a relationship between the student and courses using the NSS.

Table 6.2.2.1. Neutrosophic spherical relations between Universities and courses

	C_1	C_2	C_3	C_4	C_5	C_6
U_1	0.9 0.6 0.2	0.9 0.7 0.3	0.8 0.8 0.4	0.9 0.7 0.3	0.9 0.6 0.2	0.8 0.8 0.4
U_2	0.9 0.7 0.3	0.8 0.8 0.4	0.7 0.95 0.5	0.5 0.95 0.7	0.8 0.8 0.4	0.7 0.95 0.5
U_3	0.8 0.8 0.4	0.9 0.7 0.3	0.8 0.8 0.4	0.9 0.7 0.3	0.6 0.9 0.6	0.4 0.8 0.8
U_4	0.7 0.95 0.5	0.7 0.95 0.5	0.4 0.8 0.8	0.9 0.7 0.3	0.5 0.95 0.7	0.9 0.7 0.3
U_5	0.5 0.3 0.9	0.5 0.7 0.7	0.5 0.6 0.7	0.3 0.7 0.9	0.2 0.6 0.9	0.6 0.7 0.7

Table 6.2.2.2 Neutrosophic spherical relations between students and courses

	C_1	C_2	C_3	C_4	C_5	C_6
S_1	0.9 0.2 0.6	0.9 0.9 0.2	0.9 0.5 0.5	0.7 0.7 0.55	0.5 0.7 0.7	0.9 0.4 0.4
S_2	0.9 0.2 0.7	0.9 0.9 0.3	0.9 0.7 0.5	0.3 0.9 0.5	0.5 0.5 0.7	0.9 0.4 0.3
S_3	0.9	0.9	0.9	0.5	0.7	0.5

	0.1 0.7	0.8 0.35	0.7 0.4	0.7 0.7	0.5 0.8	0.7 0.7
S_4	0.5 0.5 0.7	0.9 0.8 0.4	0.9 0.7 0.3	0.5 0.8 0.7	0.6 0.5 0.3	0.9 0.3 0.9
S_5	0.6 0.7 0.7	0.5 0.7 0.7	0.9 0.7 0.2	0.9 0.6 0.35	0.6 0.5 0.2	0.8 0.7 0.5
S_6	0.6 0.7 0.7	0.9 0.8 0.2	0.7 0.7 0.5	0.7 0.6 0.4	0.6 0.5 0.7	0.9 0.3 0.7
S_7	0.5 0.3 0.9	0.9 0.8 0.1	0.9 0.6 0.7	0.5 0.7 0.7	0.9 0.4 0.6	0.5 0.7 0.7
S_8	0.5 0.5 0.7	0.9 0.8 0.25	0.9 0.6 0.62	0.6 0.7 0.7	0.9 0.4 0.7	0.9 0.3 0.5
S_9	0.5 0.5 0.7	0.5 0.7 0.7	0.9 0.6 0.65	0.7 0.9 0.5	0.8 0.4 0.6	0.9 0.5 0.4
S_{10}	0.8 0.7 0.5	0.9 0.7 0.65	0.8 0.7 0.5	0.9 0.5 0.7	0.6 0.4 0.6	0.8 0.4 0.8

Distance measure for the attributes

The shortest distance using Equation (1.1.2) of the ED measure $d_E(\tilde{A},\tilde{B})$ between each student (Table 6.2.2.2) and each university (Table 6.2.2.1) has been determined in Table 6.2.2.3.

Table 6.2.2.3. Euclidean Distance

$d_E(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.6512	0.6608	0.6714	0.6714	0.7444
S_2	0.7325	0.6448	0.7937	0.7566	0.6928
S_3	0.7100	0.6082	0.6512	0.8563	0.6278
S_4	0.7257	0.6576	0.7071	0.8341	0.7549
S_5	0.5780	0.6531	0.6331	0.6831	0.8548
S_6	0.6582	0.6677	0.5830	0.5909	0.7416

S_7	0.7615	0.6825	0.7257	0.8674	0.7
S_8	0.7059	0.6827	0.7604	0.7340	0.7422
S_9	0.6689	0.6350	0.7444	0.6429	0.6837
S_{10}	0.6224	0.6350	0.5894	0.6658	0.7421

Table 6.2.2.4 uses the N-ED measure $d_{n-E}(A,B)$, defined in Equation (1.1.4), to calculate the shortest distance between each student (Table 6.2.2.2) and each university (Table 6.2.2.1).

Table 6.2.2.4. Normalized Euclidean Distance

$d_{n-E}(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.2658	0.2697	0.2741	0.2741	0.3039
S_2	0.2990	0.2632	0.3240	0.3088	0.2828
S_3	0.2898	0.2483	0.2658	0.3496	0.2563
S_4	0.2962	0.2684	0.2886	0.3405	0.3082
S_5	0.2359	0.2666	0.2584	0.2788	0.3490
S_6	0.2687	0.2725	0.2380	0.2412	0.3027
S_7	0.3109	0.2786	0.2962	0.3541	0.2857
S_8	0.2881	0.2787	0.3104	0.2996	0.3030
S_9	0.2730	0.2592	0.3039	0.2624	0.2791
S_{10}	0.2541	0.2592	0.2406	0.2718	0.3029

The HD measure $d_H(\tilde{A},\tilde{B})$, using Equation (1.1.1), has been applied in Table 6.2.2.5 to determine the shortest distance between each student (Table 6.2.2.2) and each university (Table 6.2.2.1).

Table 6.2.2.5. Hamming Distance

$d_H(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	1.25	1.3333	1.3166	1.316667	1.65
S_2	1.3666	1.15	1.5	1.3833	1.5333

S_3	1.3166	1.1333	1.1166	1.7	1.25
S_4	1.4	1.25	1.4	1.75	1.5666
S_5	1.0166	1.4333	1.0166	1.3	1.55
S_6	1.2666	1.3166	1.0666	1.2166	1.5666
S_7	1.5333	1.3833	1.4	1.8166	1.1
S_8	1.3566	1.34	1.5566	1.4933	1.4566
S_9	1.4166	1.3333	1.6166	1.3333	1.35
S_{10}	1.1833	1.2333	0.9166	1.3333	1.5833

Table 6.2.2.6 uses the N-HD measure $d_{n-H}(\tilde{A}, \tilde{B})$, defined in Equation (1.1.3), to calculate the shortest distance between each student (Table 6.2.2.2) and each university (Table 6.2.2.1).

Table 6.2.2.6. Normalized Hamming Distance

$d_{n-H}(\tilde{A}, \tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.2083	0.2222	0.2194	0.2194	0.275
S_2	0.2277	0.1916	0.25	0.2305	0.2555
S_3	0.2194	0.1888	0.1861	0.2833	0.2083
S_4	0.2333	0.2083	0.2333	0.2916	0.2611
S_5	0.1694	0.2388	0.1694	0.2166	0.2583
S_6	0.2111	0.2194	0.1777	0.2027	0.2611
S_7	0.2555	0.2305	0.2333	0.3027	0.1833
S_8	0.2261	0.2233	0.2594	0.2488	0.2427
S_9	0.2361	0.2222	0.2694	0.2222	0.225
S_{10}	0.1972	0.2055	0.1527	0.2222	0.2638

Table 6.2.2.7 uses the SMSVND measure $d_{SMSVN}(\tilde{A}, \tilde{B})$, defined in Equation (1.1.7), to determine the shortest distance between each student (Table 6.2.2.2) and each university (Table 6.2.2.1).

Table 6.2.2.7. SMSVNDM Distance

$d_{SMSVN}(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.3948	0.4253	0.4057	0.4057	0.4979
S_2	0.4093	0.3565	0.4330	0.4228	0.4711
S_3	0.3980	0.3610	0.3479	0.5019	0.3920
S_4	0.4204	0.3748	0.4236	0.5140	0.4707
S_5	0.3284	0.4450	0.3306	0.3945	0.4372
S_6	0.3904	0.4087	0.3451	0.3980	0.4775
S_7	0.4552	0.4195	0.4232	0.5297	0.3219
S_8	0.4187	0.4161	0.4576	0.4635	0.4420
S_9	0.4405	0.4238	0.4866	0.4189	0.4103
S_{10}	0.3781	0.3919	0.3087	0.4233	0.4761

The TMNSDM $d_{TMNS}(\tilde{A},\tilde{B})$, using Equation (6.1.2.1) in Table 6.2.2.8, has been used to determine the shortest distance between each student (Table 6.2.2.2) and each university (Table 6.2.2.1).

Table 6.2.2.8. TMNSDM Distance

$d_{TMNS}(\tilde{A},\tilde{B})$	U_1	U_2	U_3	U_4	U_5
S_1	0.3188	0.3399	0.3361	0.3361	0.4203
S_2	0.3502	0.2934	0.3852	0.3556	0.3900
S_3	0.3370	0.2890	0.2864	0.4378	0.3182
S_4	0.3582	0.3192	0.3575	0.4482	0.4007
S_5	0.2592	0.3642	0.2592	0.3324	0.4004
S_6	0.3234	0.3360	0.2720	0.3092	0.3997
S_7	0.3925	0.3528	0.3581	0.4658	0.2833
S_8	0.3468	0.3422	0.3979	0.3813	0.3728
S_9	0.3607	0.3391	0.4124	0.3396	0.3445

S_{10}	0.3013	0.3144	0.2345	0.3400	0.4040

Evaluation

The distance between each student and each university has been calculated using six different distance measures. The shortest distance measure values for the students and each university are shown in Table 6.2.2.9.

Table 6.2.2.9 Comparison of distance measures

	U_1	U_2	U_3	U_4	U_5
S_1	0.6512	0.6608	0.6714	0.6714	0.7444
	0.2658	0.2697	0.2741	0.2741	0.3039
	1.25	1.3333	1.3166	1.3166	1.65
	0.2083	0.2222	0.2194	0.2194	0.275
	0.3948	0.4253	0.4057	0.4057	0.4979
	0.3188	0.3399	0.3361	0.3361	0.4203
S_2	0.7325	0.6448	0.7937	0.7566	0.6928
	0.2990	0.2632	0.3240	0.3088	0.2828
	1.3666	1.15	1.5	1.3833	1.5333
	0.2277	0.1916	0.25	0.2305	0.2555
	0.4093	0.3565	0.4330	0.4228	0.4711
	0.3502	0.2934	0.3852	0.3556	0.3900
S_3	0.7100	0.6082	0.6512	0.8563	0.6278
	0.2898	0.2483	0.2658	0.3496	0.2563
	1.3166	1.1333	1.1166	1.7	1.25
	0.2194	0.1888	0.1861	0.2833	0.2083
	0.3980	0.3610	0.3479	0.5019	0.3920
	0.3370	0.2890	0.2864	0.4378	0.3182
S_4	0.7257	0.6576	0.7071	0.8341	0.7549
	0.2962	0.2684	0.2886	0.3405	0.3082
	1.4	1.25	1.4	1.75	1.5666
	0.2333	0.2083	0.2333	0.2916	0.2611
	0.4204	0.3748	0.4236	0.5140	0.4707
	0.3582	0.3192	0.3575	0.4482	0.4007
S_5	0.5780	0.6531	0.6331	0.6831	0.8548
	0.2359	0.2666	0.2584	0.2788	0.3490
	1.0166	1.4333	1.0166	1.3	1.55
	0.1694	0.2388	0.1694	0.2166	0.2583
	0.3284	0.4450	0.3306	0.3945	0.4372
	0.2592	0.3642	0.2592	0.3324	0.4004
S_6	0.6582	0.6677	0.5830	0.5909	0.7416
	0.2687	0.2725	0.2380	0.2412	0.3027
	1.2666	1.3166	1.0666	1.2166	1.5666
	0.2111	0.2194	0.1777	0.2027	0.2611
	0.3904	0.4087	0.3451	0.3980	0.4775

	0.3234	0.3360	0.2720	0.3092	0.3997
S_7	0.7615	0.6825	0.7257	0.8674	0.7
	0.3109	0.2786	0.2962	0.3541	0.2857
	1.5333	1.3833	1.4	1.8166	1.1
	0.2555	0.2305	0.2333	0.3027	0.1833
	0.4552	0.4195	0.4232	0.5297	0.3219
	0.3925	0.3528	0.3581	0.4658	0.2833
S_8	0.7059	0.6827	0.7604	0.7340	0.7422
	0.2881	0.2787	0.3104	0.2996	0.3030
	1.3566	1.34	1.5566	1.4933	1.4566
	0.2261	0.2233	0.2594	0.2488	0.2427
	0.4187	0.4161	0.4576	0.4635	0.4420
	0.3468	0.3422	0.3979	0.3813	0.3728
S_9	0.6689	0.6350	0.7444	0.6429	0.6837
	0.2730	0.2592	0.3039	0.2624	0.2791
	1.4166	1.3333	1.6166	1.3333	1.35
	0.2361	0.2222	0.2694	0.2222	0.225
	0.4405	0.4238	0.4866	0.4189	0.4103
	0.3607	0.3391	0.4124	0.3396	0.3445
S_{10}	0.6224	0.6350	0.5894	0.6658	0.7421
	0.2541	0.2592	0.2406	0.2718	0.3029
	1.1833	1.2333	0.9166	1.3333	1.5833
	0.1972	0.2055	0.1527	0.2222	0.2638
	0.3781	0.3919	0.3087	0.4233	0.4761
	0.3013	0.3144	0.2345	0.3400	0.4040

In the above Table 6.2.2.9 the minimum distance measure values between each student and the corresponding universities are highlighted in yellow. Accordingly, TMNSDM $d_{TMNS}(\tilde{A}, \tilde{B})$ yields the most frequent and optimal solution, which is indicated in red.

Results and Discussion

Table 6.2.2.10 Comparison

	$d_E(\tilde{A}, \tilde{B})$	$d_{n-E}(\tilde{A}, \tilde{B})$	$d_H(\tilde{A}, \tilde{B})$	$d_{n-H}(\tilde{A}, \tilde{B})$	$d_{SMSVN}(\tilde{A}, \tilde{B})$	$d_{TMNS}(\tilde{A}, \tilde{B})$
S_1	U_1	U_1	U_1	U_1	U_1	U_1
S_2	U_2	U_2	U_2	U_2	U_2	U_2
S_3	U_2	U_2	U_3	U_3	U_3	U_3
S_4	U_2	U_2	U_2	U_2	U_2	U_2
S_5	U_1	U_1	U_1	U_1	U_1	U_1

S_6	U_3	U_3	U_3	U_3	U_3	U_3
S_7	U_2	U_2	U_5	U_5	U_5	U_5
S_8	U_2	U_2	U_2	U_2	U_2	U_2
S_9	U_2	U_2	U_2, U_4	U_2, U_4	U_5	U_2
S_{10}	U_3	U_3	U_3	U_3	U_3	U_3

Comparison of distance measures

According to the comparison Table 6.2.2.10 above, student S_1 made the same option of $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$, $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$, and $d_{SMSVN}(\tilde{A}, \tilde{B})$ which is U_1. The majority judgement is U_1, which also corresponds to the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$.

The decision of $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$, $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$, and $d_{SMSVN}(\tilde{A}, \tilde{B})$ for student S_2 is identical to U_2, which matches with the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$.

For student S_3, the decision to choose University U_5 is consistent across the distance measures $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$, and $d_{SMSVN}(\tilde{A}, \tilde{B})$. However, the measures $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$ yield different results. The majority decision in this case is "U_5", which is also supported by the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$.

The choices made by the student S_4 based on the distance measures $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$, $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$ and $d_{SMSVN}(\tilde{A}, \tilde{B})$ are the same as the choice made by S_4 using the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$ which corresponds to University U_1.

The choices made by the student S_5 based on the distance measures $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$, $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$ and $d_{SMSVN}(\tilde{A}, \tilde{B})$ are the same as the choice made by S_5 using the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$ which corresponds to University U_1.

When it comes to student S_6, the results of $d_E(\tilde{A}, \tilde{B})$, $d_{n-E}(\tilde{A}, \tilde{B})$, $d_H(\tilde{A}, \tilde{B})$, $d_{n-H}(\tilde{A}, \tilde{B})$, and $d_{SMSVN}(\tilde{A}, \tilde{B})$ are all the same, and the result is U_3, which matches with the distance measure $d_{TMNS}(\tilde{A}, \tilde{B})$.

For student S_7, the decision to choose University U_5 is consistent across the distance measures $d_H(\tilde{A},\tilde{B})$, $d_{n-H}(\tilde{A},\tilde{B})$, and $d_{SMSVN}(\tilde{A},\tilde{B})$. However, the measures $d_E(\tilde{A},\tilde{B})$, $d_{n-E}(\tilde{A},\tilde{B})$ yield different results. The majority decision in this case is "U_5", which is also supported by the distance measure $d_{TMNS}(\tilde{A},\tilde{B})$.

The choice made by students S_8, $d_E(\tilde{A},\tilde{B})$, $d_{n-E}(\tilde{A},\tilde{B})$, $d_H(\tilde{A},\tilde{B})$, $d_{n-H}(\tilde{A},\tilde{B})$, and $d_{SMSVN}(\tilde{A},\tilde{B})$ is U_2, which matches with the distance measure $d_{TMNS}(\tilde{A},\tilde{B})$.

Similarly, the choice made by students S_9 is university U_2, the usual choice of $d_E(\tilde{A},\tilde{B})$, $d_{n-E}(\tilde{A},\tilde{B})$, $d_H(\tilde{A},\tilde{B})$, and $d_{n-H}(\tilde{A},\tilde{B})$ but the choice of $d_{SMSVN}(\tilde{A},\tilde{B})$ is different. As a result, for student S_9, the choice reached by the majority is "U_2", which is also the result reached by the distance measure $d_{TMNS}(\tilde{A},\tilde{B})$.

The decision of $d_E(\tilde{A},\tilde{B})$, $d_{n-E}(\tilde{A},\tilde{B})$, $d_H(\tilde{A},\tilde{B})$, $d_{n-H}(\tilde{A},\tilde{B})$, and $d_{SMSVN}(\tilde{A},\tilde{B})$ for student S_{10} is identical to U_3, which matches with the distance measure $d_{TMNS}(\tilde{A},\tilde{B})$.

Therefore, the final judgement is that U_1 is the appropriate university for the development of students S_1 and S_5. Students in S_2, S_4, S_8, and S_9 are perfectly applied for the university "U_2", whereas S_3, S_6, and S_{10} are perfectly applied for the uviversity "U_3”. U_2 is an appropriate university for student S_7's career development.

From Table 6.2.2.10, it is observe that the most frequently occurring shortest distance is found in the tangent metric neutrosophic spherical distance measure. Hence, it can be considered the generalized distance measure among all the distance measure taken for comparision.

This observation leads us to conclude that the tangent distance measure constitutes a generalized metric for both NSS and FNS. It is also capable of handling all types of fuzzy sets, their extensions, and neutrosophic variants.

6.3 Neutrosophic Spherical TOPSIS Methodology: An Approach to Decision-Making

This section presents a novel NS-TOPSIS method that incorporates our defined aggregation operator and the TMNSDM. The method evaluates and ranks alternatives based on multiple criteria to capture subtle variations more precisely. It involves calculating the weights of ÐMs, determining the weights of each criterion from the ÐM, forming a neutrosophic spherical decision matrix, aggregating the weights with the matrix using our defined aggregation operators, and computing the TMNSDM of each alternative from the PIS and NIS. These distances are used to calculate a closeness ratio, where higher scores indicate better alternatives.

An algorithm for the neutrosophic spherical TOPSIS method is developed as follows:

Let $X = \{x_1, x_2, x_3, \ldots . x_m\}$ $(m \geq 2)$ be a discrete set of m feasible alternatives and $C = \{C_1, C_2, C_3, \ldots . C_n\}$ be the weight vector of all criteria and $L = \{l_1, l_2, l_3, \ldots . l_l\}$ represents set of decision makers. The algorithm is replaced with the following seven-step procedure.

Step 1 (Calculate the weights of ÐMs).

Linguistic terms are used to illustrate the significance of the ÐMs $D^l = [T_l, I_l, F_l]$ is the neutrosophic spherical number for l^{th} ÐM ranking. The weight of ÐM can be defined as

$$\lambda_l = \frac{\left[T_l + F_l\left(\frac{T_l}{T_l + I_l}\right)\right]}{\sum_{l=1}^{k}\left[T_l + F_l\left(\frac{T_l}{T_l + I_l}\right)\right]} \tag{6.3.1}$$

where

$$\lambda_l \in [0,1] \text{ and } \sum_{l=1}^{k} \lambda_l = \sqrt{3}/\ \lambda_l \in [0,1] \text{ and } \sum_{l=1}^{k} \lambda_l = 2 \tag{6.3.2}$$

Step 2 (Calculate the weights of criterion).

It is impossible to consider all requirements to be equally qualified. W stands for an importance levels. All ÐMs perspectives on the significance of each criterion must be combined in order to reach W.

Let $w_j^k = \left(T_j^k, I_j^k, F_j^k\right)$ denote the neutrosophic spherical number about the kth ÐM's X_j criteria. The weights of the criteria are determined using the Neutrosophic Spherical Weighted Averaging (NSWA) operator.

$$W = NSWA\, r_\lambda\left(w_j^{(1)}, w_j^{(2)}, \ldots w_j^{(l)}\right) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus \ldots . \oplus \lambda_k w_j^{(l)}$$

$$= \left\{ \begin{array}{c} \left[1 - \prod_{l=1}^{k}\left(1 - T_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}}, \\ \left[1 - \prod_{l=1}^{k}\left(1 - I_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}}, \left[1 - \prod_{l=1}^{k}\left(1 - F_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}} \end{array} \right\} \qquad (6.3.3)$$

Step 3 (Construct Neutrosophic Spherical Decision Matrix (NS D-Mx)).

Each perspective from a set of ÐMs must be integrated into a single view to produce the aggregated Neutrosophic Spherical Decision Matrix (NS D-Mx) model in order to get a precise conclusion.

Let $R^{(1)} = \left(r_{ij}\right)^{(1)}_{m*n}$ be the NS D-Mx of each ÐM. $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is the weight of the ÐM.

$$R = \left(r_{ij}\right)_{m'*n'} \qquad (6.3.4)$$

where $r_{ij} = NSWA\, r_\lambda\left(r_{ij}{}^{(1)}, r_{ij}{}^{(2)}, \ldots r_{ij}{}^{(k)}\right) = \lambda_1 r_{ij}{}^{(1)} \oplus \lambda_2 r_{ij}{}^{(2)} \oplus, \ldots, \oplus \lambda_k r_{ij}{}^{(k)}$

$$= \left\{ \begin{array}{c} \left[1 - \prod_{l=1}^{k}\left(1 - T_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}}, \\ \left[1 - \prod_{l=1}^{k}\left(1 - I_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}}, \left[1 - \prod_{l=1}^{k}\left(1 - F_{\lambda_l}{}^2\right)^{w_j^{(l)}}\right]^{\frac{1}{2}} \end{array} \right\} \qquad (6.3.5)$$

Step 4 (Aggregated weighted neutrosophic spherical decision matrix AWNSD-Mx)).

The AWNSD-Mx is calculated by combining W and R.

$$R \oplus W = \left(T'_{ij}, I'_{ij}, F'_{ij}\right) = \left\{\begin{matrix} x, \left({T_{ij}}^2 + {T_j}^2 - {T_{ij}}^2{T_i}^2\right)^{\frac{1}{2}}, \\ \left({I_{ij}}^2 + {I_j}^2 - {I_{ij}}^2{I_j}^2\right)^{\frac{1}{2}}, \\ \left({F_{ij}}^2 + {F_j}^2 - {F_{ij}}^2{F_j}^2\right)^{\frac{1}{2}} \end{matrix}\right\} \tag{6.3.6}$$

Step 5 (Calculate neutrosophic spherical positive and negative ideal solution).

Let J$_1$ represent a benefit criterion and J$_2$ represent a cost criterion. Neutrosophic spherical positive ideal solution is represented by A^+, while neutrosophic spherical negative ideal solution is represented by A^-. Next, A^+ and A^- are determined as

$$A^+ = \left(r'^+_1, r'^+_2, \dots r'^+_n\right)$$

$$r'^+_j = \left(T'^+_j, I'^+_j, F'^+_j\right), j = 1,2, \dots n \tag{6.3.7}$$

$$A^- = \left(r'^-_1, r'^-_2, \dots r'^-_n\right)$$

$$r'^-_j = \left(T'^-_j, I'^-_j, F'^-_j\right), j = 1,2, \dots n \tag{6.3.8}$$

where

$$T'^+_j = \left\{\left(\max_i\{T'_{ij}\}, j \in J_1\right), \left(\min_i\{T'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.9}$$

$$I'^+_j = \left\{\left(\max_i\{I'^+_{ij}\}, j \in J_1\right), \left(\min_i\{I'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.10}$$

$$F'^+_j = \left\{\left(\min_i\{F'_{ij}\}, j \in J_1\right), \left(\max_i\{F'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.11}$$

$$T'^-_j = \left\{\left(\min_i\{T'_{ij}\}, j \in J_1\right), \max_i\left(\{T'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.12}$$

$$I'^-_j = \left\{\left(\min_i\{I'^+_{ij}\}, j \in J_1\right), \left(\max_i\{I'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.13}$$

$$F'^-_j = \left\{\left(\max_i\{F'_{ij}\}, j \in J_1\right), \left(\min_i\{F'_{ij}\}, j \in J_2\right)\right\} \tag{6.3.14}$$

Step 6 (Obtain the separation measures between the alternatives).

The TMNSDM is used in this work to quantify the separation between potential solutions on the neutrosophic spherical set. Calculated for neutrosophic spherical positive ideal and negative ideal solutions, respectively, are S_i^+ and S_i^-, the separation measures of each choice.

$$S_i^+ = \frac{1}{\sqrt{3}n}\sum_{j=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{ij}' - T_j'^+\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{ij}' - I_j'^+\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{ij}' - F_j'^+\right|\right\}\end{bmatrix} \tag{6.3.15}$$

$$S_i^- = \frac{1}{\sqrt{3}n}\sum_{j=1}^{n}\begin{bmatrix} tan\left\{\frac{\pi}{6}\left|T_{ij}' - T_j'^-\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|I_{ij}' - I_j'^-\right|\right\} \\ + tan\left\{\frac{\pi}{6}\left|F_{ij}' - F_j'^-\right|\right\}\end{bmatrix} \tag{6.3.16}$$

Step 7 (determine the final ranking).

The relative closeness coefficient of an alternative is defined as follows:

$$CC^*{}_i = \frac{S_i^-}{S_i^+ + S_i^-} \tag{6.3.17}$$

where

$$0 \geq C^*{}_i \leq \sqrt{3} \tag{6.3.18}$$

After the relative closeness coefficient of each alternative is determined, alternatives are ranked according to descending order of $CC^*{}_i$.

The Fermatean TOPSIS methodology can be established by replacing the neutrosophic spherical aggregation operator and distance measure with the corresponding Fermatean neutrosophic operator and distance measure, while retaining the same seven-step procedure.

Illustrative Modeling of Neutrosophic Spherical TOPSIS for Decision-Making Applications

In this section, the neutrosophic spherical TOPSIS approach is applied to an illustrative model for selecting the best college for a student.

The set $X = \{X_1, X_2, X_3, X_4\}$ represents four colleges. The set $C = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ includes the criteria that colleges must satisfy: C_1 - Courses, C_2 - Placement, C_3 - Reputation, C_4 - Scholarship, C_5 - Infrastructure, C_6 - Location, C_7 - Transport, C_8 - Internship

The criteria can be ambiguous due to their subjective nature and varying importance. For example, academic reputation and faculty quality are interpreted differently by individuals, and metrics for programs and courses or campus facilities may not be standardized, and each criterion varies according to personal priorities. Research opportunities and faculty quality may overlap or influence each other, complicating consistent assessment and weighting. To address this assumption of ambiguity, using the linguistics variables for the evaluation.

Table 6.3.1 Linguistic terms for rating the alternatives.

Linguistic terms		T	I	F
Extremely High Important	EHI	0.9	0.8	0.5
Very High Important	VHI	0.8	0.8	0.6
High Important	HI	0.8	0.6	0.6
Important	I	0.7	0.7	0.7
Moderately Important	MI	0.8	0.5	0.7
Low Important	LI	0.5	0.6	0.8
Very Low Important	VLI	0.5	0.7	0.8
Extremely Low Important	ELI	0.4	0.8	0.9

The neutrosophic spherical TOPSIS model is used in this section to evaluate the options. The weight of the ĐMs was calculated in Step 1 using the linguistic words in Table 6.3.1 and equation (6.3.1) and the results are shown in Table 6.3.2. Obtain the opinions of decision-makers for each criterion listed in Tables 6.3.3, 6.3.4, and 6.3.5.

Table 6.3.2 weights of the decision makers

Linguistic terms	ĐM1	ĐM2	ĐM3
λ	Absolutely More Importance 0.81	High Importance 0.501	Equally Importance 0.421

Table 6.3.3 Judgments of ĐM1

ĐM1	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
X_1	VHI	HI	AM	I	HI	VHI	EHI	HI
X_2	EHI	AM	HI	AM	VHI	HI	LI	VHI
X_3	LI	I	VHI	ELI	I	I	HI	I
X_4	ELI	VHI	LI	HI	AM	LI	ELI	AM

Table 6.3.4 Judgments of ĐM2

ĐM2	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
X_1	EHI	HI	VHI	EHI	HI	VHI	EHI	I
X_2	LI	VHI	HI	AM	LI	EHI	VHI	LI
X_3	HI	I	I	I	LI	ELI	HI	HI
X_4	ELI	MI	VLI	VLI	VLI	MI	VHI	MI

Table 6.3.5 Judgments of ĐM3

ĐM3	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
X_1	HI	VHI	EHI	MI	EHI	HI	VHI	EHI

X_2	SL	EHI	VHI	SL	SL	VHI	HI	MI
X_3	SL	ELI	HI	HI	HI	MI	MI	MI
X_4	VLI	MI	VHI	MI	ELI	MI	VLI	VLI

Table 6.3.6 displays the important weights of the linguistic terms used to express the criteria determined by ĐMs.

Table 6.3.6 The importance weight of the criterion.

Criterion	**ĐM1**	**ĐM2**	**ĐM3**
C_1	HI	VHI	EHI
C_2	SL	EHI	VHI
C_3	EHI	HI	VHI
C_4	SL	VHI	HI
C_5	HI	MI	MI
C_6	ELI	MI	VLI
C_7	ELI	HI	ELI
C_8	MI	VHI	EL

The second step involves using Table 6.3.2 and Equation (6.3.5) to calculate the decision-makers' perceived weights of the criteria, as presented in Table 6.3.7.

Table 6.3.7 Weight of criterion

C_1	0.9540, 0.8920,0.7753
C_2	0.9086, 0.8920, 0.8695
C_3	0.9648, 0.9128, 0.7549
C_4	0.8706, 0.8554, 0.8833
C_5	0.9105, 0.8431, 0.8431
C_6	0.7795, 0.8844, 0.9569

C_7	0.7619, 0.9128, 0.9628
C_8	0.8956, 0.9033, 0.9096

To combine the ĐMs decisions into one decision, Table 6.3.8 is generated using equation (6.3.3).

Table 6.3.8 R matrix.

ĐM3	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
X_1	0.9564 0.9175 0.7712	0.9384 0.8472 0.7954	0.9540 0.8706 0.8220	0.9483 0.8719 0.7973	0.9540 0.8472 0.7753	0.9384 0.9175 0.7954	0.9751 0.9384 0.7259	0.9436 0.8677 0.8054
X_2	0.9182 0.8833 0.8706	0.9540 0.8706 0.8220	0.9384 0.8472 0.7954	0.9094 0.7259 0.8952	0.8566 0.8833 0.8920	0.9564 0.8920 0.7712	0.8706 0.8554 0.8833	0.9025 0.8718 0.8719
X_3	0.8097 0.8379 0.9128	0.8336 0.8952 0.9217	0.9384 0.8833 0.7954	0.8163 0.8989 0.9423	0.8646 0.8379 0.8855	0.8241 0.8985 0.9283	0.9269 0.8187 0.8187	0.8985 0.8564 0.8564
X_4	0.6176 0.9269 0.9751	0.9384 0.8566 0.8431	0.7953 0.8952 0.9175	0.7572 0.8757 0.9269	0.8452 0.8415 0.9361	0.8706 0.8009 0.9105	0.7795 0.9269 0.9502	0.8890 0.8120 0.8952

An aggregated weighted neutrosophic spherical decision matrix is created in Table 6.3.9 by combining the criteria weights and the aggregated neutrosophic spherical decision matrix with the help of the equations (6.3.5) and (6.3.6).

Table 6.3.9 Aggregated weighted neutrosophic spherical decision matrix.

ĐM3	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
X_1	0.9961 0.9837 0.9156	0.9895 0.9707 0.9541	0.9968 0.9796 0.9276	0.9877 0.9673 0.9591	0.9922 0.9583 0.9405	0.9763 0.9826 0.9844	0.9896 0.9900 0.9826	0.9891 0.9770 0.9692

X_2	0.9929	0.9921	0.9958	0.9788	0.9769	0.9831	0.9478	0.9814
	0.9773	0.9749	0.9761	0.9344	0.9677	0.9775	0.9773	0.9776
	0.9505	0.9596	0.9176	0.9779	0.9700	0.9827	0.9919	0.9790
X_3	0.9844	0.9730	0.9958	0.9587	0.9781	0.9349	0.9700	0.9807
	0.9691	0.9795	0.9815	0.9739	0.9559	0.9788	0.9721	0.9751
	0.9661	0.9814	0.9176	0.9876	0.9682	0.9941	0.9879	0.9767
X_4	0.9718	0.9895	0.9872	0.9469	0.9752	0.9513	0.9140	0.9790
	0.9855	0.9724	0.9833	0.9682	0.9568	0.9601	0.9882	0.9681
	0.9901	0.9640	0.9654	0.9844	0.9819	0.9927	0.9964	0.9827

Step 5 calculate neutrosophic spherical positive and negative ideal solution uses the equations (6.3.9), (6.3.10) and (6.3.11) to compute A^+ and the equation (6.3.12), (6.3.13) and (6.3.14) to calculate A^-. The values of A^+ and A^- are displayed in Tables 6.3.10 .

Table 6.3.10 A^+ and A^-

	A^+	A^-
C_1	0.9961	0.9718
	0.9855	0.9691
	0.9156	0.9901
C_2	0.9921	0.9730
	0.9795	0.9707
	0.9541	0.9814
C_3	0.9968	0.9872
	0.9833	0.9761
	0.9176	0.9654
C_4	0.9877	0.9469
	0.9739	0.9344
	0.9591	0.9876
C_5	0.9922	0.9752
	0.9677	0.9559
	0.9405	0.9819
C_6	0.9831	0.9349
	0.9826	0.9601

	0.9827	0.9941
C_7	0.9896	0.9140
	0.9900	0.9721
	0.9826	0.9964
C_8	0.9891	0.9790
	0.9776	0.9681
	0.9692	0.9827

S^+ is calculated by using the Equation (6.3.15). S^- is calculated using the Equation (6.3.16) and CC^* is calculated using the Equation (6.3.17) in this case. S^+ and S^- stand for separation measurement, while CC^* for ranking of options. The values for S^+, S^- and CC^* for the options are calculated and shown in Table 6.3.11.

Table 6.3.11 Separation measures and the relative closeness coefficient of each alternative.

Alternatives	S^+	S^-	CC^*	Ranking
X_1	0.0875	0.0883	1.0975	1
X_2	0.0942	0.0925	1.0737	2
X_3	0.0899	0.0860	1.0424	3
X_4	0.0948	0.0856	0.9879	4

In this illustrative example, the TOPSIS methodology has been extended by incorporating the proposed aggregation operator and distance measure within the framework of the Neutrosophic Spherical Set (NSS). The integration of these tools enables a more comprehensive handling of uncertainty, indeterminacy, and inconsistency inherent in real-life decision-making environments, such as the selection of an educational institution. By applying the developed approach, the ranking of the alternatives is derived as $X_4 > X_3 > X_2 > X_1$ indicating that among the considered options, X_4 represents the most suitable choice for the student. This approach can be extended to other domains where indeterminacy is a significant challenge, demonstrating its broad applicability and utility in complex decision-making scenarios.