CHAPTER - III

CHAPTER - III

ON πgb-CLOSED SETS IN TOPOLOGICAL SPACES

Chapter III dealt with the concepts of πgb-closed sets. The properties and characterizations of πgb-closed sets were analyzed. The concept of Q-set, extremely disconnected spaces, hyper connected spaces, πgb-$T_{1/2}$ spaces and πgb-spaces were discussed and some of their characterizations were studied. The properties and characterizations of π-open maps, πgb-continuous functions, πgb-irresolute functions, almost πgb-continuous functions and pre b-closed functions were analyzed. The concept of πgb-compact spaces and their behaviour under πgb-continuous and almost πgb-continuous functions were studied. The relations between πgb-compact, b-compact, gb-compact were discussed. The chapter was concluded with the definition of quasi b-normal spaces and its characterizations.

Section 3.1

Preliminaries

Definition 3.1.1

A subset A of a space (X, τ) is called **π-closed set** [54] if A is a finite intersection of regular closed sets.

Definition 3.1.2

A subset A of a space (X,τ) is called

(1) **πg-closed** [23] if cl(A) ⊂U whenever A ⊂ U and U is π-open.

(2) **πgs-closed** [6] if scl(A) ⊂ U whenever A ⊂ U and U is π-open.

(3) **πgsp-closed** [47] if spcl(A) ⊂ U whenever A ⊂ U and U is π-open.

(4) **πgα-closed** [30] if αcl(A) ⊂ U whenever A ⊂ U and U is π-open.

(5) **πgp-closed** [41] if pcl(A) ⊂ U whenever A ⊂ U and U is π-open.

Definition 3.1.3

A function f : (X, τ)⟶(Y, σ) is called **π-continuous** [23] if $f^{-1}(V)$ is π-closed in (X, τ) for every closed set V of (Y, σ).

Definition 3.1.4

A function f : (X, τ)⟶(Y, σ) is called **πg-continuous** [22] if $f^{-1}(V)$ is πg-closed in (X, τ) for every closed set V of (Y, σ).

Definition 3.1.5

A function f : (X, τ)⟶(Y, σ) is called **πgp-continuous** [42] if $f^{-1}(V)$ is πgp-closed in (X, τ) for every closed set V of (Y, σ).

Definition 3.1.6

A function f : (X, τ)⟶(Y, σ) is called **πgs-continuous** [6] if $f^{-1}(V)$ is πgs-closed in (X, τ) for every closed set V of (Y, σ).

Definition 3.1.7

A function f : (X,τ) ⟶(Y, σ) is called **π- irresolute** [6] if $f^{-1}(V)$ is π-closed in (X,τ) for every π-closed V of (Y, σ).

Section 3.2

πgb-closed sets

In this section the properties and characterizations of πgb-closed sets were studied and counter examples were substantiated. The notion of extremely disconnected spaces, hyperconnected spaces, Q-set and πgb-$T_{1/2}$ spaces were discussed. Further the properties of πgb-closure were analyzed.

Definition 3.2.1

A subset A of (X, τ) is called **πgb-closed** [49] if bcl(A) ⊂ U whenever A ⊂ U and U is π open in (X, τ).

Notation 3.2.2

The family of all πgb-closed subsets of the space (X, τ) is denoted by **πGBC(X)**

Remark 3.2.3

Finite union of πgb-closed sets need not be πgb-closed.

Example 3.2.4

Consider X = {a,b,c} , τ = { ϕ , {a} , {b} ,{a,b} , X }. Let A = {a}, B = {b}. A and B are πgb-closed sets but A ∪ B = {a,b} is not πgb-closed.

Remark 3.2.5

Finite intersection of πgb-closed sets need not be πgb-closed.

Example 3.2.6

Consider X = {a,b,c,d} , τ = { ϕ , {a} , {b} ,{a,b} , {a,b,c} ,{a,b,d} ,X }. Let A = {a,b,c}, B = {a,b,d}. A and B are πgb-closed sets but A ∩ B = {a,b} is not πgb-closed.

Theorem 3.2.7

(1) Every closed set is πgb-closed.

(2) Every g-closed set is πgb-closed.

(3) Every α-closed set is πgb-closed.

(4) Every pre-closed set is πgb-closed.

(5) Every b-closed set is πgb-closed.

(6) Every gb-closed set is πgb-closed.

(7) Every πg-closed set is πgb-closed.

(8) Every πgp-closed set is πgb-closed.

(9) Every πgα-closed set is πgb-closed.

(10) Every πgs-closed set is πgb-closed.

(11) Every πgb-closed set is πgsp-closed.

Remark 3.2.8

The converse implications of theorem 3.2.7 need not be true. It can be seen from the following example.

Example 3.2.9

Let X = {a,b,c,d} and τ = { ϕ , {a} , {d} , {a,d} , {c,d} , {a,c,d} , X }. Let A = {c}. Then A is πgb-closed but it is not closed, g-closed, α-closed, pre-closed, b-closed, gb-closed, πg-closed.

Remark 3.2.10

The converse of theorem: 3.2.7 need not be true.

Example 3.2.11

Let X = {a,b,c,d} and τ = { ϕ , {a,b} , {c,d} , {a,c,d} , X }. Let A = {a,b}. Then A is πgb-closed but it is not πgp -closed.

Remark 3.2.12

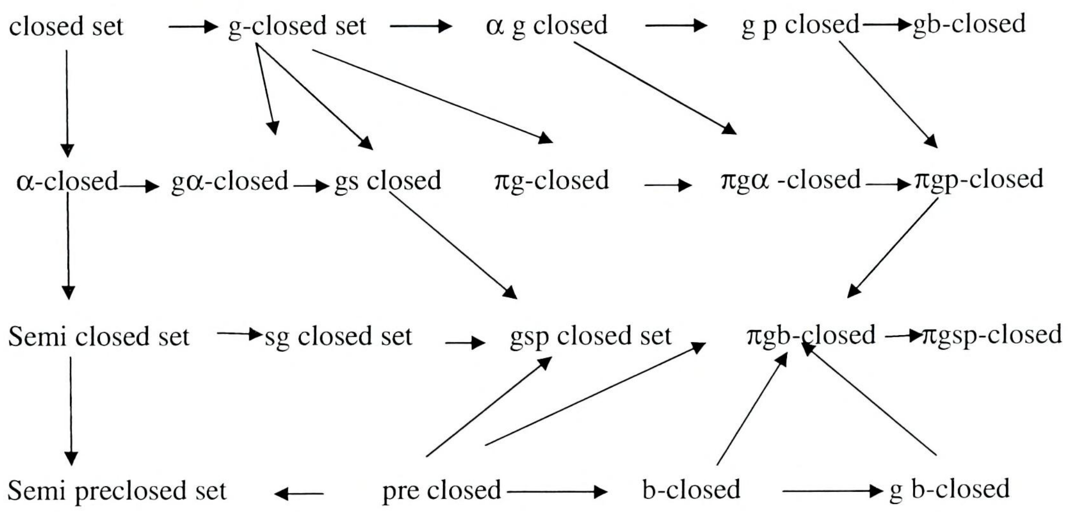
The converse of theorem: 3.2.7 need not be true can be seen from the following example.

Example 3.2.13

Let $X = \{a,b,c,d,e\}$ and $\tau = \{ \phi , \{a,b\} , \{c,d\} , \{a,b,c,d\} , X \}$. Let $A = \{a\}$. Then A is πgb-closed but it is not πgα -closed.

Remark 3.2.14

The above discussions were summarized in the following diagram.



Theorem 3.2.15

If A is π-open and πgb-closed then A is b-closed.

Proof

Let A be π -open and πgb-closed. Then,

bcl(A) ⊂ A and so A = bcl(A). Hence A is b-closed.

Theorem 3.2.16

Let A be πgb-closed in (X, τ). Then bcl(A)-A does not contain any a non empty π-closed set.

Proof

Let F be a non empty π-closed set such that F ⊂ bcl(A) -A. Then A ⊂ (X – F) and (X – F) is π-open. Since A is πgb-closed bcl(A) ⊂ (X – F)

Hence F ⊂ bcl(A) ∩ (X –bcl(A))

Implies that F = ϕ. Which is a contradiction. Therefore bcl(A) – A does not contain any non empty π-closed set.

Theorem 3.2.17

Let A be πgb-closed in (X, τ). Then A is b-closed if and only if bcl(A) - A is π-closed.

Proof

Let A be b-closed. Then bcl(A) = A. This implies bcl(A) - A = ϕ which is π-closed. Assume bcl(A) – A is π-closed. Then by theorem 3.2.12 bcl(A) – A = ϕ. Hence bcl(A) = A. Hence A is b-closed.

Theorem 3.2.18

Let A and B be πgb-closed sets in (X, τ) such that d(A) ⊂ b-d(A) and d(B) ⊂ b-d(B). Then A ∪ B is πgb-closed.

Proof

Let U be a π-open set such that A ∪ B ⊂ U. Since A and B are πgb-closed sets, bcl(A) ⊂ U and bcl(B) ⊂ U.

Since d(A) ⊂ b-d(A) and d(B) ⊂ b-d(B)

cl(A) = bcl(A) and cl(B) = bcl(B).

Hence bcl (A ∪ B) ⊂ cl(A ∪ B) = cl(A) ∪ cl(B)

= bcl(A) ∪ bcl(B) ⊂ U

Hence A ∪ B is πgb-closed.

Theorem 3.2.19

If A is a πgb-closed set and B is any set such that A ⊂ B ⊂ bcl(A), then B is a πgb-closed set.

Proof

Let B ⊂ U and U be π-open. Given A ⊂ B. Then A ⊂ U.

Since A is πgb-closed, bcl(A) ⊂ U

bcl(B) ⊂ bcl(A) ⊂ U.

Hence B is a πgb-closed set.

Theorem 3.2.20

For a subset A of X the following statements are equivalent

(1) A is π-open and πgb-closed.

(2) A is regular open.

Proof

(1)$\Longrightarrow$(2)

Let A be π-open and πgb-closed subset of X.

Then bcl(A) $\subset$A and so int(cl(A)) $\subset$ A.

Since A is open then A is pre-open. Thus A $\subset$ intcl(A).

Therefore int(cl(A)) = A $\Longrightarrow$ A is regular open.

(2)$\Longrightarrow$(1)

Suppose A is regular open.

Since every regular open set is π-open and bcl(A) = A we have bcl(A) $\subset$ A.

Hence A is πgb-closed.

Definition 3.2.21

A subset A of a topological space X is said to be **Q-set** [16] if int(cl(A)) = cl(int(A)).

Theorem 3.2.22

For a subset A of X, the following are equivalent

(1) A is π-clopen.

(2) A is π-open, Q-set, πgb-closed.

Proof

(1)$\Longrightarrow$(2)

Let A be a A is π-clopen subset of X.

Then A is π-closed and π-open. Thus A is closed and open.

Therefore, A is a Q-set .

Since every π-closed set is πgb-closed then A is πgb-closed.

(2)$\Longrightarrow$(1)

Since A is π-open and πgb-closed, A is regular open.

Since A is a Q-set, A = cl(int(A)) = int(cl(A)).

Therefore, A is regular closed $\Longrightarrow$ A is π-closed $\Longrightarrow$A is π-clopen.

Theorem 3.2.23

Let A be a subset of a topological space X. If A is semi-open then pcl(A) = cl(A).

Theorem 3.2.24

A space X is extremely disconnected if and only of every πgb-closed subset of X is πgp-closed.

Proof

Suppose X is extremely disconnected.

Let A be πgb-closed and let U be an π-open set containing A.

Then bcl(A) = A ∪ [int(cl(A)) ∩ cl(int(A))] ⊂ U

⟹ [int(cl(A)) ∩ cl(int(A))] ⊂ U

Since int(cl(A)) is closed, we have

cl(int(A)) ⊂ cl[int(cl(A)) ∩ int(A)] ⊂ [cl(int(cl(A))) ∩ cl(int(A))] ⊂ U.

Thus, pcl(A) = A ∪ cl(int(A)) ⊂ U.

Hence A is πgp-closed.

Conversely suppose every πgb-closed subset of X is πgp-closed.

Let A be regular open subset of X.

Then bcl(A) = A ∪ [int(cl(A)) ∩ cl(int(A))] = A ∪ [A ∩ cl(int(A))] ⊂ A.

Thus A is πgb-closed and so A is πgp-closed.

Since every regular open set semi-open, we have A is semi-open.

Thus cl(A) = pcl(A).

Hence cl(A) ⊂ A.

Therefore A is closed ⟹ X is extremely disconnected.

Definition 3.2.25

A topological space X is said to be **hyperconnected** if the closure of every open subset is X.

Theorem 3.2.26

Let X be a hyperconnected space. Then every πgb-closed subset of X is πgs-closed.

Proof

Assume that X is hyperconnected.

Let A be πgb-closed and let U be an π-open set containing A.

Then, bcl(A) = A ∪ [int(cl(A)) ∩ cl(int(A))] = A ∪ int(cl(A)) = scl(A).

Since bcl(A) = scl(A), scl(A) ⊂ U.

Hence A is πgs-closed.

Theorem 3.2.27

Let A be a πgb-closed set such that cl(A) = X. Then A is πgp-closed.

Proof

Suppose that A is πgb-closed set such that cl(A) = X.

Let U be an π-open set containing A.

Since $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))]$ and $cl(A) = X$,

$bcl(A) = A \cup cl(int(A)) = pcl(A) \subset U$.

Therefore, A is πgp-closed.

Definition 3.2.28

A topological space X is said to be **πgb-$T_{1/2}$ space** [49] if every πgb-closed set is b-closed.

Theorem 3.2.29

(1) X is πgb-$T_{1/2}$

(2) For every subset A of X, A is πgb-open if and only if A is b-open.

Proof

(1)⟹(2)

Let the space X be πgb-$T_{1/2}$ and let A be πgb-open subset of X.

Then (X-A) is πgb-closed and so (X-A) is b-closed.

Hence A is b-open.

Conversely, let A be a b-open subset of X.

Thus (X-A) is b-closed. Since every b-closed set is πgb-closed, (X-A) is πgb-closed.

Therefore, A is πgb-open.

(2)⟹(1)

Let A be πgb-closed subset of X.

Then (X-A) is πgb-open. By hypothesis (X-A) is b-open.

Thus A is b-closed. Since every πgb-closed set is b-closed, X is πgb-$T_{1/2}$.

Definition 3.2.30

The intersection of all πgb-closed sets, each containing a set A in a topological space X is called the **πgb-closure of A** and it is denoted by **πgb-cl(A)**.

Theorem 3.2.31

Let A be a subset of X and x ∈ X. Then x ∈ πgb-cl(A) if and only if V ∩ A ≠ ∅ for every πgb-open set V containing x.

Proof

Assume that there exists a πgb-open set V containing x such that V ∩ A = ∅ .

Since A ⊂ (X-V), πgb-cl(A) ⊂ (X-V) and then x ∉ πgb-cl(A), a contradiction.

To prove the converse, suppose that x ∉ πgb-cl(A).

Then there exists a πgb-closed set F containing A such that x ∉ F.

Since x ∈ (X-F) and (X-F) is πgb-open (X-F) ∩ A = ∅, a contradiction.

Theorem 3.2.32

Let A and B be subsets of X. Then we obtain

(a) πgb-cl(∅) = ∅ , πgb-cl(X) = X.

(b) If A is πgb-closed then πgb-cl(A) = A.

(c) πgb-cl(A) = πgb-cl(πgb-cl(A)).

(d) if A ⊂ B, then πgb-cl(A) ⊂ πgb-cl(B).

(e) πgb-cl(A∩B) ⊂ πgb-cl(A) ∩ πgb-cl(B).

(f) πgb-cl(A∪B) ⊃ πgb-cl(A) ∪ πgb-cl(B).

Remark 3.2.33

The converse of the above theorem 3.2.32 (b) need not be true as seen from the following example.

Example 3.2.34

Let X = {a, b, c, d, e, f} and τ = {X, ∅, {a, b}, {c, d}, {a, b, c, d}}. Let A = {a, b, c, d}. Then πgb-cl(A) = A but A is not πgb-closed.

Remark 3.2.35

The converse of the above theorem 3.2.32 (e) need not be true as seen from the following example.

Example 3.2.36

Let X = {a, b, c, d, e} and τ = {X, ∅, {a}, {e}, {a, e}, {c, d}, {a, c, d}, {c, d, e}, {a, c, d, e}}.

Let A = {a, c, d, e} and B = {b, c, d}.

Then A is not πgb-closed and B is πgb-closed.

Since πgb-cl(A) = X and πgb-cl(B) = B, we have πgb-cl(A) ∩ πgb-cl(B) = B = {b, c, d} but πgb-cl(A∩B) = {c, d}.

Thus πgb-cl(A) ∩ πgb-cl(B) ⊄ πgb-cl(A∩B).

Remark 3.2.37

The converse of the above theorem 3.2.32 (f) need not be true as seen from the following example.

Example 3.2.38

Let X = {a, b, c, d, e} and

τ = {X, ∅, {a}, {e}, {a, e}, {c, d}, {a, c, d}, {c, d, e}, {a, c, d, e}}.

Let A = {a, c, e} and B = {d}.

Then A is not πgb-closed and B is πgb-closed.

Since πgb-cl(A) = A and πgb-cl(B) = B, we have πgb-cl(A) ∪ πgb-cl(B) = {a, c, d, e} but πgb-cl(A∪B) = X.

Thus πgb-cl(A∪B) ⊄ πgb-cl(A) ∪ πgb-cl(B).

Section 3.3

πgb-open sets

The properties and characterizations of πgb-open sets were discussed in this section.

Definition 3.3.1

A set A ⊂ X is called **πgb-open** [49] if its complement is πgb-closed.

The family of all πgb-open subsets of the space (X, τ) is denoted by **πGBO(τ)**.

Theorem 3.3.2

A set A ⊂ X is πgb-open if and only if F ⊂ bint(A) whenever F is π-closed and F ⊂ A.

Proof

Assume that A ⊂ X is πgb-open.

Let F be π-closed such that F ⊂ A.

Then (X – A) ⊂ (X – F).

Since (X – A) is πgb-closed and (X – F) is π-open, bcl(X – A) ⊂ (X – F).

That is (X – bint(A)) ⊂ (X – F).

Hence F $\subset$ bint(A).

Conversely, assume that F is π-closed and F $\subset$ A such that F $\subset$ bint(A).

Let (X – A) $\subset$ U, where U is π-open.

Then (X – U) $\subset$ A and since (X – U) is π-closed,

(X – U) $\subset$ bint(A) $\Longrightarrow$ X – bint(A) $\subset$ U

$\Longrightarrow$ bcl(X–A) $\subset$ U.

Hence (X – A) is πgb-closed and A is πgb-open.

Theorem 3.3.3

If bint(A) $\subset$ B $\subset$ A and A is πgb-open, then B is πgb-open.

Proof

Let bint(A) $\subset$ B $\subset$ A. Then (X – A) $\subset$ (X–B) $\subset$ bcl(X–A).

Since (X–A) is πgb-closed.

(X – A) $\subset$ (X–B) $\subset$ bcl(X–A)

Hence (X–B) is πgb-closed (by Theorem : 3.2.16)

Remark 3.3.4

For any A $\subset$ X, bint(bcl(A)-A) = ϕ.

Theorem 3.3.5

If A $\subset$ X is πgb-closed then bcl(A) – A is πgb-open.

Proof

Let A be πgb-closed and F be π-closed set

Such that F bcl(A) –A by the Remark : 3.3.4 bint(bcl(A) – A) = ϕ

F = ϕ. Hence F $\subset$ bint(bcl(A) –A).

Thus, bcl(A) –A is πgb-open.

Lemma 3.3.6

Let A $\subset$ X. If A is open or dense, then πO(A , τ/A) = {V $\cap$ A : V $\in$ πO(X , τ)}.

Theorem 3.3.7

Let B $\subset$ A $\subset$ X where A is πgb-closed and π-open, then B is πgb-closed relative to A if and only if B is πgb-closed in X.

Proof

Let B $\subset$ A $\subset$ X where A is πgb-closed and π-open set.

And let B be πgb-closed in A such that $B \subset U$ where U is π-open in X.

Since $B \subset A$,

$B = B \cap A \subset U \cap A$, therefore $bcl(B) = bcl_A(B) \subset U \cap A \subset U$

Hence B is πgb-closed in X.

Conversely, let B be πgb-closed in X.

Let $B \subset O$ where O is π-open in A.

Then $O = U \cap A$ where U is π-open in X.

Therefore $B \subset O = U \cap A \subset U$.

Since B is πgb-closed in X, $bcl(B) \subset U$.

Hence $bcl_A(B) = A \cap bcl(B) \subset U \cap A = O$.

Hence B is πgb-closed relative to A.

Hence the proof.

Theorem 3.3.8

Let A be a π-open and πgb-closed set. Then $A \cap F$ is πgb-closed whenever $F \in bC(X)$.

Proof

Since A is πgb-closed and π-open, $bcl(A) \subset A$ and hence A is b-closed.

Therefore $A \cap F$ is b-closed in X.

Hence $A \cap F$ is πgb-closed in X.

Theorem 3.3.9

(1) $BO(\tau) \subset \pi GBO(\tau)$

(2) A space (X, τ) is πgb-$T_{1/2}$ if and only if $BO(\tau) = \pi GBO(\tau)$.

Proof

(1) Let A be b-open, then $(X - A)$ is b-closed so $(X - A)$ is πgb-closed.

Hence A is πgb-open.

Thus $BO(\tau) \subset \pi GBO(\tau)$.

(2) Let (X, τ) be πgb-$T_{1/2}$ space. Let $A \in \pi GBO(\tau)$.

Then $(X - A)$ is πgb-closed and hence $(X - A)$ is b-closed.

Hence $A \in BO(\tau)$. Hence $\pi GBO(\tau) = BO(\tau)$.

Conversely,

Let $BO(\tau) = \pi GBO(\tau)$. Let A be πgb-closed.

Then (X – A) is πgb-open and $(X - A) \in \pi GBO(\tau)$.

Thus, $(X - A) \in BO(\tau)$.

Hence A is b-closed.

Therefore, (X, τ) is πgb-$T_{1/2}$ space.

Proposition 3.3.10

Let A be a subset of (X, τ) and $x \in X$. Then $x \in bcl(A)$ if and only if $V \cap A \neq \phi$ for every b-open set V containing x.

Theorem 3.3.11

For a topological space (X, τ) the following are equivalent.

(1) X is πgb-$T_{1/2}$ space.

(2) Every singleton set is either π-closed or b-open.

Proof

(1) $\Longrightarrow$ (2)

Let X be πgb-$T_{1/2}$ space. Let $x \in X$ and assume that $\{x\}$ is not π-closed.

Then $X - \{x\}$ is not π-open.

Hence $X - \{x\}$ is trivially a πgb-closed set.

Since X is πgb-$T_{1/2}$ space, $X - \{x\}$ is b-closed. Therefore $\{x\}$ is b-open.

(2) $\Longrightarrow$ (1)

Assume every singleton of X is either π-closed or b-open.

Let A be a πgb-closed set. Let $\{x\} \in bcl(A)$.

Case (1)

Let $\{x\}$ be π-closed and assume that $\{x\} \notin A$.

Since $\{x\} \in bcl(A)$-A. We get a contradiction to Theorem : 3.2.16

Therefore, $\{x\} \in A$.

Hence $bcl(A) \subset A$.

Case (2)

Let $\{x\}$ be b-open. Since $\{x\} \in bcl(A)$, $\{x\} \cap A \neq \phi$ Implies $\{x\} \in A$.

Therefore $bcl(A) \subset A$. Hence A is b-closed.

Section 3.4

πgb-continuous and πgb-irresolute functions

In this section properties of πgb-continuous functions, πgb-irresolute functions, almost πgb-continuous functions and their characterizations were analyzed.

Definition 3.4.1

A function $f : X \longrightarrow Y$ is said to be **π-closed** [25] if f(V) is π-closed in Y for every π-closed set V of X.

Definition3.4.2

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called **πgb-continuous** [49]if every $f^{-1}(V)$ is πgb-closed in (X, τ) for every closed set V of (Y, σ).

Definition 3.4.3

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called **πgb-irresolute** [49] if $f^{-1}(V)$ is πgb-closed in (X, τ) for every πgb-closed set V in (Y, σ).

Proposition 3.4.4

Every πgb-irresolute function is πgb-continuous.

Remark 3.4.5

The converse of above theorem need not be true can be seen from the following example.

Example 3.4.6

Consider $X = \{a, b, c\}$, $\tau = \{ \phi, X, \{a\}, \{b\}, \{a, b\} \}$, $\sigma = \{ \phi, X, \{a\}\}$.

Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the identity map.

Then f is πgb-continuous but not πgb-irresolute.

Remark 3.4.7

Composition of two πgb- continuous functions need not be πgb-continuous.

Example 3.4.8

Let $X = \{a, b, c,d\}$, $\tau = \{ \phi, X, \{b\},\{c\}, \{b, c\} \}$, $\sigma = \{ \phi, X, \{a, b, d\} \}$.

$\tau = \{ \phi, X, \{a, d\} \}$.

Let the function $f : (X, \tau) \longrightarrow (X, \sigma)$ be defined by f(a) = a, f(b) = c, f(c) = b, f(d)= d.

And let $g : (X, \sigma) \longrightarrow (X, \tau)$ be defined by g(a) = d, g(b) = c, g(c) = b, g(d) = a.

Then f and g are πgb-continuous but $(g \circ f)$ is not πgb-continuous.

Proposition 3.4.9

(a) Every πgb-continuous function need not be gb-continuous or πg-continuous,

(b) Every πgb-continuous function need not be πgp-continuous,

(c) Every πgb-continuous function need not be πgs-continuous.

Example 3.4.10

Let X = {a, b, c, d, e}, τ = { ϕ, X, {a},{c, d}, {a, c, d},{b, c, d, e} },and Y = {x, y, z, t}, σ = { ϕ, Y, {x, y, z},{t} }.

Let the function f: X ⟶ Y be defined as follows: f(a) = z, f(b) = f(e) = t, f(c) = y and f(d) = x. Then f is πgb-continuous but it is not gb-continuous.

Example 3.4.11

Let X = {a, b, c, d}, τ = { ϕ, X, {a},{b, c}, {a, b, c},{a, b}, {b} },and Y = {x, y, z}, σ = { ϕ, Y, {x, y}, {x, z},{x} }.

Let the function f : X ⟶ Y be defined as follows: f(a) = y, f(b) = f(d) = x, f(c) = z.

Then f is πgb-continuous function which is neither πg-continuous nor πgp-continuous.

Example 3.4.12

Let X be the real numbers with the usual topology and Y = {0,1} with topology σ = { ϕ , Y, {1}}. Let the function f : X ⟶ Y defined as

$$f(x) = \begin{cases} 0, & x \in (0,2) \setminus Q \\ 1, & x \notin (0,2) \setminus Q \end{cases}$$

Then f is πgb-continuous but it is not πgs-continuous.

Theorem 3.4.13

Let f : X ⟶ Y be a function. Then the following statements are equivalent:

1. f is πgb-continuous;
2. The inverse image of every open set in Y is πgb-open in X.

Proof

(1) ⟹ (2)

Let U be a open subset of X. Then (Y – U) is closed.

Since f is πgb-continuous, f^{-1}(Y-U) = X – f^{-1}(U) is πgb-closed in X. Hence f^{-1}(U) is πgb-open in X.

(2) $\Rightarrow$ (1)

Let V be a closed subset of Y. Then (Y – V) is open and by hypothesis (2) $f^{-1}(Y\text{-}V) = X - f^{-1}(V)$ is πgb-open in X.

Hence $f^{-1}(V)$ is πgb-closed.

Therefore, f is πgb-continuous.

Theorem 3.4.14

If $f : X \longrightarrow Y$ is πgb-continuous then f(πgb-cl(A)) $\subset$ cl(f(A)) for every subset A of X.

Proof

Let A be a subset of X. Since f is πgb-continuous and A $\subset$ f^{-1}(cl(f(A))),

πgb-cl(A) $\subset$ f^{-1}(cl(f(A))

Hence f(πgb-cl(A)) $\subset$ cl(f(A)).

Remark 3.4.15

The converse of the above theorem need not be true can be seen from the following example

Example 3.4.16

Let X = {a, b, c, d}, τ = { ϕ, X, {a, b},{d}, {a, b, d} } and σ = { ϕ, X, {d} }.

Let the function $f : (X, \tau) \longrightarrow (X, \sigma)$ be defined as f(a) = c, f(b) = a, f(c) =d, f(d)=b.

Then, f(πgb-cl(A)) $\subset$ cl(f(A)) for every subset A of X.

Since {a, b, c} is closed in (X, σ) but f^{-1}({a, b, c}) = {a, b, d} is not πgb-closed in (X, τ), f is not πgb-continuous.

Theorem 3.4.17

Let $f : X \longrightarrow Y$ be a function. Then the following statements are equivalent:

(1) For each x $\in$ X and each open set V containing f(X) there exists a πgb-open set U containing x such that f(U) $\subset$ V.

(2) f(πgb-cl(A)) $\subset$ cl(f(A)) for every subset A of X.

Proof

(1) $\Rightarrow$ (2)

Let y $\in$ f(πgb-cl(A)) and let V be any open neighborhood of y.

Then there exists an x $\in$ X and a πgb-open set U such that

$f(x) = y$, $x \in U$, $x \in \pi$gb-cl(A) and $f(U) \subset V$.

By lemma: 3.2.31, $U \cap A \neq \phi$ and hence $f(A) \cap V \neq \phi$.

Therefore, $y = f(x) \in cl(f(A))$. Therefore $f(\pi gbcl\ A) \subset cl\ f(A)$.

(2) $\Rightarrow$ (1)

Let $x \in X$ and V be any open set containing f(x).

Let $A = f^{-1}(Y\text{-}V)$.

Since $f(\pi gb\text{-}cl(A)) \subset cl(f(A)) \subset (Y - V)$,

$\pi gbcl\ A \subset f^{-1}(Y\text{-}U) = A$.

Hence $\pi gb\text{-}cl(A) = A$.

Since $x \notin \pi gb\text{-}cl(A)$ there exists a πgb -open set U containing x such that $U \cap A = \phi$ and hence $f(U) \subset f(X\text{-}A) \subset V$.

Theorem 3.4.18

Let X be an extremely disconnected space and $f : X \longrightarrow Y$ be a function. If f is πgb-continuous and π-closed then f is πgb-irresolute.

Proof

Let A be a πgb-closed subset of Y. Then $f^{-1}(A) \subset U$, where U is π-open in X.

So $(X - U) \subset f^{-1}(Y\text{-}A)$.

Hence $f(X - U) \subset (Y\text{-}A)$.

Since f is π-closed, f(X-U) is π-closed.

Since $(Y - A)$ is πgb-open,

$f(X\text{-}U) \subset bint(Y\text{-}A) = Y\text{-}bcl(A)$.

Hence $f^{-1}(bcl(A)) \subset U$.

Since f is πgb-continuous and X is extremely disconnected, $f^{-1}(bcl(A))$ is πgb-closed.

Therefore, $bcl(f^{-1}(bcl(A))) \subset U$ and

hence $bcl(f^{-1}(A)) \subset bcl(f^{-1}(bcl(A))) \subset U$.

Hence $f^{-1}(A)$ is πgb-closed.

Thus f is πgb-irresolute.

Definition 3.4.19

A function $f : X \longrightarrow Y$ is said to be **almost b-continuous** [31] if $f^{-1}(V)$ is b-closed in X for every regular closed set V of Y.

Definition 3.4.20

A function $f: X \longrightarrow Y$ is said to be **almost πgb-continuous** [48] if $f^{-1}(V)$ is πgb-closed in X for every regular closed set V of Y.

Theorem 3.4.21

For a function $f: X \longrightarrow Y$, the following statements are equivalent:

1. f is almost πgb-continuous;
2. $f^{-1}(V)$ is πgb-open in X for every regular open set V of Y;
3. $f^{-1}(int(cl(V)))$ is πgb-open in X for every open set V of Y;
4. $f^{-1}(cl(int(V)))$ is πgb-closed in X for every closed set V of Y.

Proof

(1) $\Rightarrow$ (2)

Let V be a regular open subset of Y.

Since (Y – V) is regular closed and f is almost πgb-continuous $f^{-1}(Y\text{-}V) = X - f^{-1}(V)$ is πgb-closed in X.

Hence $f^{-1}(V)$ is πgb-open in X.

(2) $\Rightarrow$ (1)

Let V be a regular closed subset of Y.

Then (Y – V) is regular open.

By the hypothesis, $f^{-1}(Y\text{-}V) = X - f^{-1}(V)$ is πgb-open in X.

Hence $f^{-1}(V)$ is πgb-closed.

Hence f is πgb-continuous.

(2) $\Rightarrow$ (3)

Let V be an open subset of Y. Then int(cl(V)) is regular open.

By the hypothesis, $f^{-1}(int(cl(V)))$ is πgb-open in X.

(3) $\Rightarrow$ (2)

Let V be a regular open subset of Y. Since V = int(cl(V)) and every regular open set is open then $f^{-1}(V)$ is πgb-open in X.

(3) $\Rightarrow$ (4)

Let V be a closed subset of Y. Then (Y – V) is open. By the hypothesis $f^{-1}(int(cl(Y\text{-}V))) = f^{-1}(Y\text{-}cl(int(V))) = X - f^{-1}(cl(int(V)))$ is

πgb-open in X. Therefore $f^{-1}(cl(int(V)))$ is πgb-closed in X.

(4) $\Longrightarrow$ (3)

Let V be an open subset of Y. Then (Y – V) is closed.

By the hypothesis $f^{-1}(cl(int(Y-V))) = Y - intl(cl(V)) = X - f^{-1}(int(cl(V)))$ is πgb-closed in X.

Therefore $f^{-1}(int(cl(V)))$ is πgb-open in X.

Remark3.4.22

(a) Every πgb-continuous function is almost πgb-continuous.

(b) Every almost b-continuous function is almost πgb-continuous.

Example3.4.23

Let X = {a, b, c, d}, τ = { ϕ, X, {a, b},{d}, {a, b, d} } and σ = { ϕ, X, {d} }.

Let the function f: (X, τ) $\longrightarrow$ (X, σ) be defined as f(a) = c, f(b) = a, f(c) =d, f(d)=b. f is almost πgb-continuous.

Then f(πgb-cl(A)) $\subset$ cl(f(A)) for every subset A of X.

Since for the regular closed set {a, b, c} of (X, σ), we have f^{-1}({a, b, c}) = {a, b, d} is not πgb-closed in (X, τ), f is not πgb-continuous.

Example3.4.24

Let X = {a, b, c, d, e}, τ = { ϕ, X, {a},{c, d}, {a, c, d},{b, c, d, e} }, and Y = {x, y, z, t}, σ = { ϕ, Y, {x, y, z},{t} }.

Let the function f : X $\longrightarrow$Y be defined as follows: f(a) = z, f(b) = f(e) = t, f(c) = y and f(d) = x. f is almost πgb-continuous.

Since for the regular closed set {x, y, z} of (Y, σ), we have f^{-1}({x, y, z}) = {a, c, d} is not b-closed in (X, τ).

Theorem 3.4.25

Let X be a πgb-$T_{1/2}$ topological space. Then f : X $\longrightarrow$Y is almost πgb-continuous if and only if f is almost b-continuous.

Proof

Necessity

Let A be a regular closed subset of Y and f : X $\longrightarrow$Y be an almost πgb-continuous function.

Then $f^{-1}(A)$ is πgb-closed in X. Since X is $\pi gb\text{-}T_{1/2}$ space, $f^{-1}(A)$ is b-closed in X. Hence f is almost b-continuous.

Sufficiency

Suppose that f is almost b-continuous and A be a regular closed subset of Y. Then $f^{-1}(A)$ is b-closed in X.

Since every b-closed set is πgb-closed then

$f^{-1}(A)$ is πgb-closed.

Therefore, f is almost πgb-continuous.

Definition 3.4.26

A function f: X ⟶Y is said to be **pre b-closed** [49] if f(U) is b-closed in Y for each b-closed set U in X.

Theorem 3.4.27

Let f : (X, τ) ⟶ (Y, σ) be π-irresolute and pre b-closed map.

Then f(A) is πgb-closed in Y for every πgb-closed set A of X.

Proof

Let A be πgb-closed in X. Let $f(A) \subset V$ where V is π-open in Y. Then $A \subset f^{-1}(V)$ and A is πgb-closed in X. Implies $bcl(A) \subset f^{-1}(V)$.

Hence $f(bcl(A)) \subset V$.

Since f is pre b-closed, $bcl(f(A)) \subset bcl(f(bcl(A))) = f(bcl(A)) \subset V$.

Hence f(A) is πgb-closed in Y.

Definition3.4.28

A topological space X is a **πgb-space** [49] if every πgb-closed set is closed.

Propoistion3.4.29

Every πgb space is $\pi gb\text{-}T_{1/2}$ space.

Theorem3.4.30

(1) If f is πgb-irresolute and X is $\pi gb\text{-}T_{1/2}$ space, then f is b-irresolute.

(2) If is πgb-continuous and X is $\pi gb\text{-}T_{1/2}$ space, then f is b-continuous.

Proof

(1) Let V be b-closed in Y. Since f is πgb-irresolute, $f^{-1}(V)$ is πgb-closed in X. Since X is $\pi gb\text{-}T_{1/2}$ space, $f^{-1}(V)$ is b-closed in X. Hence f is b-irresolute.

(2) Let V be closed in Y. Since f is πgb-continuous, $f^{-1}(V)$ is πgb-closed in X.

By assumption, it is b-closed. Therefore f is b-continuous.

Definition3.4.31

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is **π-open map** [25] if f(F) π-open in Y for every π-open set in X.

Theorem 3.4.32

If the bijective function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a b-irresolute and π-open map, then f is πgb-irresolute.

Proof

Let V be πgb-closed in Y. Let $f^{-1}(V) \subset U$ where U is π-open in X.

Then $V \subset f(U)$ and f(U) is π-open implies $bcl(V) \subset f(U)$.

Since f is b-irresolute, $(f^{-1}(bcl(V)))$ is b-closed.

Hence $bcl(f^{-1}(V)) \subset bcl(f^{-1}(bcl(V))) = f^{-1}(bcl(V)) \subset U$.

Therefore, f is πgb-irresolute.

Theorem 3.4.33

If $f : X \longrightarrow Y$ is π-open, b-irresolute, pre b-closed surjective function and if X is πgb-$T_{1/2}$ space, then Y is πgb-$T_{1/2}$ space.

Proof

Let F be a πgb-closed set in Y.

Let $f^{-1}(F) \subset U$ where U is π-open in X.

Then $F \subset f(U)$ and F is a πgb-closed set in Y implies $bcl(F) \subset f(U)$.

Since f is b-irresolute, $bcl(f^{-1}(F)) \subset bcl(f^{-1}(bcl(F))) = f^{-1}(bcl(F)) \subset U$.

Therefore $f^{-1}(F)$ πgb-closed in X. Since X is πgb-$T_{1/2}$ space,

$f^{-1}(F)$ is b-closed in X. Since f is pre b-closed, $f(f^{-1}(F)) = F$ is b-closed in Y.

Hence Y is πgb-$T_{1/2}$ space.

Section 3.5

πgb-compact spaces

This section includes the definition, properties and characterizations of πgb-compact spaces and nearly compact spaces.

Definition 3.5.1

A collection{ U_i : i ∈I} of πgb-open sets in a topological space X is called a **πgb-open cover** of a subset A of X if A ⊂ { U_i : i ∈I} holds.

Definition 3.5.2

A topological space X is ***π*gb-compact** if every *π*gb-open cover of X has a finite subcover.

Definition 3.5.3

A subset A of a topological space X is said to be ***π*gb-compact relative to X** if, for every collection { U_i : i ∈I} of *π*gb-open subsets of X such that A ⊂ ∪ { U_i : i ∈I} there exists a finite subset I_0 ⊂ I such that A ⊂ ∪ { U_i : i ∈ I_0 }.

Definition 3.5.4

A subset A of a topological space X is said to be ***π*gb-compact** if A is *π*gb-compact as a subspace of X.

Theorem 3.5.5

Every πgb-closed subset of a πgb-compact space is πgb-compact space relative to X.

Proof

Let A be a πgb-closed subset of a *π*gb-compact space X.

Let { U_i : i ∈I} be a *π*gb-open cover of X.

Thus A ⊂ $\bigcup_{i\in I} U_i$ and then (X-A) ∪ ($\bigcup_{i\in I} U_i$) = X.

Since X is *π*gb-compact, there exists a finite subset I_0 of I such that

(X-A) ∪ ($\bigcup_{i\in I_0} U_i$) = X.

Then A ⊂ $\bigcup_{i\in I_0} U_i$.

Hence A is *π*gb-compact relative to X.

Definition 3.5.6

A topological space (X, τ) is said to be **nearly compact space** if every cover X by regular open sets has a finite subcover.

Theorem 3.5.7

The surjective πgb-continuous image of a πgb-compact space is compact.

Proof

Let { U_i : i ∈I} be any cover of Y by open subsets.

Since f is πgb-continuous, then {$f^{-1}(U_i)$: i ∈I} is πgb-open cover of X.

By πgb-compactness of X, there exists a finite subset I_0 of I such that X = $\bigcup_{i\in I_0} f^{-1}(U_i)$.

Since f is surjective, we obtain Y= $\bigcup_{i\in I_0} U_i$.

Hence Y is compact.

Theorem 3.5.8

The surjective almost πgb-continuous image of a πgb-compact space is nearly compact.

Proof

Let { U_i : i ∈I} be any cover of Y by regular open subsets.

Since f is almost πgb-continuous, then {$f^{-1}(U_i)$: i ∈I} is πgb-open cover of X.

By πgb-compactness of X, there exists a finite subset I_0 of I such that X = $\bigcup_{i\in I_0} f^{-1}(U_i)$.

Since f is surjective, we obtain Y= $\bigcup_{i\in I_0} U_i$.

Hence Y is nearly compact.

Theorem 3.5.9

If f : X→ Y is πgb-irresolute and a subset A of X is πgb-compact relative to X, then the image f(A) is πgb-compact relative to Y.

Proof

Let { U_i : i ∈I} be any collection of πgb-open subsets of Y such that f(A) ⊂ ∪ { U_i : i ∈I}.

Then A ⊂ ∪ {$f^{-1}(U_i)$: i ∈I} holds.

Since by hypothesis A is πgb-compact relative to X,

there exists a finite subset I_0 of I such that A ⊂ ∪ {$f^{-1}(U_i)$: i ∈ I_0}.

Therefore, we have f(A) ⊂ ∪ { U_i : i ∈ I_0 }, which shows that f(A) is πgb-compact relative to Y.

Definition 3.5.10

A space X is said to be

(1) **πgp-compact** if every πgp-open cover of X has a finite subcover.

(2) **gb-compact** if every gb-open cover of X has a finite subcover.

(3) **b-compact** if every b-open cover of X has a finite subcover.

Remark 3.5.11

Since every regular open set is open, b-open, gb-open, πgb-open and πgp-open, for a space X the following implications hold.

πgb-compact ⟶ gb-compact ⟶ b-compact ⟶ compact ⟶ nearly compact

↓

πgp-compact

Definition 3.5.12

A function $f : X \to Y$ is said to be **πgb-open** if f(U) is πgb-open in Y for every πgb-open set U of X.

Theorem 3.5.13

If $f : X \to Y$ is πgb-open bijection and Y is πgb-compact space then X is a πgb-compact space.

Proof

Let $\{ U_i : i \in I\}$ be a πgb-open cover of X, thus $X = \bigcup_{i\in I} U_i$.

Then $Y = f(X) = f(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f(U_i)$.

Since f is πgb-open, for each $i \in I$, $f(U_i)$ is πgb-open.

By πgb-compactness of Y, there exists a finite subset I_0 of I such that $Y = \bigcup_{i\in I_0} f(U_i)$.

Therefore, $X = f^{-1}(Y) = f^{-1}(\bigcup_{i\in I_0} f(U_i)) = \bigcup_{i\in I_0} f^{-1}(f(U_i)) = \bigcup_{i\in I_0} U_i$.

Hence X is πgb-compact.

Theorem 3.5.14

If $f : X \to Y$ is πgb-irresolute bijection and X is πgb-compact space then Y is a πgb-compact space.

Proof

Let $\{ U_i : i \in I\}$ be a πgb-open cover of Y, thus $Y = \bigcup_{i\in I} U_i$.

Then $X = f^{-1} (Y) = f^{-1} (\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}(U_i)$.

Since f is πgb-irresolute, for each $i \in I$, $f^{-1}(U_i)$ is πgb-open set.

By πgb-compactness of X, there exists a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(U_i)$.

Therefore, $Y = f(X) = f(\cup_{i\in I_0} f^{-1}(U_i)) = \cup_{i\in I_0} f(f^{-1}(U_i)) = \cup_{i\in I_0} U_i$.

Hence Y is πgb-compact.

Section 3.6

Quasi-b-normal spaces

In this section properties and characterizations of quasi b-normal spaces and almost continuous functions were analyzed.

Definition 3.6.1

A space X is said to be **quasi-normal** [54] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$.

Definition 3.6.2

A space X is said to be **b-normal** [25] if for every pair of disjoint closed subsets A, B of X, there exist disjoint b-open subsets U, V of X such that $A \subset U$ and $B \subset V$.

Definition 3.6.3

A space X is said to be **quasi-b-normal** [48] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint b-open subsets U, V of X such that $A \subset U$ and $B \subset V$.

Remark 3.6.4

For a topological space X the following implications holds:

normal ⟶ quasi-normal ⟶ b-normal ⟶ quasi-b-normal

Theorem 3.6.5

The following statements are equivalent for a space X.

(1) X is qausi-b-normal.

(2) For any disjoint π-closed subsets A, B of X, there exist disjoint gb-open subsets U, V of X such that $A \subset U$ and $B \subset V$.

(3) For any closed set A and any π-open set B containing A, there exist a gb-open set U such that $A \subset U \subset bcl(U) \subset B$.

(4) For any disjoint π-closed subsets A, B of X, there exist disjoint πgb-open subsets U, V of X such that $A \subset U$ and $B \subset V$.

(5) For any π-closed set A and any π-open set B containing A, there exist a πgb-open set U such that $A \subset U \subset bcl(U) \subset B$.

Proof

(1) $\Rightarrow$(2)

Suppose X is quasi-b-normal. Since every b-open set is gb-open, by definition for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$.

(2) $\Rightarrow$(3)

Let A be any π-closed subset of X and B any π-open subset of X such that $A \subset B$.

Then, A and (X-B) are disjoint π-closed subsets of X.

Therefore, there exist disjoint gb-open sets U and V such that $A \subset U$ and $(X\text{-}B) \subset V$.

By the definition of gb-open sets,

$(X\text{-}B) \subset bintV$ and $U \cap bintV = \emptyset$.

Therefore, we obtain $bcl(U) \cap bint(V) = \emptyset$

Hence $A \subset U \subset bcl(U) \subset B$.

(3) $\Rightarrow$(4)

Let A and B be any disjoint π-closed subsets of X.

Then $A \subset (X\text{-}B)$ and (X-B) is π-open.

Hence there exist a gb-open subset G of X such that $A \subset G \subset bcl(G) \subset (X\text{-}B)$.

Since every gb-open set is πgb-open, G is πgb-open and (X-bcl(G)) is b-open, thus (X-bcl(G)) is πgb-open.

Now let V = (X-bcl(G)).

Then, G and V are disjoint πgb-open subsets of X such that $A \subset G$ and $B \subset V$.

(4) $\Rightarrow$(5)

Let A be any closed subset of X and B any π-open subset of X such that $A \subset B$.

Since every closed set is π-closed, A and (X-B) are disjoint π-closed subsets of X.

Therefore, there exist disjoint πgb-open sets U and V such that $A \subset U$ and $(X\text{-}B) \subset V$.

By the definition of gb-open sets,

(X-B) ⊂ bintV and U ∩ bintV = ∅.

Therefore, we obtain bcl(U) ∩ bint(V) = ∅

Hence A ⊂ U ⊂ bcl(U) ⊂ B.

(5) ⟹(1)

Let A and B be any disjoint π-closed subsets of X.

Then A ⊂ (X-B) and (X-B) is π-open.

Hence there exist a πgb-open subset G of X such that A ⊂ G ⊂ bcl(G) ⊂ (X-B).

Now let U = bint(G) and V = (X-bcl(G)).

Then, U and V are disjoint b-open subsets of X such that A ⊂ U and B ⊂ V.

Therefore, X is quasi-b-normal.

Definition 3.6.6

A function f : X ⟶ Y is said to be **almost πgb-closed** [48]if for each regular closed subset F of X, f(F) is πgb-closed subset of Y.

Theorem 3.6.7

A surjection f : X ⟶ Y is almost πgb-closed if and only if for each subset G of Y and each U ∈ RO(X) containing f^{-1}(G), there exists a πgb-open subset V of Y such that G ⊂ V and f^{-1}(V) ⊂ U.

Proof

Suppose f is almost πgb-closed.

Let G be a subset of Y and U ∈ RO(X) containing f^{-1}(G),

If V = (Y-f(X-U)), then V is a πgb-open set of Y such that G ⊂ V and f^{-1}(V) ⊂ U.

To prove the converse, let F be any regular closed set of X.

Then (X-F) ∈ RO(X) and f^{-1}(Y-f(F)) ⊂ (X-F).

There exist a πgb-open set V of Y such that (Y-f(F)) ⊂ V and f^{-1}(V) ⊂ (X-F).

Therefore, (Y-V) ⊂ f(F) and F ⊂ f^{-1}(Y-V).

Hence, we obtain f(F) = (Y-V) and f(F) is πgb-closed in Y.

Thus f is almost πgb-closed.

Theorem 3.6.8

Let f : X ⟶ Y be a continuous, almost πgb-closed surjection. If X is normal, then Y is quasi-b-normal.

Proof

Let A and B be disjoint π-closed subsets of Y.

Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X.

By the normality of X, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$.

Let G = int(cl(U)) and H = int(cl(V)).

Then, G and H are disjoint regular open subsets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$.

By theorem 3.6.7 there exist πgb-open subsets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$.

Since $G \cap H = \emptyset$ and f is surjective, $K \cap L = \emptyset$.

It follows that Y is quasi-b-normal.

Theorem 3.6.9

Let $f : X \longrightarrow Y$ be a π-irresolute , almost πgb-closed surjection. If X is quasi-normal, then Y is quasi-b-normal.

Proof

Let A and B be disjoint π-closed subsets of Y.

Since f is π-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed subsets of X.

By the quasi-normality of X, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$.

Let G = int(cl(U)) and H = int(cl(V)).

Then $f^{-1}(A) \subset U \subset G$, $f^{-1}(B) \subset V \subset H$, $G \cap H = \emptyset$ and $G, H \in RO(X)$.

Since f is almost πgb-closed, thus there exist πgb-open subsets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$.

Since f is surjective, we have $K \cap L = \emptyset$.

This shows that Y is quasi-b-normal.