Chapter - I

CHAPTER I

PRELIMINARY DEFENITIONS AND RESULTS

Definition: 1.1

A **semiring** consists of a set S and two binary operations on S, addition (+) and multiplication (•), such that:

(i) (S, +) is an Abelian monoid (identity denoted by 0)

(ii) (S, •) is a monoid ((identity denoted by 1)

(iii) multiplication distributes over addition:

(iv) $s \bullet 0 = 0 \bullet s = 0$ for all $s \in S$; and

(v) $1 \neq 0$.

Note: 1.2

Usually S denotes both the semiring and the set. Thus all rings with identity are semirings.

Definition: 1.3

A semiring S is **commutative** if (S, •) is abelian.

Definition: 1.4

A semiring S is **antinegative** if 0 is the only element to have an additive inverse.

Note: 1.5

From the definition no ring is antinegative expect {0}. Algebraic terms such as **unit** and **zero-divisor** are defined for semirings as for rings.

Definition: 1.6

Let $\mathbb{C}$ be any chain with lower bound 0 and upper bound 1, then $(\mathbb{C},+,\bullet) = (\mathbb{C}, \max, \min)$ is a semiring, a **chain semiring**.

Definition: 1.7

If $\mathbb{F}$ is the real interval [0, 1], then ($\mathbb{F}$, max, min) is a semiring, the **fuzzy semiring.**

Definition: 1.8

Let S be a semiring S is called an **antiring** if $a + b = o$ implies that $a= b=o$ for any $a, b \in S$.

Example: 1.9

If $\mathbb{P}$ is any subring with identity, of $\mathbb{R}$, the reals (under real addition and multiplication) and $\mathbb{P}_+$ denotes the non negative part of $\mathbb{P}$, then $\mathbb{P}_+$ is a semiring. In particular, $\mathbb{Z}_+$ (resp. $\mathbb{R}_+$) the nonnegative integers (resp., reals), is a semiring.

Definition: 1.10

Let $\mathbb{B} = \{0, 1\}$, then $(\mathbb{B}, +, \bullet)$ is a Boolean algebra if

$0+0 = 0\bullet0 = 0\bullet1 = 1\bullet0 = 0$ and

$1+1 = 1\bullet1 = 1$

Hence Boolean algebra is a semiring.

Note: 1.11

Algebraic terms such as units and zero divisors are defined for semirings as rings, 0 and 1 denote the additive identity and the multiplicative identity respectively.

Notation: 1.12

Let S be a semiring and a $\in$ S. We denote by a^k the k^{th} power of a and by ka the sum a+ a+ a+ ....+a (k times) for any positive integer k.

Definition: 1.13

For a $\in$ S, a is called an **idempotent element** in S if a^2 = a. The set of all idempotent elements in S is denoted by I(S), i.e., I(S) = {a $\in$ S: a^2 = {a}.

Result: 1.14

If S is an incline and I(S) is the set of all idempotent elements in S, then I(S) is a distributive lattice.

Definition: 1.15

An element a $\in$ S is called **invertible** in S if there exists and element b $\in$ S such that ab = ba = 1. The element b is called an **inverse** of a in S. The inverse of a in S is unique. The inverse of a in S is denoted by a^{-1}. Let U(S) denote the set of all invertible elements in S. Then U(S) forms a group with respect to the multiplication of the semiring S.

Definition: 1.16

A commutative semiring S is called an **incline** if a+1 = 1 for all a $\in$ S.

Result: 1.17

Any incline is a commutative antiring.

Proof:

If S is an incline and a, b $\in$ S such that a + b = 0, then

a = a + 0 = a + (a +b) = (a+a)+b = (1+1) a+ b = a+ b = 0 and so b = 0.

Result: 1.18

If S is an incline then U(S) = {1}.

Proof:

If S is an incline and $a \in U(S)$, then there exists and element b is S such that ab = 1 and so a= a•1 = a(1+b) = a+ab = a+1 = 1. Then U(S) = {1}.

Definition: 1.19

For a fixed positive integer k, let $\mathbb{B}_k$ be the (general) Boolean algebra of subsets of a k-element set S_k and σ_1, σ_2, ……..σ_k denote the singleton subsets of S_k. Union is denoted by + and intersection by juxtaposition; 0 denotes the null set and 1 the set S_k. Under these two operations, $\mathbb{B}_k$ is a commutative semiring (that is, only 0 has an additive inverse) all of its elements, except 0 and 1, are zero-divisors. In particular, if k = 1, $\mathbb{B}_1$ is called the **binary Boolean algebra**.

Notation: 1.20

Let $M_{m,n}(S)$ denote the set of all m ×n matrices with entries in a semiring S. If m= n, we use the notation $M_n(S)$ instead $M_{n,n}(S)$. The matrix I_n is the n × n identity matrix $J_{m,n}$ is the m ×n matrix all of whose entries are 1, and $O_{m,n}$ is the m ×n zero matrix.

Definition: 1.21

A zero- one matrix in $M_{m,n}(S)$ with only one entry equal to 1 is called **a cell**. If the nonzero entry occurs in the i^{th} row and the j^{th} column, we denote the cell by E_{ij}. We say E_{ij} is **an off-diagonal cell**; $E_{i,i}$ is a **diagonal cell**. For any cells E_{ij} and $E_{u,v}$, we have $E_{ij}\,E_{u,v} = E_{i,v}$ or O_n according as $j = u$ or $j \neq u$.

Note: 1.22

Let $\underline{n}$ denote the set $\{1, 2, \ldots\ldots n\}$ for any positive integer n and S_n denote the symmetric group on the set $\underline{n}$.

Definition: 1.23

For $A \in M_n(S)$ if $a_{ij} = 0$ for all i and j provided that $i \neq j$ then A is called a **diagonal matrix** and denoted by diag $(a_{11}, a_{12}, \ldots\ldots, a_{nm})$. In particular if

$$a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

for all $i, j \in \underline{n}$, then A is called the **identity matrix** and denoted by I_n; if $a_{ij} = 0$ for all $i, j \in \underline{n}$, then A is called the **zero matrix** and denoted by O_n.

Definition: 1.24

Let $V_n(S)$ denote the set of all $m \times n$ matrices over S. For $\alpha \in V_n(S)$, we denote by α_i the element of S corresponding to the i^{th} co-ordinate of α. If $\alpha_i = 1$ for all $i \in \underline{n}$ then α is called the **universal vector** of $V_n(S)$ and denoted by e; if $\alpha_i = 0$ for all the $i \in \underline{n}$, the α is called the **zero vector** and denoted by 0. For any $i \in \underline{n}$, we denote by e_i the vector in $V_n(S)$ with 1 as the i^{th} co-ordinate, 0 otherwise.

Notation: 1.25

For $A \in M_n(S)$. We denote by $A(i_1 \ldots\ldots i_r \mid j_1 \ldots\ldots j_r)$ the $(n-r) \times (n-r)$ submatirix of A obtained from A by deleting rows $i_1, \ldots\ldots, i_r$ and columns $j_1, \ldots j_r$, where $\{i_1, i_2, \ldots\ldots i_r\}$ and $\{j_1, \ldots j_r\} \subseteq \underline{n}$ with $i_s \neq i_t$ $(s \neq t)$ and $j_s \neq j_t$ $(s \neq t)$.

Definition: 1.26

Given A, B $\in M_{m\times n}(S)$ and C $\in M_{n\times l}(S)$, we define:

$$A+B=\left(a_{ij}+b_{ij}\right)_{m\times n};$$

$$AC=\left(\sum_{k\in \underline{n}} a_{ik}C_{kj}\right)_{m\times l};$$

$$\lambda A=\left(\lambda a_{ij}\right)_{m\times n}, \lambda \in S.$$

Definition: 1.27

For A $\in M_n(S)$, the powers of A are defined as follows: $A^0 = I_n$, $A^l = A^{l-1} \bullet A$, where l is any positive integer. The (i, j)th entry of A^l is denoted by $a_{ij}^{(l)}$.

Remark: 1.28

The following properties are derived immediately from these definitions:

(i) $M_n(S)$ is an Abelian monoid with the identity element O_n with respect to the matrix addition;

(ii) $M_n(S)$ is another monoid with the identity element I_n, with respect to the matrix multiplication;

(iii) The distributivity holds, i.e. A(B+C) =AB+ AC and (A+B)C = AC+BC for any A, B, C$\in M_n(S)$.

(iv) The absorption property holds, i.e., $O_nA = AO_n = O_n$ for all $A \in M_n(S)$. Therefore, $(M_n(S), +, \bullet, O_n, I_n)$ is a semining.

Note: 1.29

For any matrix A, A^T denoted by the transpose of A.

Definition: 1.30

A matrix $P \in M_n(S)$ is called a permutation matrix if and only one entry of its every row and every column is 1 and the other entries are 0.

Definition: 1.31

A matrix $A \in M_n(S)$ is called **idempotent** if $A^2 = A$. The matrices O_n, I_n and J_n are clearly idempotents in $M_n(S)$. Furthermore we can easily show that all diagonal cells are idempotents, but all off-diagonal cells are not idempotents.

Definition: 1.32

A matrix A in $M_n(S)$ is said to be **right invertible** (left invertible) in $M_n(S)$ if $AB = I_n$ ($BA = I_n$) for some $B \in M_n(S)$. The matrix B is called a **right inverse** (left inverse) of A in $M_n(S)$. If A is both right and left invertible in $M_n(S)$ then it is called **invertible** in $M_n(S)$.

Note: 1.33

If A is invertible then its right inverse coincides with its left inverse which is called its inverse. The inverse of A is denoted by A^{-1}.

Note: 1.34

Any permutation matrix P is invertible and its inverse is P^T.

Notation: 1.35

Let $GL_n(S)$ denote the set of all invertible matrices in $M_n(S)$. Then $GL_n(S)$ forms a group with respect to the multiplication of $M_n(S)$.

Result: 1.36

Let A, B $\in M_n(S)$. If $AB = I_n$ then $BA = I_n$.

Result: 1.37

(i) Let $a_i \in S$, $i = 1, 2 \ldots\ldots, n$. Then the matrix diag $(a_1, a_2, a_3, \ldots\ldots a_n)$ is invertible in $M_n(S)$ iff a_i is invertible in S for each $i \in \underline{n}$.

(ii) The inverse of any invertible diagonal matrix in $M_n(S)$ is a diagonal matrix.

Definition: 1.38

Let $A \in M_{m\times n}(S)$ and $m \leq n$. Then the permanent per A of A is defined by

Per A = $\sum_{\sigma\in S_{m,n}} a_{1\sigma(1)} a_{2\sigma(2)} \ldots\ldots\ldots\ldots a_{m\sigma(m)}$, where $S_{m,n}$ is the set of all injective mappings from the set $\underline{m}$ to the set $\underline{n}$.

Result: 1.39

Let $A \in M_n(S)$. Then

(i) Per A = $\Sigma_{j\in\underline{n}}\, a_{ij}$ Per A (i | j) for any $i \in \underline{n}$;

(ii) Per A = $\Sigma_{i\in\underline{n}}\, a_{ij}$ Per A (i | j) for any $j \in \underline{n}$;

Result: 1.40

Let S be a commulative antiring. Let A, B$\in M_n(S)$. If $AB = I_n$, then

(i) $a_{ij}\, a_{ik} = a_{ji}\, a_{ki} = b_{ij}\, b_{ik} = b_{ji}\, b_{ki} = 0$ for any i, j, $k \in \underline{n}$ with $j \neq k$;

(ii) $a_{ik}\, b_{kj} = a_{ki}\, b_{jk} = 0$ for any i, j, $k \in \underline{n}$ with $i \neq j$;

(iii) $(\Sigma_{k\in\underline{n}}\, a_{ik})\, (\Sigma_{\ell\in\underline{n}}\, b_{\ell i}) = (\Sigma_{k\in\underline{n}}\, a_{kj})\, (\Sigma_{\ell\in\underline{n}}\, b_{j\ell}) = 1$ for any i, $j \in \underline{n}$.

Definition: 1.41

Let $A \in M_n(S)$. The **adjoint matrix** adj $A \in M_n(S)$ of A is the matrix whose $(i, j)^{th}$ entry is per $A\,(j \mid i)$ for all $i, j \in \underline{n}$.

Definition: 1.42

A **bideterminant** of a matrix $A = [a_{i,\ j}] \in M_n(S)$ is a pair bidet $A = (\|A\|^{+}, \|A\|^{-})$, where

$$\|A\|^{+} = \sum_{\sigma \in A_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots\cdots\cdots a_{n\sigma(n)}\ ,$$

$$\|A\|^{-} = \sum_{\sigma \in S_n / A_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots\cdots\cdots a_{n\sigma(n)}\ ,$$

where S_n denotes the symmetric group on the set {1,......,n), A_n denotes its subgroup of even permutations.